The Nature and Effect of Idiosyncratic Examples in Student Reasoning about Limits of Sequences

Preliminary Research Report

Catherine Hart-Weber
Arizona State University
catherine.hart@asu.edu

Michael Oehrtman
University of Northern Colorado
michael.oehrtman@unco.edu

Jason Martin
Arizona State University
jason.h.martin@asu.edu

Craig Swinyard
University of Portland
swinyard@up.edu

Kyeong Hah Roh
Arizona State University
kyeonghah.roh@asu.edu

Abstract

We apply a Vygotskian perspective on the interplay between spontaneous and scientific concepts to identify and characterize calculus students' idiosyncratic use of examples in the process of trying to formulate a rigorous definition for convergence of a sequence. Our data is drawn from a larger teaching experiment, but analyzed for this study to address questions of the origins, nature, and implications of students' nonstandard ways of reasoning. We observed two students interpreting a damped oscillating sequence as divergent, drawing from considerations from an initial, intuitively-framed definition, but remaining persistent and consistent over the duration of multiple sessions. We also trace some of the implications of their idiosyncratic reasoning for their reasoning and ultimately for their definition of convergence. We conclude by posing several questions about the nature of such example use in terms of our Vygotskian perspective.

Keywords: Limits, Definition, Examples, Spontaneous and Scientific Concepts

Introduction and Research Questions

The research literature on students' understanding of limit concepts in introductory calculus courses is replete with idiosyncratic imagery related to informal notions and ontological commitments regarding infinity (Sierpinska, 1987; Tall, 1992; Tirosh, 1991), infinitesimals (Artigue, 1991; Tall, 1990, Oehrtman, 2009), the structure of the real numbers (Cornu, 1991; Tall & Schwarzenberger, 1978), incidental and misleading aspects of graphical representations (Monk, 1994; Orton, 1983), nonmathematical concepts such as speed limits, physical barriers, and motion (Davis & Vinner, 1986; Frid, 1994; Oehrtman, 2009; Tall, 1992; Tall & Vinner, 1981; Thompson, 1994; Williams, 1991), and epistemological beliefs about mathematics in general (Sierpinska, 1987; Szydlick, 2000; Williams, 1991, 2001). Little research, however, has provided an in-depth look at the origins, nature, and implications of a single idiosyncratic image over the course of significant reasoning and problem-solving activity. As part of a larger teaching experiment, we established protocols to attempt to identify idiosyncratic images, should they appear, as calculus students try to formulate a precise definition of convergence of a sequence. The aim of this study was to address the following research questions:

1. What idiosyncratic examples do students construct as the wrestle with reinventing a formal definition for sequence convergence?
2. What are the origins of these idiosyncratic examples?
3. What are the effects of students' idiosyncratic examples on their emerging understanding of a formal definition for sequence convergence?

Theoretical Perspective

We base our study design and analysis on Lev Vygotsky's (1978, 1987) characterization of conceptual development as a complex interplay between intuitive (spontaneous) and formally structured (scientific) thought. These two types of thought are distinguished by their relationship to the objects of reference and by the nature of thought available to them. Spontaneous concepts develop first through a direct encounter with the object and form the basis of experiential knowledge developed informally over long periods of time. They are intuitive in nature and can be applied spontaneously, without conscious reflection on their meaning, but are not available for application to problems in non-concrete situations. Scientific concepts emerge later through a mediated relationship to the object, such as a verbal definition. They are expressed and initially applied only in abstract ways affording quick mastery of operations and relationships, but they are disconnected from personal experience or meaning.

Especially within a field as structurally rich as mathematics, scientific concepts are distinguished by their systematicity. Within a spontaneous concept system, where the only relationships possible are relationships between objects (and not between concepts), verbal thinking is governed by the logic of graphic imagery and thus is highly dependent on perception. Corresponding concepts are presyncretic, that is, they are not tied to other concepts in meaningful ways. It is the appearance of higher order concepts that allows this to change; the unification of concepts within a single structure allows for the comparison and analysis of subordinate concepts. To recognize contradictions or evaluate one conceptualization against another, the individual must understand two different concepts as relating to the same thing within a single superordinate structure. Comprehending the structure of a scientific concept, therefore, requires the learner to develop higher levels of reasoning, to form new categories of relationships, and to generalize.

The strengths of the scientific concept are the weaknesses of the spontaneous concept, and vice-versa. By means of their complementarities, each one lays the foundation for the development of the other. The development of the scientific concept is mediated by the spontaneous concept as intuitive modes of analysis become available to it. The spontaneous concept is in turn transformed through this mediation much in the same way that one's native language is transformed when it mediates the learning of a foreign language. The structure provided by the scientific concept enables the spontaneous concept to grow and become more available to abstract functioning.

