An Investigation of Students' Proof Preferences: The Case of Indirect Proofs

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Abstract: This paper reports findings from an exploratory study regarding undergraduate natural sciences students' proof preferences, as they relate to indirect proof. While many agree that students dislike indirect proofs and fail to find them convincing, quantitative studies of students' proof preferences have not been conducted. The purpose of this study is to build on the existing qualitative research base and to determine if the identified preferences and conviction levels can be established as general tendencies among undergraduates. Specifically, the aim of the study is to explore two common claims: (1) students experience a lack of conviction when presented with indirect proofs; and (2) students prefer direct and causal arguments, as opposed to indirect arguments. The purpose of this preliminary report is to share findings from the proof preference pilot study.

> *"Why do I have to start with something that is not? … … However, the final gap is the worst, … … it is a logical gap, an act of faith that I must do, a sacrifice I make. The gaps, the sacrifices, if they are small I can do them, when they all add up they are too big."*
>
> (Fabio quoted in Antonini & Mariotti, 2008)

> *"The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons"*
>
> (G. H. Hardy, *A Mathematican's Apology*, 2005/194, p. 19)

Research Goals & Questions

Research on undergraduate mathematics students' understandings of proof has continuously demonstrated that proof represents a significant barrier within the undergraduate mathematics curriculum (Selden & Selden, 2003; Harel & Sowder, 1998). Research suggests that students experience a variety of difficulties, including but not limited to difficulties moving between syntactic and semantic proofs (Weber & Alcock, 2004), transitioning from computational courses to proof-centered courses and topics (Moore, 1995), reading mathematical proofs (Selden & Selden, 2003), understanding the logical structure of proofs (Selden & Selden, 1995), developing appropriate proof schemes (Harel and Sowder, 1998), and interpreting proofs (Weber, 2001). Beyond difficulties producing and understanding proof, research suggests that not all forms of proof are equally difficult for students, with the most problematic being Mathematical Induction (Dubinsky, 1986, 1989; Movshovitz-Hadar, 1993a, 1993b; Fischbein & Engel, 1989; Harel, 2001; Brown, 2003; Harel and Brown, 2008) and Proof by Contradiction (Harel & Sowder, 1998; Antonini & Mariotti, 2008). Thus, little has changed since Robert and Schwarzenberger (1991) noted, "Research into students' ability to follow or produce proofs … confirms that students find proof difficult, with proofs by (mathematical) induction and proofs by contradiction presenting particular difficulties" (p. 130). Interestingly, the two forms of proof that Robert and Schwarzenberger (1991) highlight – *proof by mathematical induction* and *proof by contradiction* – are likely to be the two most ubiquitous forms of mathematical proof in the undergraduate mathematics curriculum. Thus, it seems likely that identifying factors contributing to students' difficulties with these particular forms might be critical to developing instructional innovations that foster students' transition to proof.

The Case of Indirect Proof

In relation to *indirect proof*, which we will take to include both proof of the contrapositive and proof by contradiction, qualitative studies have demonstrated that students experience a lack of conviction with respect to such proofs (Harel & Sowder, 1998; Antonini & Mariotti, 2008; and Leron, 1985). For example, Harel and Sowder (1998) found that many students in their teaching experiments preferred constructive proofs – proofs that directly construct mathematical objects rather than solely justify existence – and dislike proofs by contradiction. The following remark, made by a student, Dean, is provided as a typical example of students' views towards this form of proof: "I really don't like proof by contradiction. I have never understood proofs by contradiction, they never made sense" (Harel & Sowder, 1998, p. 272). In more recent work by Antonini and Mariotti (2008), involving clinical interviews with Italian secondary school and university students, it is argued that students' dislike of indirect proofs may be tied to a lack of intuitive acceptance regarding the equivalence of a particular mathematical statement and its contrapositive; that is, while students recognize the contrapositive and can evaluate the proof of the contrapositive, they find it difficult to accept such proofs as proofs of the original theorem, as indicated by Fabio's remarks at the beginning of this paper. Such an interpretation fits well with others' comments regarding indirect proofs. For instance, Leron (1985) argued that when engaging in such proofs "we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered" (p. 323). Antonini and Mariotti (2008) suggest that in the case of statements for which there exists a direct proof, students may find the direct proof more intuitively acceptable.

While many agree that students dislike such proofs and fail to find them convincing, quantitative studies of students' proof preferences and conviction levels have not been conducted. The purpose of this study is to build on the existing qualitative research base and to determine if the identified preferences and conviction levels can be established as general tendencies among undergraduate natural sciences students. Specifically, the aim of the study is to conduct a quantitative validation study of two claims: (1) Students experience a lack of conviction when presented with indirect proofs; and (2) Students prefer direct and causal arguments rather than indirect arguments. The purpose of this preliminary report is to share findings from the pilot survey, which explored claim (2).

