Concepts Fundamental to an Applicable Understanding of Calculus
Contributed Research Report

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Abstract: Calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Unfortunately, many students leave calculus with an exceptionally primitive understanding and are ill-prepared for discipline courses. This study seeks to identify the fundamental calculus concepts necessary for successful academic pursuits outside the undergraduate mathematics classroom, describe appropriate understanding of these concepts, and collect tasks that elicit, document, and measure this understanding. Data were collected through a series of interviews with select undergraduate mathematics and other discipline faculty members. The data were used to build descriptions of and frameworks for understanding the calculus concepts and generate the pool of tasks. Implications of these findings for calculus curriculum are presented.

Keywords: Calculus, understanding, design research

Introduction

According to Ganter and Barker (2004):

> *Mathematics can and should play an important role in the education of undergraduate students. In fact, few educators would dispute that students who can think mathematically and reason through problems are better able to face the challenges of careers in other disciplines—including those in non-scientific areas. Add to these skills the appropriate use of technology, the ability to model complex situations, and an understanding and appreciation of the specific mathematics appropriate to their chosen fields, and students are then equipped with powerful tools for the future.*
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> *Unfortunately, many mathematics courses are not successful in achieving these goals. Students do not see the connections between mathematics and their chosen disciplines; instead, they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings and whose importance to their future careers is not appreciated. Indeed, the mathematics many students are taught often is not the most relevant to their chosen fields. For these reasons, faculty members outside mathematics often perceive the mathematics community as uninterested in the needs of non-mathematics majors, especially those in introductory courses.*
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> *The mathematics community ignores this situation at its own peril since approximately 95% of the students in first-year mathematics courses go on to major in other disciplines. The challenge, therefore, is to provide mathematical experiences that are true to the spirit of mathematics yet also relevant to students' futures in other fields. The question then is not whether they need mathematics, but what mathematics is needed and in what context.* (p. 1)

These claims detail the rationale for The Mathematical Association of America's (MAA) Curriculum Foundations Project (http://www.maa.org/cupm/crafty/cf_project.html). This project studied the first two years of undergraduate mathematics curriculum. Portions of the mathematics community and its partner disciplines (e.g., biology, business, chemistry, computer science, several areas of engineering) worked together to generate a set of recommendations that have assisted mathematics departments plan their programs to better serve the needs of its partner or client disciplines (Ferrini-Mundy & Gücler, 2009).

The push to better serve the needs of client disciplines stemmed from the calculus reform efforts. Between the mid 1980's and the early 1990's, the undergraduate mathematics community engaged in a concentrated effort to overhaul the teaching and curriculum of beginning calculus (Ferrini-Mundy & Gücler, 2009). The heart of the reform was the concern over the depth and breadth of students' understanding of calculus (Douglas, 1986). This lack of understanding became especially apparent when students were asked to apply calculus in unfamiliar situations (Hughes Hallett, 2000).

As Ganter and Barker (2004) implied, client department faculty often complain that students are unable to apply calculus in the client coursework. Sometime this coursework asks students to use the calculus concepts in ways not familiar to them. For example, the minimization of average cost is done symbolically in calculus, whereas it is usually done graphically in economics (Lovell, 2004). At other times, even when the concept is used in a similar fashion, differences in notation or a lack of familiar cues (e.g., "maximum" or "minimum" in an optimization problem) derails students. Such difficulties in transferring knowledge between disciplines are stark indicators of a lack of understanding (Hughes Hallett, 2000). Thus, the reform called for fundamental changes in curriculum and pedagogy of beginning calculus. These changes emphasized conceptual understanding rather than procedural skills [1] (Ferrini-Mundy & Gücler, 2009).

Description of Study

The changes that have taken place during the reform years have placed greater emphasis on conceptual understanding (Hughes Hallett, 2000), but as Ganter and Barker (2004) point out, it has not been enough. So the question remains: what mathematics is needed and in what context? Following in the footsteps of the MAA's Curriculum Foundations Project, this study began exploring the potential disconnect between the calculus taught in the mathematics classrooms and the calculus needed outside the mathematics classroom at a particular undergraduate institution. Through exploring the disconnect, this study was able to identify some fundamental calculus concepts students need for successful academic pursuits outside the undergraduate calculus classroom, describe what it means to understand these concepts, and collect tasks that elicit, document, and measure student understanding of these concepts.

Describing the fundamental/core calculus concepts and creating the pool of tasks/activities constituted a design research study (Collins, 1992). In design research, the goal

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[1] Ferrini-Mundy an d Gücler did not define either *conceptual understanding* or *procedural skills*. To establish a common definition, for the purposes of this discussion, I refer the reader to the definitions offered by the MAA's Curriculum Foundations Project. Conceptual understanding is defined as the "broad understanding encompassing logical reasoning, generalization, and abstraction" (Kasube & McCallum, 2004, p. 109). Procedural skills are equated with computational ability (Kasube & McCallum, 2004, p. 109).

is to put people with different perspectives into situations that require them to express not only how they think about a concept, but to express it in a way that requires them to test and revise their way of thinking (Lesh, 2002). As such, each cycle included divergent ways of thinking, selection criteria for the most useful ways of thinking, and sufficient means of carrying forward the ways of thinking so they may be tested during the next cycle. Diversity, selection, and accumulation are necessary for iterative revisions to be passed forward.

Select faculty members at an engineering undergraduate institution participated in an iterative series of interviews during which they expressed, tested, and revised the descriptions of the fundamental calculus concepts, frameworks for understanding each concept, and associated tasks/activities. At this institution, all students are required to take two semesters of calculus and several calculus-based science and engineering courses. Mathematics and client department faculty were selected based on their proximity to the calculus courses and the client courses.

The interviews were designed around a series of concept descriptions, frameworks, and tasks developed by the researchers and/or adapted from the research of others. The intention was to provide scaffolding for the faculty to evaluate and recognize not only the necessary calculus concepts, but the ways in which the concepts need to be understood. Additionally, the tasks provided to and elicited from the faculty themselves served to provide a means to elicit, document, and measure the understanding students have of these concepts.

Results

The rounds of interviews addressed content and understanding. When faculty from mathematics and client departments were asked questions such as:[2]

- What conceptual calculus concepts must students master to be success in disciplines outside mathematics?
- What calculus (or mathematical) problem solving skills must students master to be success in disciplines outside mathematics?
- What broad mathematical topics must students master in the first two years? What priorities exist between these topics?
- What is the desired balance between theoretical understanding and computational skill? How is this balance achieved?

a dialogue centered on the fundamental calculus concepts emerged. This study will report on the blossoming of this dialogue into descriptions of essential calculus concepts and frameworks used to assess understanding of these concepts. For some of the concepts, tasks that elicit, document, and measure student understanding of the concepts were discussed and analyzed.

Implications

As stated before, calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Therefore, this study offers a collective vision to focus the content of beginning calculus courses on the meeting the needs of client disciplines. In the end, it is the mathematicians that have the responsibility to create courses and curricula that embrace the spirit of this vision while maintaining the intellectual integrity of mathematics. By explicitly

[2] Questions adapted from the MAA's Curriculum Foundations Project workshop questions (Ganter & Barker, 2004).

knowing what and how students should be prepared for client courses, teachers and curriculum developers of both calculus and client disciplines can work together to prepare students for academic success.

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