Individual and Collective Analysis of the Genesis of Student Reasoning Regarding the Invertible Matrix Theorem in Linear Algebra

Megan Wawro
San Diego State University &
University of California, San Diego

Contributed Research Report

Abstract: I present research regarding the development of mathematical meaning in an introductory linear algebra class. In particular, I present analysis regarding how students–both individually and collectively–reasoned about the Invertible Matrix Theorem over the course of a semester. To do so, I coordinate the analytical tools of adjacency matrices and Toulmin's (1969) model of argumentation at given instances as well as over time. Synthesis and elaboration of these analyses was facilitated by microgenetic and ontogenetic analyses (Saxe, 2002). The cross-comparison of results from the two analytical tools, adjacency matrices and Toulmin's model, reveals rich descriptions of the content and structure of arguments offered by both individuals and the collective. Finally, a coordination of both the microgenetic and ontogenetic progressions illuminates the strengths and limitations of utilizing both analytical tools in parallel on the given data set. These and other results, as well as the methodological approach, will be discussed in the presentation.

Key words: linear algebra, individual and collective, genetic analysis, argumentation, Toulmin scheme, adjacency matrices.

The Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay, & Porter, 1993) named the following as topics necessary to be included in any syllabus for a first course in undergraduate linear algebra: matrix addition and multiplication, systems of linear equations, determinants, properties of $\mathbf{R}^n$, and eigenvectors and eigenvalues. Some of the specific concepts involved in the aforementioned topics are: (a) span, (b) linear independence, (c) pivots, (d) row equivalence, (e) determinants, (f) existence and uniqueness of solutions to systems of equations, (g) transformational properties of one-to-one and onto, and (h) invertibility. These concepts, in addition to others, are the very ones addressed and linked together in what is referred to as the Invertible Matrix Theorem (see Figure 1). The Invertible Matrix Theorem (IMT), which consists of seventeen equivalent statements, is a core theorem for a first course in linear algebra in that it connects the fundamental concepts of the course.

I take the perspective that the emergence and development of mathematical ideas occurs not only for each individual student but also for the classroom as a collective whole. Many researchers acknowledge in the role of the collective on the mathematical development of a learner and vice versa (Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Rasmussen & Stephan, 2008; Saxe, 2002). Through this viewpoint, the interrelatedness of the individual and the collective come to the fore, highlighting how the activity of one necessarily affects that of the other. These two forms of knowledge genesis—on an individual and on a collective level—are inextricably bound together in their respective developments. Therefore, in order to gain the most fully developed understanding of the emergence, development, and spread of ideas in a particular classroom, analysis along both individual and collective levels, over the course of the semester, is warranted and necessary.

This presentation will highlight portions of my dissertation research, which has two main aspects: (a) research into the learning and teaching of linear algebra, and (b) research into analyzing the development of mathematical meaning for both students and the classroom over time. The two research questions that guide my dissertation work are the following:

1. How do students –both individually and collectively—reason about the Invertible Matrix Theorem over time?
2. How do students—both individually and collectively—reason with the Invertible Matrix Theorem when trying to solve novel problems?

The first research question investigates the connections that are made, on both the individual and the collective level, between the various statements in the IMT. The second research question investigates the ways in which students, on both the individual and the collective level, use the IMT as a tool for reasoning about new problems. During my presentation, I will discuss results from both individual and collective-level analyses from question one.

Background and Methodology

The theoretical perspective on learning that undergirds my work is the emergent perspective (Cobb & Yackel, 1996), which coordinates psychological constructivism (von Glasersfeld, 1995) and interactionism (Forman, 2003; Vygotsky, 1987). In honoring the importance of both psychological and social processes, the emergent perspective posits that:

> The basic relationship posited between students' constructive activities and the social processes in which they participate in the classroom is one of reflexivity in which neither is given preeminence over the other...A basic assumption of the emergent perspective is, therefore, that neither individual students' activities nor classroom mathematical practices can be accounted for adequately except in relation to the other." (Cobb, 2000, p, 310)

From the perspective that learning is both an individual and a social process, investigating the mathematical development of students necessarily involves considering the individual development of students as well as the collective activity and progression of the community of learners in which the individuals learners participate. Thus, in studying the development of reasoning regarding the Invertible Matrix Theorem, both levels of development will be analyzed.

