Using The Emergent Model Heuristic to Describe the Evolution of Student Reasoning regarding Span and Linear Independence

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Contributed Research Report

Abstract: A prominent problem in the teaching and learning of undergraduate mathematics is how to build on students' current ways of reasoning to develop more generalizable and abstract ways of reasoning. A promising aspect of linear algebra is that it presents instructional designers with an array of applications from which to motivate the development of mathematical ideas. The purpose of this talk is to report on student reasoning as they reinvented the concepts of span and linear independence. The reinvention of these concepts was guided by an innovative instructional sequence known as the Magic Carpet Ride problem, whose creation was framed by the emergent models heuristic (Gravemeijer, 1999). During our talk we will: explain how this instructional sequence differs from a popular "systems of equations first" approach, present the instructional sequence via the framing of the emergent models heuristic; and provide samples of students' sophisticated thinking and reasoning.

Key Words: Linear algebra, Student Reasoning, Realistic Mathematics Education, Inquiry-Oriented Instruction

A prominent problem in the teaching and learning of K-16 mathematics is how to build on students' current ways of reasoning to develop more generalizable and abstract ways of reasoning. This problem is particularly pressing in undergraduate courses that often serve as a transitional point for students as they attempt to progress from more computationally based courses to more abstract courses that feature proof construction and reasoning with formal definitions. One such course is that of introductory linear algebra. A promising aspect of linear algebra, however, is that it presents an array of applications to science, engineering, and economics, providing instructional designers with opportunities to use these applications to motivate and develop mathematical ideas. The purpose of this talk is to report on student reasoning as they reinvented the concepts of span and linear independence. The reinvention of these concepts was guided by an innovative instructional sequence known as the Magic Carpet Ride problem, whose creation was framed by the emergent models heuristic (Gravemeijer, 1999) of the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991). The sequence makes use of an experientially real problem setting (in the sense that students can readily engage in the task) and aids students in developing more formal ways of reasoning about vectors and vector equations. Thus, during our talk we will:

1. Explain how this instructional sequence differs from a popular "systems of equations first" approach and why this conscious change was made;
2. Present the instructional sequence via the framing of the emergent models heuristic; and
3. Provide samples of students' sophisticated thinking and reasoning.

Literature Review

In addition to research that categorizes student difficulties in linear algebra (e.g., Dorier, 1995; Harel, 1989; Hillel, 2000), more recent work has examined the productive and creative ways that students are able to interact with the ideas of linear algebra. For instance, Possani, Trigueros, Preciado, and Lozano (2010) analyzed the use of a teaching sequence that began with a real life problem and reported on student progress as they advanced through different solution strategies. In a similar spirit, Larson, Zandieh, and Rasmussen (2008) reported a key idea that emerged as a central and powerful way in which students came to reason and eventually develop the formal ideas and procedures for eigenvalues and eigenvectors. Complementary to these two veins of research, we report on students' activity as they both reinvent and reason with the notions of span and linear independence.

The instructional sequence that was developed to foster student reinvention of these ideas does so within the first five days of the course, prior to any explicit treatment of Gaussian elimination. This is in contrast to a widespread tendency to begin the semester with systems of linear equations and Gaussian elimination (e.g., Anton, 2010; Lay, 2003). One possible reason for beginning the course in this manner is to build from students' prior experiences with solving systems of linear equations. We strongly agree with beginning a course with content that has an intuitive basis for students. Our instructional sequence, however, relies on a different intuitive background from which to build and structure an introductory linear algebra course. Our approach begins by focusing on vectors, their algebraic and geometric representations in $\mathbf{R}^2$ and $\mathbf{R}^3$, and their properties as sets. We contend that this switch not only fosters the development of formal ways of reasoning about the 'objects' of linear algebra, namely vectors and vector equations, but also instigates an intellectual need (Harel, 2000) for sophisticated solution strategies, such as Gaussian elimination. These aspects will be elaborated upon during the presentation.

Theoretical Background

Drawing on the work of Freudenthal (1991) and the instructional design theory of Realistic Mathematics Education (RME), we take the perspective that mathematics is first and foremost a human activity of organizing mathematical experiences in increasingly sophisticated ways. A central RME heuristic that captures this perspective is referred to as "emergent models." This heuristic offers researchers and teachers a way to design and trace ways that students can build on their current ways to reasoning to develop rather formal mathematics. In RME the term model has a specific meaning. In particular, Zandieh and Rasmussen (2010) define models as student-generated ways of organizing their activity with observable and mental tools. Observable tools refer to things in the environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. Mental tools refer to ways in which students think and reason as they solve problems—their mental organizing activity. Following Zandieh and Rasmussen, we make no sharp distinction between the diversity of student reasoning and the things in their environment that afford and constrain their reasoning.

The emergent model heuristic involves the following four layers of increasingly sophisticated mathematical activity: Situational, Referential, General, and Formal. Situational activity involves students working toward mathematical goals in an experientially real setting. Referential activity involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting. General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity. The model-of/model for transition is therefore concurrent with the creation of a new mathematical reality.

Methods

The classroom sessions analyzed for this presentation come from a classroom teaching experiment (Cobb, 2000) conducted in the spring of 2010 at a southwestern research university. This classroom was the third iteration of a semester-long classroom teaching experiment in linear algebra. Video-recordings were made of each classroom episode. Transcriptions were then made from the videos. Daily reflections and homework were also collected.

