Title: Assessing Active Learning Strategies in Teaching Equivalence Relations

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Contributed Research Report

Abstract: In this study, students in transition-to-proof courses were introduced to equivalence relations either using a traditional classroom lecture or using small group learning activities. Students' understanding of equivalence relations were then assessed using task-based interviews aimed at assessing concept image, concept definition, as well as concept usage in terms of writing proofs. The students involved in small group activities made stronger connections to partitions and were more successful in writing proofs. In addition, the concrete learning activities gave many participants a strong prototypical example that aided in encapsulating the essential features of an equivalence relation.

Key words: transition to proof, classroom teaching experiment, concept definition

The notion of equivalence plays a role in understanding relationships between a wide variety of mathematical objects, such as fractions, equations, and vectors. This fundamental idea is formalized in the notion of an equivalence relation. Students are typically introduced to the formal definition of an equivalence relation in a transition-to-proof course. In a study involving a transitions course, Chin and Tall (2001) point out that the idea of relations is one of the more difficult concepts for students to understand. In particular, while many students were able to recall that equivalence relations must be reflexive, symmetric, and transitive, the standard definition of a relation as a subset of the cross product had little concrete meaning. Further, for most students, the conceptual link between equivalence relations and partitions was fairly limited. In terms of proofs, Chin and Tall (2000) also observed that many students attempted demonstrate that a relation was an equivalence relation using informal rather than formal, definition-based arguments. The goal of this study is to assess the impact of introducing equivalence relations and partitions using small group learning activities. In particular, when compared to a standard lecture, do small group activities improve student understanding of equivalence relations and partitions? Further, does an understanding developed through small group interaction lead to greater success in writing formal proofs?

Background

In his study of a transitions course, Moore (1994) identified several difficulties that students had in writing proofs. These included not knowing the concept definition, having inadequate concept images, and not knowing how to use the definitions to structure their proofs. Building on the distinction between concept definition and concept image, Moore introduced the notion of concept usage. Concept usage refers to the way one operates with the concept in generating examples or in writing proofs. Taken together, Moore referred to the concept definition, image, and usage as the concept-understanding scheme. He noticed that students had difficulty remembering formal, abstract definitions without an informal understanding of the concept: "The students often needed to develop their concept images through examples, diagrams, graphs and other means before they could understand the formal verbal or symbolic definitions" (Moore, 1994, p. 262). In terms of constructing an understanding of a new concept, Dahlberg and Housman (1997) outlined four basic strategies that students use when presented with a new concept definition: example generation, reformulation, decomposition and synthesis, and memorization. Their research indicated that the strongest evoked concept image arose from example generation. With this in mind, the group activities to introduce equivalence relations were designed to help students develop an informal understanding of the concept definition through example generation and exploration.

Methodology

Participants in the study were undergraduate students enrolled in either a lower division discrete mathematics course or an upper division transition-to-proof course. During the first year of the study, students in both courses were introduced to equivalence relations and partitions using a traditional lecture format, with one fifty minute class period devoted to equivalence relations and another devoted to partitions. Using the standard definition, equivalence relations were introduced as a subset of the cross product satisfying the reflexive, symmetric, and transitive properties. The instructor used several examples to illustrate the meaning of each requirement and how it could fail. There was good classroom interaction, with a variety of questions and discussion involving both the instructor and the students. During the second lecture, after defining and illustrating the notion of a partition, the instructor proved the standard theorem connecting equivalence relations and partitions. During the second year of the study, students worked in small groups during the two class periods. Activities on the first day related to equivalence relations while the second day focused on partitions. Students were given the formal definitions, and the quantifiers in each requirement were verbally emphasized. After this introduction, the rest of the class period was devoted to small group interactions. Beginning with a variety of colored shapes, the students were asked to determine whether "same color" and "differ in exactly one attribute" were equivalence relations. Using these same shapes, students were asked to formulate other examples and non-examples of equivalence relations. Students then considered several relations defined symbolically, such as aRb if and only if $a+b=2n$ for $n \in \mathbb{Z}$, and were asked to convince each other that the given relations were or were not equivalence relations. Students participated in similar activities regarding partitions and, using some concrete examples, convinced themselves that equivalence classes naturally give rise to a partition. In addition, they discovered that a collection of sets that do not form a partition do not give rise to an equivalence relation. The theorem relating equivalence relations and partitions was not proven, but was stated as a result that generalized the examples they had investigated. During the group activities, the instructor spoke with individual groups to help clarify any questions or to encourage rigor in their arguments.