Considerations Regarding Indirect Proof

Indirect proofs, according to many (e.g., see Polya, 1957), occur in two forms: (a) *proof by contraposition*; and (b) *proof by contradiction*. The two forms of proof prove different yet logically equivalent statements. In the case of proof by contraposition, one proves the contrapositive of a statement rather than the original statement; i.e., one proves $\sim Q \Rightarrow \sim P$, rather than $P \Rightarrow Q$. Proof by contradiction, also referred to as *reductio ad absurdum*, entails proving $P \wedge \sim Q \Rightarrow Q \wedge \sim Q$ or that $P \wedge \sim Q \Rightarrow P \wedge \sim P$. Others studying indirect proof have opted to group the two forms of proof together (See Antonni and Mariotti, 2008). In the context of this study, the two forms of indirect proof are viewed as distinct in terms of their structure. This is not to say that the proofs do not overlap but rather that they do not lie in complete bijection. This follows from consideration of what one can assume at the outset of constructing such proofs. Here we see that one can assume ~Q, when constructing a proof by contraposition, while one can assume P∧~ Q, when constructing a proof by contradiction.

It is possible that being able to assume more initially (P∧~ Q) might better enable novices to construct such proofs. On the other hand, such proofs require one to assume "the absurd," which might make such proofs especially difficult for novices, for they require purely hypothetico-deductive thinking in the sense of Piaget. Another complicating factor is that indirect proofs take multiple and varied forms and are sometimes the only apparent or feasible approach. Take, for example, the task of proving the irrationality of the √2. One can either prove that for every pair of integers, p and q, √2 ≠ p/q or one can assume there exists integers p and q such that √2 = p/q and arrive at a contradiction. To the experienced proof writer, the later may seem easier.

Finally, historically, the mathematics community has held discrepant views of proof by contradiction or "indirect proof." For instance, the famous mathematician, G. H. Hardy describes *reducio ad absurdum* as "one of a mathematician's finest weapons" (p. 19). In contrast, Polya (1957), in his discussion of "Objections" to indirect proof, states, "we should be familiar both with 'reductio ad absurdum' and with indirect proof. When, however, we have succeeded in deriving a result by either of these methods, we should not fail to look back at the solution and ask: *Can you derive the result differently*" (Polya, p. 169). Taking a more extreme stance, mathematicians, such as L. E. J. Brouwer, who in the early twentieth century were part of the intuitionist movement, rejected the law of the excluded middle and thus, proof by contradiction. Thus, as also noted by (Antonini & Mariotti, 2008) and (Mancuso, 1996), the mathematics community does not appear to be in harmony in terms of a general preference for indirect proof.

Methodology

To explore the claim that students prefer direct and causal arguments rather than indirect arguments an 8-item indirect proof survey was developed. This survey included three types of proof comparison tasks. Type I tasks as participants to compare a direct proof to an indirect proof and to indicate which argument they found more convincing. For example, a participant might be asked to compare a proof by induction (direct proof) to a proof that relied on the Well-ordering Principal (proof by contradiction). Type II tasks asked participants to compare a *Proof by Construction*, in which a mathematical object is constructed, to an *Existence Proof*; that is, to a non-constructive, indirect proof of existence. Type III tasks explored the idea that there might be psychological distinctions to be made between the two forms of indirect proof, and asked participants to compare a proof by contraposition to a proof by contradiction. In addition to the comparison tasks, Type IV tasks asked participants to select from among three statements which statement they would choose to prove. Specifically, students were asked to indicate: (a) if a given theorem could be proved by proving an alternative statement of the theorem; and, (b) which among the potential alternative statements they would choose to prove. Alternative statements were of the form ~ Q ⇒ ~P and "there exists no P such that, P∧~ Q." Participants of the study were undergraduate mathematics students, enrolled in post-calculus collegiate mathematics courses such as differential geometry, linear algebra, and knot theory. Responses were anonymous, with respondents simply indicating their major and year in school.

Findings

Preliminary findings from a cohort of students (n = 20) drawn from four advanced mathematics courses indicates that advanced, undergraduate mathematics students' proof preferences are not consistent across comparison type. In comparison tasks of Type I, participants preferred direct proofs, which relied on the Principal of Mathematics Induction, when such proofs were contrasted with a proof by contradiction, which relied on the Well-Ordering Principal. In comparison tasks of Type II, students overwhelmingly selected an existence argument, with an implicit proof by contradiction, when compared with a constructive proof. This finding conflicts with prior qualitative research. No trends were observed in Type III comparison tasks. Finally, regarding Type IV survey items, students overwhelming selected direct statements; that is, statements which did not include a negation. This finding aligns with prior qualitative research on indirect proofs.

Discussion

This paper presents a novel approach to studying students' proof preferences related to indirect proof. The findings reported in this paper are preliminary and should be viewed as such, in part, because the result are drawn from a small sample of advanced students and because the work is preliminary. Further work is needed both with this population and with other populations. Indeed, novice proof writers, students at the beginning of the undergraduate studies, may exhibit different proof preferences. It is interesting, however, that much of the qualitative work on indirect proof has stressed that students prefer direct and constructive arguments (Harel & Sowder, 1998; Antonini & Mariotti, 2008) yet, in the context of the survey, students responses indicated a preference for the existence proof. Finally, variations in students' proof preferences across task type suggest that students' proof preferences may be more nuanced than indicated by prior characterizations.

Audience Questions

1. *Students' proof preferences appear to be linked, in some cases, to students' self-reported "comfort level" with particular forms of proof (e.g., induction proofs), as indicated by students' survey comments. In such cases, is "preference" an appropriate characterization of students' responses?*

2. *Several students noted in the comment section that they prefer direct proofs to indirect proofs. Yet, these same students selected the existence argument, with an implicit indirect proof, over a direct, constructive proof. What can we infer from instances in which students' comments do not align with trends in their proof preferences?*

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