How Mathematicians Use Diagrams to Construct Proofs

1. Introduction

One of the primary goals of undergraduates' upper-level mathematics courses is to improve their abilities to construct formal proofs. Unfortunately, numerous studies reveal that mathematics majors have serious difficulties with this task (e.g., Moore, 1994; Weber, 2001). While there has been extensive research documenting undergraduates' difficulties with proof construction, research on how undergraduates can or do successfully construct proofs has been limited.

One approach that several researchers recommend is for students to base their formal proofs on diagrams and other informal arguments (e.g., Gibson, 1998; Raman, 2003). These recommendations are supported by the theoretical advantages afforded by visual reasoning (Alcock & Simpson, 2004; Gibson, 1998), successful illustrations of students using visual arguments as a basis for formal arguments (e.g., Alcock & Weber, 2010; Gibson, 1998), and the fact that mathematicians claim to use diagrams extensively in their own work.

However, for this to be useful pedagogical advice, more research is needed on *how* students can effectively use diagrams in their proof construction. Researchers such as Pedemonte (2007) and Alcock and Weber (2010) have noted that students find it difficult to translate an informal visual argument a formal proof. Also, several studies have failed to find a correlation between students' propensity to use diagrams and their success in proof-writing (e.g., Alcock & Simpson, 2004; Alcock & Weber, 2010; Pinto & Tall, 1999). If undergraduates are to successfully use diagrams as a basis for their proofs, they need to have a better understanding of how diagrams can be useful in proof construction and the skills needed to express and justify inferences drawn from a diagram in the language of formal mathematical proof. The goal of this presentation is to investigate these issues by analyzing ten mathematicians' behavior as they complete a non-trivial proof construction task that invites the construction and use of a graph.

2. Theoretical assumptions

This paper is based on the assumption that a goal of instruction in advanced mathematics courses is to lead students to reason like mathematicians with respect to proof (a position endorsed by Harel & Sowder, 2007), realizing these goals requires having a more accurate understanding of mathematical practice than we currently have (a position argued by the RAND Mathematics Study Panel, 2003), and we can improve our understanding of mathematical practice by carefully observing mathematicians engaged in mathematical tasks (see Schoenfeld, 1992).

3. Research Methods

Data collection. Ten mathematicians participated in a study in which they were asked to "think aloud" as they proved that the sine function was not injective on any interval of length greater than π. They were told to produce a proof suitable for an undergraduate textbook for second and third year mathematics majors. This task was chosen because we anticipated the participants would likely draw a graph of the sine function, quickly become convinced that the theorem was true as a result of inspecting this graph (or prior to constructing it), but nonetheless have some difficulty producing a formal argument that this was true. We note in the results section that our assumptions proved to be accurate. All interviews were videotaped.

Analysis. Analysis was conducted in the style of Weber and Mejia-Ramos (2009). We first noted every inference the participant made while constructing the proof, where an inference could be a mathematical assertion (e.g., $\sin(x + \pi) = -\sin x$), a proving approach (e.g., use a proof by cases, use a calculus-based derivative argument), or an evaluation of either (e.g., a conjecture is not true, $\sin(x + \pi) = -\sin x$ is true but not useful to prove the claim). For each inference, we coded whether the inference was made from inspecting the appearance of the graph, a logical deduction from some other inference, recall, or from some other source (e.g., a metaphor, some other diagram they constructed). Also, for each inference, we noted what previous inferences that the new inference was based upon. Once this was coded, we looked at the final proof and determined the chain of inferences used to produce this written argument. Consequently, for each inference we coded, we determined whether it was part of a chain of argumentation that led to the final proof or constituted a "dead-end" (i.e., was not directly used to produce the final argument). Finally, for each inference that was based on a graph, we used an open-coding scheme to categorize how the graph was used to support this inference.

4. Results

This was a surprisingly challenging task for mathematicians. One participant was unable to complete it successfully and several other mathematicians produced invalid proofs. Nine of the ten participants spent between 9 and 40 minutes in completing this task. During their proof construction processes, most drew inferences or suggested proof approaches that did not play a role in the construction of the proofs they wound up producing, suggesting that translating the conviction they obtained from the graph to a formal proof was not direct or straightforward.

The participants used the graph for six purposes:

(a) noticing properties and generating conjectures of the sine function that might be useful for the proof (e.g. sine is periodic with period 2π),

(b) representing or instantiating an assertion or an idea on the graph,

(c) disconfirming conjectures that are not true (e.g., one participant initially conjectured $\sin(\pi + x) = \sin x$ and used the graph to reject this conjecture)

(d) verifying properties that they deduced through logic,

(e) suggesting proving techniques (such as using the periodicity of the sine function or forming a case-based argument) to prove the theorem,

(f) using the graphs as a justification for claims they wished to make (e.g., noting that a student could see that a claim was true by inspecting the graph).

The extent of participants' graph usage. The extent of graph usage varied greatly by participant, with some frequently interacting with the diagram and others making little use of the diagram after it was drawn. Some proofs were not based on any inferences that were derived from the graph, suggesting that not all mathematicians write their proofs based on the visual arguments they used to obtain conviction.

Skills needed to use visual diagrams in proof-writing. We identified a number of skills that the participant used to utilize the graphical inferences they made into their formal proofs. These skills included access to a number of domain specific proving strategies (e.g., for a continuous function, proving injectivity and monotonicity are equivalent), fluency in algebraic manipulations, and translating logical statements into equivalent statements that are easier to work with.

The limited use of the graph in the final product. Only one participant included the diagram in the proof that he would present in the textbook. This illustrates how mathematicians may, perhaps unintentionally, mask the informal processes they use to create formal arguments when presenting proofs to their students. When this was pointed out to them, some viewed the lack of a graph as a shortcoming of their presentations while others did not.

5. Significance

The participants' difficulties with this task shows how challenging it is to base a formal proof on visual evidence. Hence, it should be no surprise that students also find this process difficult. This study describes the specific ways in which the visual diagrams were used by the participants to construct their proofs. It can be beneficial for instructors to make students aware of these purposes. The variance in the extent of graphical usage is consistent with the arguments of others that there is no single way that mathematicians engage in doing mathematics; some mathematicians use diagrams regularly in their mathematical work while others do not (e.g., Pinto & Tall, 1999; Alcock & Inglis, 2008). Finally, the skills that we outlined are important for students to master if they are to successfully use diagrams in their own proof-writing.

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