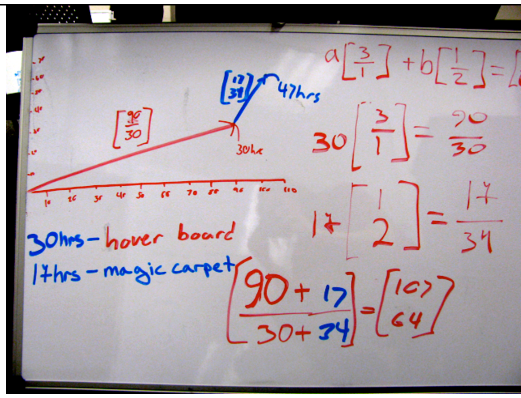
Results

This section discusses how student reasoning progressed through each of the four levels of activity throughout the semester, but especially in relationship to the tasks that students worked on during the first five days of class. Given space limitations, we provide more detail on student reasoning at the beginning of the task sequence. Note that we spent approximately one day per task during the semester.

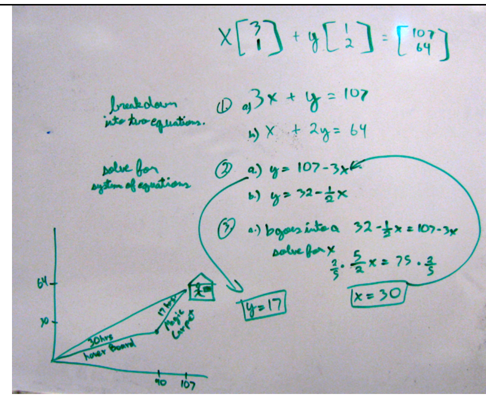
Situational and Referential Activity. The student thinking on the first two tasks was primarily Situational activity in that students focused on engaging in solving problems in the Magic Carpet Ride task setting. However, even at this level students were developing symbolic and graphical inscriptions that were models of their thinking and that the teacher was able to label with the terminology of the mathematical community such as linear combination and span. During the third and fourth tasks, student reasoning was more explicitly Referential as students used their experience in the Magic Carpet Ride setting to create a definition for the linear dependence of two vectors and as they worked to interpret the definition of linear independence in terms of the Magic Carpet Ride scenario.

TASK 1. You are given a hover board and a magic carpet. The hover board can move according to <3, 1> and the magic carpet according to <1, 2>. If Old Man Gauss lives in a cabin 107 miles East and 64 miles North, can you get there with the board and carpet?

This activity helped students explore the notion of a linear combination of one or two vectors in $\mathbf{R}^2$, including its symbolic and graphical representations. The figure below provides two examples of student thinking on this problem. On the left students use a non-standard symbolic vector notation and a guess and check methodology. On the right the students converted their vector equation into a system of equations and solved for the appropriate weights.



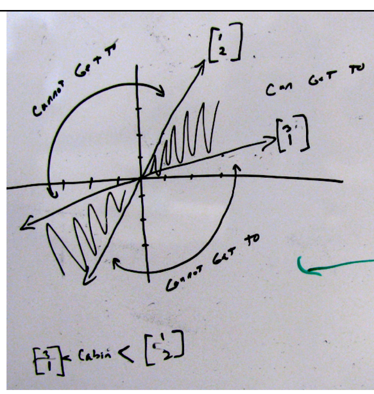
Guess and check via vector weighting



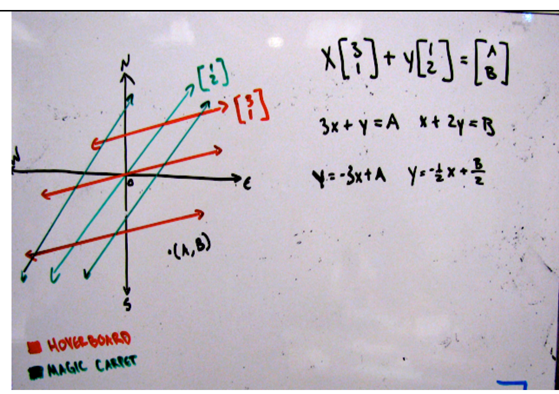
Vector equation then system

TASK 2. Are there some locations where Gauss can hide and you cannot reach him from your home with these two modes of transportation?

This extension pushes students to explore how a linear combination of two vectors can encompass all points in $\mathbf{R}^2$ and introduces the term span. The figure below provides two examples of student thinking on this problem. Notice that the board on the left indicates that this group of students thought that they could only get to points within the double funnel using the two modes of transportation, whereas the group on the right used a grid to illustrate that they could reach any point on the plane.



The Double Funnel



The Grid

TASK 3. You still have two modes of transportation, but now you cannot get everywhere. What are the possible vectors for the movement of the hover board and magic carpet now?

In discussing which sets of vectors span all of $\mathbf{R}^2$ and which do not, students defined linear dependence for pairs of vectors. In particular, students determined that if two vectors are multiples of each other, then they are linearly dependent.

> *TASK 4.* You may travel each mode of transportation only once. Can you start and end back at home?

This activity allows for the introduction of the formal definition of linear independence. Students were asked to interpret this formal definition in terms of the Magic Carpet Ride task.

General Activity. In task 5, students are given a series of questions that asks them to create a linearly independent (or dependent) set of 2 (or 3 or 4) vectors in $\mathbf{R}^3$. Some students were able to develop conjectures about what must be true a set of vectors to span a space. One such conjecture was that to span $\mathbf{R}^n$, one must have n vectors and they must be linearly independent. This is General activity since the students are now working with vectors without referring back explicitly to the Magic Carpet activity as they explore properties of these sets of vectors.

Formal Activity. Formal activity occurs much later in the term as students are able to use definitions of span or linear independence in the service of making other arguments without having to explicitly recreate or reinterpret those definitions.

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