The Treatment of Composition the Secondary and Early College Mathematics Curriculum

Aladar Horvath
Michigan State University

While many studies have focused on student knowledge of function, few studies have focused on composition. This report describes a curriculum analysis of the treatment of composition in the secondary (algebra, geometry, algebra 2, precalculus) and early college (precalculus, calculus) mathematics curriculum. In this study composition is conceptualized as a sequence of functions and as a binary operation on functions. The curriculum analysis utilizes a framework of conceptual, procedural, and conventional knowledge elements as well as representations and types of functions. Preliminary data will be presented during the session and a discussion will center on conceptual, procedural, and conventional knowledge elements for composition.

Keywords: composition, curriculum analysis, conceptual and procedural knowledge, representations

The function concept is an essential topic to the undergraduate mathematics curriculum and to mathematics, in general. Freudenthal (1983) stated, "The strength of the function concept is rooted in the new operations - composing and inverting functions" (p. 523). The operation of composition is vastly different than the arithmetic operations (i.e., addition, subtraction, multiplication, division) that students encounter in elementary school. For example, composition is applied to mathematical objects such as functions (including constant functions), relations and transformation, but not to numbers. While educational researchers have extensively studied student knowledge of function (Carlson, 1998; Even, 1990, 1998; Ferrini-Mundy & Graham, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Monk, 1994; Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989), research on the teaching and learning of the operation of composition has received little attention (Engelke, Oehrtman, and Carlson, 2005).

The existing research on composition has documented that the learning of composition is nontrivial for students. Research on the learning of topics built upon composition (i.e., chain rule) has reported that students' difficulties are related to a weak foundation of composition (Clark et al., 1997; Horvath, 2008). Studies of composition have focused on student knowledge or the output of learning. None have reported on the teaching of composition or on the written curriculum or the input of learning except for one study that focused on the genetic decomposition (Ayers et al., 1988). The study reported here is a beginning to fill this gap through a curriculum analysis on the treatment of composition in secondary (algebra, geometry, algebra 2, precalculus) and early collegiate (precalculus, calculus) mathematics textbooks. While the written curriculum does not determine what teachers teach or what students learn, the written curriculum influences both (Remillard, Herbel-Eisenmann, & Lloyd, 2009). The research questions guiding this work are: In what ways is the concept of composition developed across the algebra to calculus curriculum? Is there a difference between the way composition is treated in the secondary curriculum and college precalculus and calculus curriculum and if so, what are the characteristics of that gap?

In this study composition is conceptualized in two ways which relate to the notations of $g(f(x))$ and $(g \circ f)(x)$. First is the *sequence view of composition*. In this view, $g(f(x))$ denotes a sequence of functions where f corresponds x to $f(x)$ and g corresponds $f(x)$ to $g(f(x))$. Thus, the output of f, $f(x)$, is the input of g and it is the elements x and $f(x)$ that are being acted upon. In general, this view of composition describes composition as a sequence of recursive relations (including functions) where the input of the *n*th term is the output of the (*n-1*)th term. Carlson, Oehrtman, and Engelke (2010) described this as a process view of composition, while Harel and Kaput (1991) referred to this process of "acting on individual elements of [the] domain" and called it a *point-wise operation* (p. 84).

The other conceptualization of composition is the *operation view of function*. In this view, $(g \circ f)(x)$ is a binary operation on two functions, f and g, resulting in a new function $g \circ f$. In this case functions are the objects being acted upon and not simply their domain and range elements. Others who have written about replacing processes with objects include Asiala et al. (1996) using the term encapsulation, Sfard (2008) using the term reification, and Martin (1991) using the term of nominalization which is a specific case of Halliday's (1985, 1995) grammatical metaphor. The common feature among these perspectives is that processes (or verbs) are treated as entities (or nouns) which become the objects of other actions and procedures (or verbs). For example, the function $f(x) = x^2$ can be viewed as the process of corresponding any number to its square. Composing the function $g(x) = 2x + 4$ with $f(x)$ can be viewed as "plugging in" $f(x)$ into the x's in the $g(x)$ function resulting in $g(f(x)) = 2(f(x)) + 4$ or $g(f(x)) = 2(x^2) + 4$. In this situation, $f(x)$ is treated as an object and not as a correspondence between its domain and range.