The overarching structure of my analysis is influenced by a framework of genetic analysis that delineates multiple levels of investigation. Saxe (2002) and his colleagues (Saxe & Esmonde, 2005; Saxe, Gearhart, Shaughnessy, Earnest, Cremer, Sitabkhan, et al., 2009) investigated knowledge development through the notion of cultural change. Particular to development in the classroom, the authors investigated how researchers could collect data (how much, from what sources, etc.) and conduct analyses that would allow them to make descriptions of how individuals' ideas develop in the classroom over time, given that the classroom is also changing over time. As a response, they suggested analyzing human development over time from three different strands, providing researchers a way to account for some of the complex factors of development. *Microgenesis* is defined as the short-term process by which individuals construct meaningful representations in activity, *ontogenesis* as the shifts in patterns of thinking over the development of individuals, and *sociogenesis* as the reproduction and alteration of representational forms that enable communication among participants in a community (Saxe et al., 2009, p. 208). I focus on and adapt the first two strands in my own analysis.

The data for this study comes from a semester-long classroom teaching experiment (Cobb, 2000) conducted in a linear algebra course at a large university in the southwestern United States. Students enrolled in the course had generally completed three semesters of calculus and were in their second, third, or fourth year of university. Furthermore, the majority of students enrolled in the course had chosen engineering (computer, mechanical, or electrical), mathematics, or computer science as their major course of study at the university.

In order to address the *individual* components in the proposed research questions, I focused on five of the students enrolled in the linear algebra course. All five sat at the same table during class, which is one of three tables that are videorecorded during every class period for the duration of the semester. In order to collect data relevant to these five individuals and their establishment of meaning regarding the IMT, I collected four sources of data: video and transcript of whole class discussion, video and transcript of their small group work, video and transcript from their individual interviews, and various written work. Individual interview data comes from two semi-structured (Bernard, 1988) interviews, one conducted midway through the semester and one conducted at the end of the semester.

In order to collect data relevant to the *collective* establishment of meaning regarding the IMT, I collected video and transcript of whole class discussion and small group work, photos of whiteboard work, and written work from in-class activities. As stated, portions of 12 class days are analyzed, which were the days that the IMT was explicitly addressed during whole class discussion.

In order to investigate how students reasoned about the IMT over time, I utilize five analytical phases, and each has both an individual and a collective level. The five phases are: 1) Microgenetic analysis via the construction of adjacency matrices; 2) Microgenetic analysis via the construction of Toulmin schemes of argumentation; 3) Ontogenetic analysis of constructed adjacency matrices; 4) Ontogenetic analysis of constructed Toulmin schemes,; and 5) Coordination of analysis across the two analytical tools. As highlighted in the five phases, I employ two main analytical tools: adjacency matrices and Toulmin's (1969) model of argumentation. Adjacency matrices are representational tools from graph theory used to depict how the vertices of a particular graph are connected (e.g., Frost, 1992). These matrices can be used to represent data from a variety of graph forms. In my dissertation, I create adjacency matrices that correspond to directed graphs in which the vertices are the statements in the Invertible Matrix Theorem (or students' explanations of those statements) and the edges are directed in such a way as to match the implication offered by the student. The developed adjacency matrices are n x n, where n is the number of recorded relevant yet distinct statements made by students in any given explanation. The rows are the 'p' and the columns are the 'q' in statements of the form "p implies q" or "another way to say p is q." Adjacency matrices are used as a tool to analyze explanations that explicitly address how students connect the ideas of the Invertible Matrix Theorem, as well as to analyze arguments made at the collective level during whole class discussion. These arguments are comprised of statements from one or many students in the class as meaning is negotiated collectively through participation in the classroom.

The second main analytical tool I use is Toulmin's (1969) model of argumentation, which describes six main components of an argument: claim, data, warrant, backing, qualifier, and rebuttal. The first three of these—claim, data, and warrant—are seen as the core of an argument. According to this scheme, the claim is the conclusion that is being justified, whereas the data is the evidence that demonstrates that claim's truth. The warrant is seen as the explanation of how the given data supports the claim, and the backing, if provided, demonstrates why the warrant has authority to support the data-claim pair. This work has been adapted by many in the fields of mathematics and science education research as a tool to assess the quality or structure of a specific mathematical or scientific argument and to analyze students' evolving conceptions by documenting their collective argumentation (Erduran, Simon, & Osborne, 2004; Krummheuer, 1995; Rasmussen & Stephan, 2008; Yackel, 2001). While the Toulmin model has proven a useful tool for documenting mathematical development at a collective level (e.g., Stephan & Rasmussen, 2002), I utilize Toulmin's model to analyze structure of individual and collective exchanges both in isolation and as they shift over time.