Assessment

Approximately three to four weeks after the classroom lecture or group activities, students' understanding of equivalence relations was assessed using task-based interviews. 17 students from the lecture sections and 21 students from the active learning sections participated in the interviews. These occurred outside of class, lasted approximately a half an hour, and were videotaped for further review. Participants were first asked "what is an equivalence relation" and, in attempting to get a better picture of their concept image, they were asked to describe any other related concepts, ideas, or illustrations that they thought of when hearing the term "equivalence relation." In terms of concept usage, students were given two relations and asked to determine whether or not they formed equivalence relations. For each relation, participants were first asked to give an example of two elements that were related and two other elements that were not related. This was to ensure that there was no confusion regarding the notation or definition of the relations. Students were asked to talk out loud as they worked on the problems, and they were prompted for additional clarification when their reasoning was unclear.

Results

In terms of being able to define an equivalence relation, the vast majority of students in both courses remembered the three words reflexive, symmetric, and transitive. When asked to describe each requirement, most students gave symbolic definitions that were largely correct. The most common mistakes involved the use of quantifiers. This is similar to the results in Chin and Tall (2000, 2001), and there was little difference between the written definitions given by students in the traditional lecture sections versus those with small group activities. When asked about any other related concepts or ideas that popped into their head when they heard the term "equivalence relation," there was a difference observed between the lecture sections and the active learning sections. Partitions were mentioned by only 4 of the 17 participants in the traditional lecture sections, while 10 of the 21 students in the active learning sections made a connection with partitions. Finally, several students made comments about the examples used in their initial encounter with equivalence relations. Two students in the lecture sections indicated that equivalence relations involved ordered pairs, while almost a quarter of the students in the active learning sections described sorting colored shapes into related groups of objects. For example, in describing other ideas related to equivalence relations, one student remarked that "Well, definitely the shape and colors concept is the strongest ... it does help to remember kind of the flavor of relation that we're describing."

In terms of concept usage, students were asked to prove or disprove that two relations formed equivalence relations. The first relation, aRb if and only if $a \cdot b \geq 0$ for $a, b \in \mathbb{R}$, is not transitive (take $a = -1$, $b = 0$ and $c = 1$ so that $ab \geq 0$ and $bc \geq 0$ but $ac \ngeq 0$). Students in the active learning sections were more successful at noticing that transitivity fails, with several students immediately considering the transitive property as they suspected it might be problematic. In contrast, all students in the traditional lecture proceeded by checking reflexive first, symmetry second, and transitive third. Only one student, a student in the active learning section, made a connection with partitions before demonstrating that transitivity fails: "Obviously, the negative numbers are related to each other and also the positive numbers are related to each other. The problem is zero itself ... it doesn't create a partition because zero is related to any positive or any negative." The second relation participants considered, mRn if and only if $m^2 = n^2$ where $m, n \in \mathbb{Z}$, does form an equivalence relation. The students in the active learning sections were more successful at writing a correct proof of this fact. In particular, only 4 of 17 participants in the lecture sections presented a correct proof compared with 14 out of 21 in the active learning sections. Finally, in terms of proof schemes (as described in Harel and Sowder, 2007), almost half of the students in the traditional lecture sections gave only an empirical argument while almost all of the students in the active learning sections attempted a deductive argument.

Discussion

When compared with a traditional lecture, small group learning activities involving equivalence relations and partitions led students to a more interconnected concept image as well as greater success in writing proofs. In addition, sorting concrete shapes into groups of equivalent objects seemed to provide many students with a prototypical example that embodied the formal definition while making a clear link to partitions. Typically, one of the goals of a transitions course is to help students in working with abstract mathematical ideas, where meaning is often derived from the formal definition. In this sense, classroom activities where students are given an opportunity to explore formal definitions in small groups supports the goals of a transitions course. Further, this study demonstrates that these same types of activities can help students in taking a more formal, deductive approach to mathematical argument and proof. Finally, in terms of limitations of this study, it is important to note that the small group learning activities occurred during two days of a semester where almost all classes were taught in a modified lecture format. It is unclear how students would perform in a course where most classes involved active group learning. However, this study does suggest that small group activities where students generate and explore mathematical definitions can be an effective tool for teaching certain concepts within a lecture style transitions course.

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