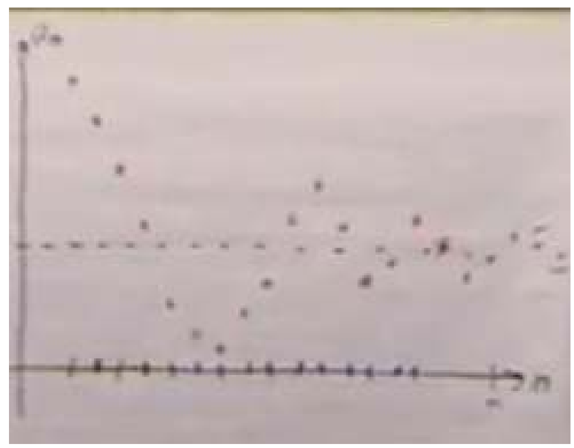
Outpacing of development, one purpose of instruction is to encourage in the student conscious awareness and volitional use of their spontaneous knowledge. This occurs as thinking is modeled within a system that is just beyond the current comprehension of the student but within their ability to imitate. Vygotsky argues that imitation is not an act of thoughtless mimicry but rather requires a beginning grasp of the structure of the system, noting that animals cannot imitate except through training. As opposed to performing a trained behavior, a student can only spontaneously imitate if the task lies within the zone of his or her own intellectual potential, the so-called "zone of proximal development."

Methods

The authors conducted a multi-day teaching experiment with two calculus students at a large, southwest, urban university. For this presentation, we report results from the first two 90-minute sessions of the study. The central objective of this portion of the teaching experiment was for the students to generate rigorous definitions of sequence convergence based on initial activity in which they generated examples of sequences that converge to 5 and sequences that do not converge to 5. These examples were then intended to i) motivate a need to generate a statement that was true of all sequences that converge to 5 but false of all sequences that do not converge to 5, ii) serve as images of their intuitive understanding of convergence against which they could test their definition, and iii) identify ways in which their definition needed refinement. The sessions were videotaped and transcribed, and we analyzed the data to first identify any examples of sequences that were improperly classified relative to their convergence or that were characterized in non-standard ways. We then identified all passages in the transcripts where students referred to these examples and looked for evidence of the origins of their non-standard interpretations, details of the ways in which they used these examples, and implications of their use of these examples.

Results

The students participating in the teaching experiment, pseudonyms Megan and Belinda, described several examples of sequence in non-standard ways during the teaching experiment, but most of these descriptions were not persistent. Nor did they use most of their idiosyncratic images to explicitly draw inferences about their definitions. One example, however, persisted over time and had wide-ranging implications for their reasoning and ultimately for their definition of a convergent sequence. Megan and Belinda both agreed that their example of a damped oscillating series (see Figure) would not converge because they noted



Megan: No matter how close it gets though there's going to be a point, you know where
Belinda: Where it's still moving away
Megan: It's still gonna come away every so often. And that, that coming away feels like it's not convergent.

Megan first suggested that this series would be divergent after trying to apply an early definition including the phrase "at some point *N*, it becomes closer and closer" to the example and concluding that since it was originally in their list of examples that converge to 5, it must eventually become monotonic. When one of the researchers asked "What if it continued to go away and come back and go away, but always going away less, would it be convergent?" she insisted it must then be placed in the category of examples not converging to 5, to which Belinda strongly agreed. Subsequently, both students continually returned to this example during the first two days of the teaching experiment and cited it's divergence as a reason to include statements like "and always gets closer to 5" or "$|5 - a_{n+1}| < |5 - a_n|$" in their definition of a sequence $\{a_n\}$ converging to 5. Even though they recognized that "for any chosen acceptable error range, there would be some point after which $|5 - a_n|$ does not exceed [that bound]," they still insisted that this sequence was divergent since the "errors don't get smaller."

Megan and Belinda were amazingly consistent in their nonstandard interpretation of this example and raised logical counterarguments to any suggestions made by the researchers why one might consider the damped oscillating series to be convergent. They even suggested adding the phrase "or is always 5" to their definition to include the constant sequence a_n=5 as convergent but still be able to keep their statement "$|5 - a_{n+1}| < |5 - a_n|$." Thus we see that although their reasoning is highly idiosyncratic, it is also systematically structured and applied volitionally with conscious awareness.

Questions

In our presentation, we will show video clips tracing the origins of Megan and Belinda's idiosyncratic interpretation of the convergence for a damped oscillating sequence, the nature of their arguments about the sequence, the effects they had on their definitions, and finally the method by which we convinced the students to alter their definition to include this sequence as convergent. We are interested on feedback from the audience on the following questions:

1. Is the reasoning of these students appropriately characterized as spontaneous, scientific, or neither? What are the implications of this?
2. Given the idiosyncratic but logical and consistent reasoning illustrated, what is the nature of the zone of proximal development for these students?
3. How might instruction mediate a productive interaction for these students with the standard interpretation held by the mathematics community?

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