Title: Evaluating Mathematical Quality of Instruction in Advanced Mathematics Courses By Examining the Enacted Example Space

Abstract: In advanced undergraduate mathematics, students are expected to make sense of abstract definitions of mathematical concepts, to create conjectures about those concepts, and to write proofs and exhibit counter-examples of these abstract concepts. In all of these actions, students must be able to draw upon a rich store of examples in order to make meaningful progress.

We have created a methodology to evaluate what students might learn from a particular course by describing and analyzing the enacted example space (Mason & Watson, 2008) for a particular concept. This method will both give a means to create testable hypotheses about individual student learning as well as provide a way to compare disparate pedagogical treatments of the same content. Here, we describe and assess the enacted example space by studying the teaching of abstract algebra.

Keywords: example spaces, classroom research, teaching, evaluation, mathematical quality of instruction

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0 Studying teaching and the quality of instruction

There is a desire to evaluate the quality of instruction and to compare teachers based on their classroom effectiveness. The interest in evaluating the quality of instruction has engendered papers exploring the concept of quality of teaching via conceptual and empirical processes (i.e., Fenstermacher, & Richardson, 2005), and handbooks for undergraduate faculty that present ways that they may evaluate their teaching (i.e., Angleo & Cross, 1993).

The desire to evaluate the quality of instruction is made more difficult when the pedagogical techniques that different teachers use are vastly different. Consider the case of two instructors teaching an undergraduate abstract algebra course; one employs a standard lecture method, whereas the other adopts an inquiry-based pedagogy. One series of assessment instruments has been created that measures how closely teachers follow the tenets of process-product instruction (e.g., Brophy & Good, 1986); these can inform the quality of lecture-based teaching. A second set of instruments measures how closely teachers follow reform-oriented practices (e.g., Horizon, 2000; Sawada & Pilburn, 2000). However, neither the measures of the process-product tradition nor those of the reform tradition would allow for meaningful judgments to be made about the quality of instruction offered to these two groups of students, who are learning under instructors with contrasting pedagogies. Yet, given the proliferation of inquiry-based curricula for undergraduate courses and the continuing predominance of the lecture method (Pemberton, et al., 2004), this is exactly the situation that we are faced with.

The Learning Mathematics for Teaching Project (*LMTP*) has argued that we should shift our attention away from these instruments, due to their failure to take into account "one critical aspect of mathematics instruction: its mathematical quality" (2010, p. 2). Following the LMTP definition, when we refer to the mathematical quality of instruction we mean "the nature of the mathematical content available to students during instruction" (LMTP, 2010, p, 6). This is meant to be independent of the instructional format, classroom environment, or level of discourse. We do not believe that these are unimportant to student learning. We believe that the quality and range of mathematical ideas that comprise a classroom experience have direct bearing on students' ability to develop a rich understanding of mathematics regardless of the instructor's pedagogical preferences.

1 Example spaces—Students' range of thought, knowing what can vary, knowing what must stay the same

In advanced undergraduate courses, especially proof-based courses, increasing emphasis is placed on using examples as a pedagogical tool. For example, Alcock and Inglish (2008) examined doctoral students' use of examples in evaluating the truth value of claims, Dahlberg and Housman (1997) found that students who generated their own examples were more likely to develop initial understandings of concepts, and Mason and Watson (2008) described ways to make use of the range of possible variation for pedagogical purposes.

We draw upon the enacted example space to measure and compare mathematical quality of instruction and resulting potential for student learning. We argue that this is an appropriate measure of quality due to the importance of examples for student understanding in proof-based courses, and assert that this measure is meaningful across pedagogical styles. We outline a methodology for using the enacted example space to describe potential and probable student learning, and finally we show the value of this methodology by using it to analyze one aspect of instruction in an introductory abstract algebra courses.

An example space is the "experience of having come to mind one or more classes of mathematics objects together with construction methods and associations" (Goldenberg & Mason, 2008, p. 189). This example space may include relatively frequently accessed members of the classes and less accessed members of the classes, and, via the construction methods, may include new members of the classes. The first important feature of an example space is that it purposefully includes construction methods and associations such as links to important theorems and relations to other constructs. These allow mathematicians to create new examples that meet specific criteria of theorems and to determine which classes of objects are most relevant in particular situations.

Mason and Watson (2008) point out two other important features of example spaces: what aspects of the examples the learner realizes can be varied, and what range of variation the learner believes is appropriate. For example, in the case of the definition of a group, when thinking about the possible aspects of a group it is possible to think about characteristics of the underlying set, the group itself, or of the behavior of specific elements.

