Classroom Activity with Vectors and Vector Equations: Integrating Informal and Formal Ways of Symbolizing $\mathbf{R}^n$

Contributed Research Report

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Instructional design based upon realistic problems and scenarios allow students to examine the mathematics from a variety of mathematical positions, and create meaning that integrates geometric, algebraic, and formal linear algebra. However, a potential consequence of researching student work on complex activities in difficult mathematics is that classroom mathematical activity from this perspective requires examining how meaning for mathematical objects gets generated over time as a process of collective action and negotiation. In this talk, I will answer two questions: What are the activities that students engage in as they learn to symbolize vector spaces in $\mathbf{R}^n$ using realistic situations? And, what is the process by which the classroom community developed these activities? Answering these questions can provide teachers ways being responsive to student needs and thinking as they lead their classrooms in symbolizing vectors and vector equations.

Keywords: Linear Algebra, Symbolizing, Sociocultural Perspectives

Introduction

Several researchers (Dorier, Robert, Robinet, & Rogalski, 2000; Harel, 1989, 1990; Harel & Kaput, 2002; Hillel, 2000; Sierpinska, 2000) have indicated the need to integrate students understanding of algebra, geometry and symbolic formalism in order to help students use linear algebra to solve problems and do proofs. The results from these studies provide powerful evidence of students' difficulties and the challenges inherent in learning linear algebra. Recently, researchers (Larson, Nelipovich, Rasmussen, Smith, & Zandieh, 2008; Possani, Trigueros, Preciado, & Lozano, 2009) have used modeling and instructional design based upon realistic situations in order to deal with integrating the algebraic, geometric and formal aspects of linear algebra . These approaches to teaching linear algebra allow for students to interact with one another, examine the situation from a variety of mathematical positions, and create meaning that is integrated and deep. In this talk, I will answer two questions: What are the activities that students engage in as they learn to symbolize vector spaces in $\mathbf{R}^n$ using realistic situations intended to promote the integration of formal linear algebra, algebraic symbolism and geometric intuition? And, what is the process by which the classroom community developed these activities and how does this process reflect the moment-to-moment and context dependent needs of that community? Answering these questions can provide teachers the ability to be responsive to student needs and thinking as they lead their classrooms in symbolizing vectors and vector equations. As well, it can provide instructional designers with valuable insight into how classroom communities integrate informal and formal aspects of linear algebra.

Theoretical Perspective

A potential consequence of researching student work on complex activities in complex mathematics is that classroom mathematical activity from this perspective requires examining how meaning for mathematical objects gets generated over time as a process of collective action and negotiation. In undergraduate mathematics, several studies have examined how classroom communities generate meaning for mathematics (Rasmussen & Blumenfeld, 2007; Rasmussen, Zandieh, King, & Teppo, 2005; Rasmussen, Zandieh, & Wawro, 2009; Stephan & Rasmussen, 2002) and the role that gesture plays in argumentation(Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004). Because of the multiple voices present in the classroom, meaning from a collective perspective is never really fixed. At a given moment, for a given task, the researcher might be able to say that students are utilizing a certain meaning or engaging in a certain activity, but that meaning is undergoing a constant process of construction and deconstruction. According to Wenger (1998), the process by which members of a community come to understand a particular artifact or concept is via the process of the negotiation of meaning. Negotiation of meaning implies that meaning is created over time as a process of give and take between members of the community. The classroom community I examined spent several class periods discussing and arguing about the creation, use and interpretation of symbols in the classroom.

In this analysis, I use the term activity to signify the collections of meanings and practices that students created, used and yielded as interpretations when working with vector spaces in $\mathbf{R}^n$**.** The use of the term activity is purposeful here as it indicates a frame for action that is both goal directed and the product of cultural mediation(Lave & Rogoff, 1984). As well, any set of activities has associated with a set of goal directed actions that make up that particular activity. Hence, when characterizing an activity, it is essential to indicate not only what is being done, but also to what end is that action being done.

