Counting Problem Strategies of Preservice Secondary Teachers

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"Counting problems" are a class of problems in which the solver is asked to determine the number of possible ways a set of requirements can be satisfied. Students are often taught to use combinatorial formulas, such as permutation or combination formulas, to solve such problems. However, it is common for students to incorrectly apply such formulas. Heuristics, such as "looking for whether or not order matters," can be unhelpful or misleading. We will discuss an ongoing analysis of preservice and inservice secondary and community-college level teachers' responses to six counting problems in order to determine the strategy or formula used in attempting to solve the problem. We are particularly interested in whether or not an explicit statement about order "mattering" helps or hinders the participants' ability to choose an appropriate strategy.

Counting problems, which ask the solver to determine the number of possible ways to fulfill a set of requirements, are an important type of problem in combinatorics and discrete probability. Solving such problems involves combinatorial ideas such as the multiplication principle (also known as the basic counting principle), combination formula, permutation formula, or a mixture of any of these. These topics are included in the typical introductory material of probability and combinatorics courses at the undergraduate level, and of second-year algebra and statistics courses at the secondary school level.

Solving such counting problems can be quite troublesome for students. It can be very difficult to determine the combinatorial ideas that are appropriate for solving a given problem. (This problem is compounded by the fact that there are several different ways to solve most counting problems.) Suppose we consider the number of arrangements of r objects selected from a set of n distinct objects. Students are often taught a common heuristic for solving counting problems: for ordered arrangements, use the permutation formula $n!/(n-r)!$; for selections that are unordered, use the combination formula $n!/(n-r)!r!$. However, this heuristic can be unhelpful or even misleading in certain contexts. In order to successfully and consistently solve counting problems, students must have a much richer knowledge of these combinatorial ideas.

Therefore, it is important that preservice teachers, in particular, be able to confidently solve counting problems. Furthermore, it is critical that they be able to choose an appropriate strategy for solving a given problem. By *strategy*, in this context, we mean the combinatorial constructions (such as the aforementioned basic counting principle, combination formula, or permutation formula) applied in order to solve the problem. Currently, there is very little research that has been done in this area. Therefore, we are currently conducting a study to investigate the counting problem strategies used by preservice teachers. We are currently in the beginning stages of this project, but we will have collected a significant amount of data by the end of the current semester. We propose to speak on the design and preliminary results of this study at the upcoming conference on RUME.

There is a small body of recent literature on the strategies used in solving counting problems. Annin and Lai (2010) have identified some common errors made by students when solving particular types of counting problems. Godino et al. (2005) have looked at counting problems through an ontological-semiotic model. Two of the same authors examined the responses of secondary school students' responses in an earlier paper (Batanero et al., 1997).

At our institution (a large public university in the Southwest United States), there are several mathematics and education classes intended for preservice teachers at the secondary or community college levels. During the current semester, these courses include two undergraduate "capstone" courses for students in the teaching option of the mathematics major, two graduate courses in the master's degree program in teaching mathematics, two courses in the post-baccalaureate teaching credential program, and a workshop for current teaching assistants at the university. We will give a short "quiz" of six counting problems to students in each class. (There are some students who are enrolled in more than one of these courses; we will ask these students to take the quiz more than once, if they are willing. This will give us data on the consistency of the students' strategies.) We expect that all of the participants will have encountered the basic combinatorial ideas necessary to solve counting problems in at least one high-school level course and at least one college-level course.

The six problems chosen for the quiz are given below:

1) A scientist has six test tubes, labeled A-F. Each tube contains one liquid: water, sugar solution, hydrochloric acid, or chocolate milk. In how many ways can the scientist place liquids in the tubes so that exactly two tubes contain water?

2) A bag contains 26 marbles, labeled A through Z. In how many ways can six marbles be chosen, where each of the six chosen marbles is different and the order in which they are chosen matters?

3) Six college freshmen must each be assigned to one of ten available academic advisors. If each student is to receive exactly one advisor, and each is assigned to a different advisor, in how many ways can these assignments be made?

4) A painter has twelve colors of paint available. When painting a house, she needs to choose a main color, trim color, accent color, and siding color, and all of these colors must be different from one another. How many ways are there for the painter to pick colors for the house?

5) A youth hockey team has twelve members. How many ways are there to choose a starting lineup of center, left wing, right wing, left defense, right defense, and goalie, if the order in which these positions are filled does not matter?

6) A toddler is stacking colored blocks, which can be red, white, blue, or green. If the toddler makes a stack of eight blocks, how many ways are there to stack the blocks so that exactly three blocks are red? (The order in which the blocks are arranged matters.)

The problems chosen are intended to be unfamiliar to the participants, as we did not want the participants to rely on previously memorized strategies. Each of these six problems can be solved using several different strategies. Three of the problems make an explicit statement of whether or not "order matters" (that is, whether or not rearrangements of objects are to be counted separately), but are otherwise "pairwise isomorphic" to the three problems that do not include a specific statement about order: problems 1) and 6) are essentially the same, as are the pair 2) and 3) and the pair 4) and 5). All six problems are really asking for a number of possible permutations; Problems 1) and 6) allow for repetition. The statements regarding order in problems 5) and 6) are deliberately misleading. In this way, we intend to investigate how the context and wording of a counting problem, particularly the inclusion of an explicit statement about order, affect the strategies used by the participants. We expect that at least some of the participants will rely heavily on the heuristic of using a permutation formula when explicitly told that order matters, and a combination formula when explicitly told that order does not matter. We are interested to see how widespread the use of this heuristic is in our data, and the strategies used when an explicit statement of order is not given by the problem.

The challenges of teaching combinatorics, particularly the strategies for solving counting problems, are not surprising– few formulas and set procedures can be blindly applied without a careful understanding of the delicacies that exist. Subtle differences in wording or interpretation of the questions can lead to vastly different solution techniques and answers. This research attempts to gain a foothold on some of the challenges in this area with a vision of enhancing the quality of instruction in the area of counting problems and bringing to light some new ideas on how teachers can help students avoid the pitfalls described above.

Questions for the Audience:

- What other studies have examined the strategies used by students and teachers to solve counting problems?
- What other difficulties are often encountered in solving counting problems?
- What other heuristics are used to solve counting problems?

References

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