Research on the sequence view of composition has reported students have interpreted the composition statement of f (g(3) as the multiplication statement of $f(3) \cdot g(3)$ (Engelke et al., 2005, Meel 1999). These studies have reported students interpreting composition as multiplication while using formulas, graphs, and tables. Research has also shown that students have different success rates on composition problems in different representations. When asked to evaluate $g(f(2)$, Carlson et al. (2010) reported that 94% of students were successfully given two algebraic functions, 50% with graphical functions and 47% with tabular functions. Hassani (1998) reported students' success rates as 84%, 10%, and less than 50% for algebraic, graphical, and tabular, respectively. When the task was rephrased to evaluate $(g \circ f)(2)$ the success rates of students in Hassani's study changed to 35%, 25% and 33%, respectively. In an interview with a student in a developmental algebra course DeMarois & Tall (1996) reported that he was able to complete a composition task using the table with considerable guidance from the interview, was then unable to begin graphical composition task, but following that he was successful with minimal guidance on the algebraic composition task. This research implies that algebraic composition tasks are easier for students than other representations. One explanation has claimed that this is due to a curriculum that is heavily algebraic and that students have had more exposure and experience with dealing with the algebraic representation (Hitt, 1998). However, a curriculum analysis has not been conducted to empirically validate such claims.

Research on the operation view of composition has reported that students frequently implement this view by plugging in or substituting the one function for a variable in the other function (Ayers, et al., 1988; Carlson, 1998; Horvath, 2010; Uygur & Ozdas, 2007) or by interpreting composition as multiplication (Horvath, 2010; Meel, 1999). The difference in the multiplication between the sequence view and the operation view is that students not only multiply numbers, but are also multiplying objects such as functions. This interpretation appears symbolically as $(f \circ g)(x) = f(x) \cdot g(x)$.

This study uses the conceptual knowledge and procedural fluency framework to study curriculum materials. Many scholars have participated in the debate of conceptual and procedural knowledge. Piaget, Tulving, Anderson, Scheffler, and Skemp are a few who have done so. Hiebert and Lefevre (1986) described conceptual knowledge as knowledge that is rich in relationships and is like a network where both the vertices and the edges (words taken from Graph Theory) are essential and of equal importance. "In fact, a unit of conceptual knowledge cannot be an isolated piece of information" and the individual must consciously recognize links to other information (p. 4). They described procedural knowledge in two components. "One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6).

For this study of curriculum conceptual knowledge includes definitions and properties of composition such as the associativity and commutativity (in rare situations), the non-uniqueness of decomposition, etc. Procedural fluency elements performance of procedures and algorithms such as evaluate the composition, find the domain, decompose a function, etc. Vocabulary of important terms and notation is placed under the separate category called Conventional Knowledge Elements. This would include items such as the parenthetic $f(g(x))$ and circle, $f \circ g$, notations and what objects are described as being "composite." Other major categories in this study's framework are Representation (i.e., algebraic, graph, table) and Function Type (i.e., polynomial, trigonometric, exponential, logarithmic, piece-wise).

Method

In order to better understand the potential influence of the written curriculum on what opportunities to learn students have, this study analyzes the development of the concept of function composition in written curriculum over the span from Algebra to Calculus. High school curricula will be analyzed to study examples of the ways in which students are introduced to composition in high school (CCSS-M, 2010). The texts to be analyzed will include entire series of Algebra 1 and 2, Geometry, and Precalculus. The notion of composition is developed further in calculus which many students study in college. Thus, collegiate Precalculus and Calculus texts will also be analyzed. The duplication of the precalculus text at both the high school and college level will help identify any differences between the preparation for calculus at the different levels.

The two secondary mathematics curriculum series to be analyzed are *Glencoe/McGraw Hill Mathematics* (2010/2011) and the *CME Project* (2009). Glencoe Mathematics was chosen due to its large share of the secondary school market (see Dossey et al., 2008). The *CME Project* materials were chosen to be the second series because it has been developed more recently and have different features that provide a broader view of the treatment of composition across curricula. At the collegiate level, a widely used precalculus and calculus series was determined by surveying approximately 100 Department of Mathematics' websites and identifying the texts used for calculus and precalculus courses. The institutions chosen for the survey are those classified as very research intensive in the Carnegie Classification.This survey was conducted in June 2010. The survey results identified *Calculus: Early Transcendentals, 6th edition* (2008) by Stewart and *Precalculus: Mathematics for Calculus, 5th edition* (2006) by Stewart, Redlin, and Watson as the most widely used calculus and precalculus texts, respectively. The second precalculus and calculus text to be analyzed is *Functions Modeling Change: A Preparation for Calculus, 4th edition* (2011) by Connally, Hughes-Hallett, Gleason, et al. and *Calculus, 5th edition* (2009) by Hughes-Hallett, Gleason, McCallum, et al., respectively.