Methods

The following analysis is based upon data gathered during a classroom teaching experiment (Cobb, 2000) conducted at a southwestern research university. This study was part of a larger study that followed an introductory linear algebra course over the course of an entire semester. The study examined eight days from that semester-long class, focusing on classroom sessions that dealt with material germane to the study, including vectors, vector equations, linear dependence/ independence, span and basis. Each classroom session was videotaped and student work and daily reflections were collected and used for triangulation purposes. . The classroom sessions in this study focused on two sets of tasks. The first set of tasks, which took place over the first 3 weeks, involved an imagined scenario involving two or three modes of transportation, symbolized by vectors, and the ability of a rider to get around in two and three dimensions using these modes. This scenario was used to teach the symbolic system of vectors and vector equations, solution methods using Gaussian elimination, linear independence and dependence, and span. It also served as a springboard for formalized linear algebra. The second set of tasks, which took place in the 13^{th} and 14^{th} weeks, focused on basis and change of basis and integrated the language and imagery from the first set of tasks into class discussion. Furthermore, in the third and fourth week of the semester, 3 focus group interviews were conducted, and in the final week of semester, three more were conducted. The focus groups had students address the norms of the classroom and their understanding and use of symbolic expressions. Focus group participants were chosen reflect various ability levels and because of their membership in various small groups in the class. Video-recordings were made for these focus groups and written work was collected

Analysis of this data was three-phased. In phase-one, the six focus group interviews were analyzed with regards to students' creation, use and interpretation of vectors and vector equations. Then, whole class and small group episodes were coded using a modified Toulmin scheme (Rasmussen & Stephan, 2008). The use of this scheme was intended to locate meanings that were functioning-as-if-shared in the classroom community by identifying data, claims, warrants or backings that either shift roles within a chain of arguments or cease to require further justification by members of the classroom as they are used in later arguments. The analysis of argumentation focused on activity with students' symbolizing, but also included meanings generated by students with regard to these symbols. This analysis was compared against the focus group analysis creating a narrative for the meanings that these students developed for the symbolic system. This narrative illustrated what the meanings were, how they came to be and the ways that students used them to solve problems in linear algebra. Finally, the whole class analysis was compared against the focus group analysis in order to insure that the two analyses were consistent.

Results and conclusions

Analysis of the focus group interviews and whole class sessions yielded three distinct, but integrated activities for the symbols for vector spaces in $\mathbf{R}^n$.

- Drawing and Interpreting Lines In Space
- Coordinating Slopes
- Generating Linear Combinations

The first activity, called "drawing and interpreting lines in space" was utilized as students were constructing geometric intuitions and was most prevalent when working directly with the geometry of $\mathbf{R}^2$ and $\mathbf{R}^3$. When engaged in this activity, students coordinated the lines in space in order to reach specific destinations or to generally specify where on the plane a set of vectors could reach. The directionality of the vector specifies where on the plane or in 3-space a vector allows the student to reach a destination. The use of this meaning was prevalent when discussing the parallelogram rule for vector addition and early in the class when solving for scalar multiples. Scalars were used to represent numbers of iterations of these vectors, while addition of the vectors is used in order to coordinate discrete distances in potentially differing directions.

The second activity: "coordinating slopes": reflects the use of vectors component-wise and grouping them together by common ratios. Frequently, the goal for this activity was to create relationships between two or more vectors and draw conclusions based upon those relationships. Although the term slopes often indicates geometric interpretations, for this class the term was more algebraic in its connotation. A slope was the specific relationship between the components of a vector. However, students did not find these slopes for individual vectors, but rather established a vectors slope as an equivalence class. If one vector could be expressed as a scalar times another vector, then those two vectors were members of the same equivalence class, called like "slopes." Vectors with like ratios between their components supplied redundant information, as they did not allow for movement in differing directions. This redundancy of information became a precursor for students' meanings for linear dependence, as students noticed that vectors that had like slopes allowed for movement away from the origin in one direction and movement back to the origin by multiplying by a negative scalar.

The classroom community developed the third activity, generating linear combinations, when they needed to create more generalized meanings and communicate those meanings with others. From a student perspective, algebraic relationships or geometric interpretations are either too imprecise or lack the ability to communicate an entire range of possibilities that a set of vectors might provide. Thus, the language of linear combinations provided a precise and fully generalized way of expressing mathematical solutions and relationships. It is important to note, however, that these relationships did not begin formal in nature, but instead became formal as students developed meaning for formal definitions and notation. When engaged in this activity, students used scalar multiplication and addition in conjunction with one another to identify specific properties of sets of vectors, including whether or not the set of vectors was linearly dependent or independent and what space the set might span.

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