How Do Mathematicians Make Sense of Definitions?
Preliminary Report

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ABSTRACT: It seems clear that students' activity while working with definitions differs from that of mathematicians. The constructs of concept definition and concept image have served to support analyses of both mathematicians' and students' work with definitions (c.f. Edwards & Ward, 2004; Tall & Vinner, 1981). As part of an ongoing study, we chose to look closely at how mathematicians make sense of definitions in hopes of informing the ways in which we interpret students' activity and support their understanding of definitions. We conducted interviews with mathematicians in an attempt to reveal their process when making sense of definitions. A striking observation relates to the role of examples. We will share a preliminary analysis of these interviews and engage the audience in reflecting on the ideas.

KEY WORDS: mathematical definitions, advanced mathematical thinking, mathematicians' practice, examples

How do we come to understand mathematical definitions? Is the process different for students than it is for mathematicians? What can be learned from the practice of mathematicians that could support students' learning? In their chapter on advanced mathematical thinking, Harel, Selden, & Selden (2006) identified mathematical definitions as one area of focus when comparing the activity of students with the practice of mathematicians. The constructs of concept definition and concept image have served to support analyses of both mathematicians' and students' work with definitions (c.f. Edwards & Ward, 2004; Tall & Vinner, 1981). Our current research attempts to bring such ideas together into explanatory models for mathematicians' and students' activity. In this presentation we will focus on mathematicians and how their ability to build adequate concept images might develop.

Mathematicians encounter definitions in their work in a variety of ways. There are definitions included in courses they teach, definitions proposed by other mathematicians, and perhaps even new definitions created in the course of their own research work. In instructional settings, mathematicians must decide how to present definitions to students. In the context of current mathematical work, mathematicians must judge the clarity and appropriateness of stated definitions. In preparing to share proposed definitions, mathematicians must also consider presentation, clarity, and usefulness. Each of these settings requires some level of making sense of a given definition within a mathematical setting. We set out to create an interview context in which aspects of this activity were brought out and thus became accessible for analysis. The interviews provided opportunities for the mathematicians to articulate their perspectives on making sense of definitions and to participate in definition-related tasks (Watson & Mason, 2005).

In the interviews, participants were first asked to describe how they make sense of new mathematical definitions and to provide a recent example of doing this, if possible. The second interview question asked participants to share how they support students' work with definitions. Participants were then asked to engage in an example-generation activity, and finally were given a formal definition from an unfamiliar context and asked to share how they would go about making sense of the definition. We will focus on participants' articulated and observed sense-making process and possible connections to their articulation of instructional practice.

Participants described their process of making sense of a new definition by sharing insights from a range of activities, including reading papers or textbooks and creating definitions within their research work. When presented with the unfamiliar definition, participants reflected on their thinking, identifying what they found challenging about the task, and what next steps they would take. Participants were told that additional information was available and would be provided if requested.

One of our main observations concerns the use of examples. When asked how they make sense of a new definition, participants immediately referred to the use of examples and repeated the importance of examples when describing their teaching. Given their emphasis on examples, it was reasonable to expect participants to use an example when presented with a new definition. Rather, participants focused on specific terms or notations within the statement, either working through these on their own or asking for supporting information. Given the immediacy of their reference to examples previously in the interviews, we found the tendency to *not* ask for an example initially surprising. Our analysis needed to account for this expressed relevance of examples and the seemingly contradictory behavior of not actually asking for an example. Two themes emerged from the analysis that served to coordinate our observations. We will present these themes first as an explanatory model and then compare the model to what the mathematicians said about their instructional practice.

Mathematicians make sense of definitions by situating the definition within a particular mathematical setting and considering the usefulness of the definition within that setting. Placing a definition within a setting involves a progression of previous definitions, notations, and examples. When presented with the unfamiliar definition, participants began by sorting through the specific terms and notations within the statement. This involved requesting and receiving various supporting definitions. Within this process, participants made references to their own previous knowledge or to contexts with which they were familiar. In particular, participants questioned things such as why some terms were presented in a specific way, or whether the definition needed to be as general as it appeared to be. In most cases, participants did not ask to see an example as part of this process. We see this as the mathematicians needing to situate the statement of the definition clearly within a mathematical setting to judge the value or usefulness of the definition.

Mathematicians use examples as a tool for understanding definitions. In discussing both their own work and their work with students, participants spoke about examples as key to building understanding. Examples should be chosen carefully so that they serve to draw attention to important aspects. In making sense of definitions, the participants said they use examples to confirm their understanding, often choosing "messy" examples to be sure they had not introduced inappropriate assumptions. Creating or considering non-examples was considered an essential component of understanding: "and the only way to get there is look at concrete examples and look at concrete non-examples." The use of examples and non-examples seemed critical for their own understanding and how they support students' understanding.

In this study, mathematicians placed definitions within a particular mathematical setting; in their general practice and in their instruction, this setting is already set. When teaching, they attend to presenting ideas in a logical progression so that students have the necessary pieces to understand definitions. In their own work, they are familiar with current terms and notations within their field, so the setting and progression are understood. Within the setting, examples and non-examples serve to spotlight key features of the definition; using a non-example can help to clarify why certain aspects of the definition are needed (why does the function need to be continuous here?). When we presented mathematicians with an unfamiliar definition, it was not situated within a particular setting. Therefore, they needed to understand the key components of the statement before they could use examples as a tool. That is, examples do not "carry" the definition entirely.

In the presentation we will engage the audience with the topic by asking them to reflect on their own practice, provide background on our study, share our preliminary analysis, and finally, ask the audience to provide feedback. The following prompts will be used.

1. Take a minute to think about what you do when you want to make sense of a new definition. How do you know when you understand a definition?
2. Take a minute to think about what you do to support students' understanding of mathematical definitions. How does this compare to what you do for yourself?
3. Do these themes resonate with your experience? How does this help us interpret students' mathematical activity and inform instructional practice?

References

Edwards, B. S. & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly, 111*(5), 411-424.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology mathematics education: Past, present, future* (pp. 147-172). Rotterdam, The Netherlands: Sense Publishers.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Watson, A. & Mason, J. (2005). *Mathematics as a Constructive Activity: Learners Generating Examples*. Mahwah, N.J.: Lawrence Erlbaum Associates.