2 Our methodology

Video data was digitized and Transana was used to code all incidents where an example or non-example was shown, constructed or analyzed in class. We created an example log, similar to Stephan and Rasmussen's (2008) argument log which characterized each example or non-example in four columns.

- Column 1: each example or non-example of the particular construct (in this case, an algebraic group).
- Column 2: counts the number of class meetings since the formal definition of a group (a written homework assignment was coded as occurring on the day that it was assigned).
- Column 3: description of the qualities of the example or non-example. In the case of examples, the third column described any additional qualities that the example possessed from a list that would be known to first semester algebra students by the midpoint of the semester (e.g., being a commutative group, a finite group, or a cyclic group). In the case of non-examples we described any properties of the construct that were missing as well as additional properties that the non-example possessed from the list above.
- Column 4: description of the manner in which the example or non-example was made part of the classroom discourse.

2.1 Our theory of measuring the enacted example space

We use three filters to assess the enacted example space and to describe the set of examples in that space: (1) *example neighborhood*, (2) *example construction*, and (3) *example function*. We define the *example neighborhood* as the entire collection of examples that the students are exposed to during the course of their studies. These may be concrete examples or relevant non-examples of a given concept. We analyze how the examples are organized on four levels: (1) who's on first? (2) temporal proximity (3) permissible variation and (4) variation constraint. We pay particular attention to the first few examples as instructors believe they are often the ones that students most closely link with a concept (Zodik & Zaslavsky, 2008). Dienes (1963) argued that students should see examples that vary only in a constrained manner so that they are able to determine what is structural and what is allowed to vary as well as to comprehend the range of permissible variation. Then, they should see other examples that vary along a different dimension. As a result, we argue that early examples that vary along too many dimensions may actually lower the mathematical quality of instruction. Similarly, a collection of examples that fails to support student construction of critical aspects of the construct will also lead to lower mathematical quality of instruction.

Secondly, we examine the *example construction* to support a particular concept. Example construction focuses on the range of possible variation to be included in the neighborhood of a particular example space. The analysis of example construction focuses on how particular examples are created and examines the tools for creating additional examples that students may derive from the creation of examples. The construct of example construction also makes possible mapping from concrete examples to a broad description of the example space that students may have the (perhaps untapped) ability to populate for themselves. In this way, the example space explicitly includes both the examples and the means of construction (Goldenberg & Mason, 2008).

Finally, *example function* situates the example in a particular area of the example space based on its frequency of use and exemplar status. In short, example function analyzes and describes how frequently a particular example or set of examples it called upon and in what contexts. In particular, we examine which examples are most frequently called upon. Frequently used examples may obtain "ready access" status for students (linked to Vinner's (1991) concept of *evoked concept image*). The frequency of use not only gives us a means to assess or predict the student's perception of the relative importance of each of the examples, but also a means to predict which examples can most readily function as an example for them. We assess separately using each filter, and then read them together to analyze the example space.

3 Using the method

The presentation will include a preliminary analysis of the teaching of one abstract algebra class. While data analysis has begun, it is not yet complete. Preliminary results include the fact that in one traditionally taught abstract algebra course, the example neighborhood for *group* was: $(\mathbf{Z},+)$, $(\mathbf{Q},+)$, $(\mathbf{Q}^*,\cdot)$, $(\mathbf{R}^*,\cdot)$, $(\mathbf{Z}_{12},+_{12})$ and $(\mathbf{Z}_n,+_n)$. For both of the multiplicative groups, the instructor initially proposed using the complete set of rational or real numbers and then noted that zero does not have a multiplicative inverse. He then demonstrated the construction of a new set, without zero, such that all elements have multiplicative inverses. Similarly, he introduced the set $A = \mathbf{R} - \{-1\}$ and as part of the class, constructed an operation, *, such that (A, *) is a group. We claim that the instructor demonstrated examples of groups as well as two different construction methods that are likely to have become part of the students' example spaces. But, we claim that the example space will not strongly support evaluation of conjectures because all of the examples are commutative groups.

4 Questions for discussion

1) While we believe this a helpful methodology for assessing the quality of instruction and, potentially, comparing different pedagogical treatments, we wonder if it is too narrow of a lens?
2) Similarly, is it too time-intensive to be useable?
3) Besides glaringly obvious teaching suggestions like, "include non-commutative examples early and often," what potential does this have for affecting instruction of either lecture or Inquiry based teaching? Further research?
4) What more should we be doing?
5) Can this methodology be adapted to other topics such as teaching proof?

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