The content included in the analysis was determined by the following criteria. These criteria include both the explicit development and implicit use of composition. Any lesson that includes exposition regarding function composition in the student or teacher edition is considered to be explicitly developing the concept of composition. In those situations the entire lesson was included in the analysis. Lessons on function operations, inverse function, and composition of geometric transformations are examples of explicit development. For implicit uses of composition, only the sentence (if in the exposition) or the example will be included in the analysis. A few examples of implicit use of composition include translations of graphs, solving equations involving trigonometric functions with non-trivial angles. Exercises and review problems that explicitly or implicitly use composition were also included.

The results of this study will be important for secondary and university mathematics teachers as well as curriculum developers. It may reveal aspects of composition that are over- or under-emphasized. It will also inform college instructors on how students have been prepared by the secondary curriculum with respect to what is expected in the early calculus curriculum.

Questions to be posed to the audience

What do you consider to be a conceptual knowledge element or task for composition?
What topics (e.g., inverse function, chain rule) do you consider as explicit use of composition?

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education,* 2, 1-32.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education, 19*(3), 246-259.

Carlson, M. P. (1998). A cross sectional investigation of the development of the function concept. CBMS Issues in Mathematics Education Vol 7. 114-162.

Carlson, M., Oehrtman, M., and Engelke N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145.

Clark, J., F. Cordero, J. Cottrill, B. Czarnocha, D. DeVries, D. St. John, T. Tolias, & D. Vidakovic. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior*. 16(4), 345-364.

Common Core State Standards for Mathematics (CCSS-M) (2010).

DeMarois, P. & Tall, D. (1996). Facets and layers of the function concept. In ---------- (Eds.) Proceedings of PME 20, vol 2. (pp. 297-304). Valencia.

Dossey, J., Halvorsen, K., & McCrone, S. (2008). *Mathematics education in the United States 2008: A capsule summary factbook*. Reston: National Council of Teachers of Mathematics.

Engleke, N., Oehrtman, M., & Carlson, M. (2005). Composition of function: Precalculus students' understandings. In Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Roanoke, VA.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521-544.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior*. 17(1), 105-121.

Ferrini-Mundy, J. & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*. 98(7), 627-635.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel Publishing Company.

Halliday, M. A. K. (1985). *An introduction to functional grammar*. London: Edward Arnold.

Halliday, M. A. K. (1995). Language and the theory of codes. In A. R. Sadovnik (Ed.) *Knowledge and Pedagogy: The Sociology of Basil Bernstein* (pp. 127-144). Norwood, NJ: Ablex Publishing Corporation.

Harel, G. & Kaput, J. (1991). The role of conceptual entities in the construction of advanced mathematical concepts and their symbols. In D. O. Tall (ed.). *Advanced Mathematical Thinking*. (pp. 82-94).

Hassani, S. (1998). *Calculus students' knowledge of the composition of functions and the chain rule.* Unpublished doctoral dissertation, Illinois State University, Normal.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*. 17(1), 123-134.

Horvath, A. (2008). Looking at calculus students' understanding from the inside-out: The relationship between the chain rule and function composition. Proceedings of the 11th Annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA.

Horvath A. K. (2010). Calculus students, function composition, and the chain rule. *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Columbus, OH.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Martin, J. R. (1991). Nominalization in science and humanities: Distilling knowledge and scaffolding text. In E. Ventola (Ed.) *Functional and Systemic Linguistics* (pp. 307-337). Berlin: Mouton de Gruyter.

Meel, D. E. (1999). Prospective teachers' understandings: Function and composite function. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*. 1, 1-12.

Monk, G. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*. 9, 21-24.

Oehrtman, M., M. Carlson, and P. W. Thompson. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In *Making the Connection: Research and Practice in Undergraduate Mathematics, MAA Notes Volume 73*, eds. M. Carlson and C. Rasmussen, 27-41. Washington, DC: Mathematical Association of America.

Remillard, J. T., Herbel-Eisenmann, B. A., & Lloyd, G. M. (Eds.) (2009). *Teachers at work: Connecting curriculum materials and classroom instruction*. New York: Routledge Taylor, and Francis.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge, UK: Cambridge University Press.

Uygur T., & Ozdas, A. (2007). The effect of arrow diagrams on achievement in applying the chain rule. *Primus, 17*(2), 131-147.

Vinner & Dreyfus, (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.