While Phases 1 and 2 are comprised of many discrete analyses, Phases 3 and 4 are compiled from the results of Phases 1 and 2. In Phase 3, shifts in form and function of how students reason about reason with the various concepts in the IMT over time are analyzed by considering qualitative changes in constructed adjacency matrices from Phase 2. This type of analysis is what Saxe (2002) refers to as *ontogenetic analysis*. Phase 4, on the other hand, considers the individually constructed Toulmin schemes from Phase 2 as a whole. This sort of analysis, at the collective level, is consistent with the work of Rasmussen and Stephan (2008) in identifying classroom mathematics practices. Finally, Phase 5 combines the work done in parallel with adjacency matrices and Toulmin schemes on both the microgenetic level (comparing the results of Phases 1 and 2) and the ontogenetic level (Phases 3 and 4). In other words, Phase 5 consists of cross-comparative analyses, for any given argument or collection of arguments, of the results from both analytical tools (adjacency matrices and Toulmin schemes).

Results

The cross-comparison of results from the two analytical tools, adjacency matrices and Toulmin's model, provides a rich way to investigate the content and structure of arguments offered by both individuals and the collective. A coordination of both the microgenetic and ontogenetic progressions illuminates the strengths and limitations of utilizing both analytical tools in parallel on the given data set. Analysis reveals rich student reasoning about the IMT that may not be apparent through use of only one analytical tool. For instance, adjacency matrices proved an effective analytical tool on arguments consisting of multiple connections that were for explanation, whereas Toulmin models proved illuminating for arguments with complex structure for the purposes of conviction. These and other results, as well as my methodological approach, will be discussed during my presentation.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following are equivalent:

a. The columns of A span $\mathbf{R}^n$.
b. The matrix A has n pivots.
c. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$.
d. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a way to write $\mathbf{b}$ as a linear combination of the columns of A.
e. A is row equivalent to the $n \times n$ identity matrix.
f. The columns of A form a linearly independent set.
g. The only solution to $A\mathbf{x}=\mathbf{0}$ is trivial solution.
h. A is invertible.
i. There exists an n x n matrix C such that $CA = I$.
j. There exists an n x n matrix D such that $AD = I$.
k. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
l. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.
m. Col $A = \mathbf{R}^n$.
n. Nul $A = \{\mathbf{0}\}$.
o. The column vectors of A form a basis for $\mathbf{R}^n$.
p. Det $A \neq 0$.
q. The number 0 is not an eigenvalue of A.

Figure One: The Invertible Matrix Theorem

References

Bernard R. H. (1988). *Research Methods in Cultural Anthropology*. London: Sage.

Carlson, D., Johnson, C., Lay, D., & Porter, A. D. (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *College Mathematics Journal, 24,* 41-46.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–330). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist,* 31, 175-190.

Erduran, S., Simon, S., & Osborne, J. (2004). TAPping into argumentation: Developments in the application of Toulmin's Argument Pattern for studying science discourse. *Science Education, 88*(6), 915-933.

Forman, E. (2003). A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 333-352). Reston, VA: NCTM.

Frost, R.B. (1992). Directed graphs and their adjacency matrices: Misconceptions and more efficient methods. *Engineering Optimization, 20*(3), 225-239.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's "shared knowledge." *Mathematics Education Research Journal, 19*(2), 41-68.

Krummheuer, G. (1995). The ethnology of argumentation. In P. Cobb & H. Bauersfeld

(Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Erlbaum.

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A.E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.). *Handbook of innovative design research in science, technology, engineering, mathematics (STEM) education* (pp. 195 - 215). Taylor and Francis.

Saxe, G. B. (2002). Children's Developing Mathematics in Collective Practices: A Framework for Analysis. *Journal of the Learning Sciences, 11*, 275-300.

Saxe, G. B., & Esmonde, I. (2005). Studying cognition in flux: A historical treatment of fu in the shifting structure of Oksapmin mathematics. *Mind, Culture, and Activity Special Issue: Combining longitudinal, cross-historical, and cross-cultural methods to study culture and cognition, 12*(3–4), 171–225.

Saxe, G., Gearhart, M., Shaughnessy, M., Earnest, D., Cremer, S., Sitabkhan, Y., et al. (2009). A methodological framework and empirical techniques for studying the travel of ideas in classroom communities. In Schwarz, B., Dreyfus, T., & Hershkowitz, R. (Eds.), *Transformation of knowledge through classroom interaction* (pp. 203-222*)*. New York: Routledge.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior, 21,* 459-490.

Toulmin, S. (1969). *The uses of argument*. Cambridge: Cambridge University Press.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: Falmer Press.

Vygotsky, L. (1987) *Thought and language*. Cambridge, MA: The MIT Press.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th international conference on the psychology of mathematics education*, Vol. 1 (pp. 9–23). Utrecht, Holland, IGPME.