Student Use of Set-Oriented Thinking in Combinatorial Problem Solving
Preliminary Research Report

This study seeks to contribute to research on the teaching and learning of combinatorics at the undergraduate level. In particular, the authors draw upon a distinction characterized in combinatorial texts between set-oriented and process-oriented definitions of basic counting principles. The aim of the study is to situate the dichotomy of set-oriented versus process-oriented thinking within the domain-specific combinatorial problem-solving activity of students. The authors interviewed post-secondary students as they solved counting problems and examined alternative solutions. Data was analyzed using grounded theory, and a number of preliminary themes were developed. The primary theme reported in this study is that students showed a strong tendency to utilize set-oriented thinking during the problem-solving phase that Carlson & Bloom (2005) refer to as checking, especially when they engaged in the evaluation of alternative solutions.

Keywords: combinatorics, counting, problem-solving, grounded theory

Introduction and Motivation. In spite of the seemingly elementary nature of "counting," students tend to experience a great deal of difficulty as they encounter increasingly complex counting problems. These difficulties are well-documented in the mathematics education research literature (Batanero, Navarro-Pelayo, & Godino, 1997; English, 2005; Kavousian, 2006). Also well-established is the relevance of combinatorics in the K-12 and undergraduate curricula (Batanero, Navarro-Pelayo, et al., 1997; English, 1991; NCTM, 2000), particularly because of its applications in probability and computer science. English (1993) emphasizes the value in studying combinatorics education, noting that "the domain of combinatorics is a particularly fertile field for research in mathematics education" (p. 451). Attempts have been made to improve the implementation of combinatorial topics in the classroom (Kenney & Hirsch, 1991; NCTM, 2000), but in spite of such efforts, students overwhelmingly struggle with understanding the concepts that underpin this growing field. Batanero, Godino and Navarro-Pelayo (1997, p. 182) make the following claim:

> …[C]ombinatorics is a field that most pupils find very difficult. Two fundamental steps for making the learning of this subject easier are understanding the nature of pupils' mistakes when solving combinatorial problems and identifying the variables that might influence this difficulty.

This call by Batanero, Godino et al. acknowledges the difficulties described above, and it also highlights a need for a deeper look at students' ways of thinking that will help researchers comprehend the nature of their mistakes.

This preliminary report stems from the first author's doctoral dissertation work, which examines two particular ways of combinatorial thinking. The aim of the study is to situate the dichotomy of set-oriented versus process-oriented thinking within students' domain-specific combinatorial problem-solving activity. Combinatorics textbooks (e.g., Brualdi, 2004; Tucker, 2002) tend to formulate two foundational counting principles – the addition and multiplication principles – in one of two different ways: either they employ *set-theoretic* language or they describe them using *process-oriented* language. For example, as found in Tucker (p. 170, emphasis in original) the exact statements of each principle are:

> <u>The addition principle</u>: If there are r_1 different objects in the first set, r_2 different objects in the second set, …, and r_m different objects in the mth set, and *if the different sets are disjoint*, then the number of ways to select an object from one of the m sets is $r_1 + r_2 + ... + r_m$.
>
> <u>The multiplication principle</u>: Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage,

…, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages and *if the composite outcomes are all distinct*, then the total procedure has $r_1 \times r_2 \times ... \times r_m$ different composite outcomes.

In Tucker's definition of the addition principle, his language involves sets explicitly. The definition reflects a fundamental conception of counting as the enumeration of the number of objects in a set. In his definition of the multiplication principle, however, counting is framed as the completion of a task consisting of successive stages. Other authors reflect this distinction as well; some (e.g., Brualdi, 2004; Rosen, 2007) include two different definitions of each principle, one in terms of sets, and the other in terms of processes.

This dichotomy in the way mathematicians present these basic principles suggests that a relevant distinction could be manifested in student approaches to counting problems. The literature does not address this issue – only a handful of studies in combinatorics education (English, 1991; Hadar & Hadass, 1981) refer to the distinction between sets and processes at all, but no study has explicitly addressed this phenomenon and its potential bearing on students' counting. Noticing this distinction between sets and processes has led the authors to study whether these two formulations indicate any differences in the ways students think about and approach counting problems.

Design and methodology. In designing the study, based on her experiences, the first author suspected that students may draw more heavily on set-oriented thinking when asked to justify whether an answer is right or wrong. Therefore, in an attempt to narrow the scope, she purposefully put students in situations in which they had to evaluate alternative solutions (thus engaging in error detection and correction). Furthermore, for efficiency, she focused on counting problems that are commonly susceptible to errors – problems that have incorrect solutions that frequently seem correct to students. Problems were drawn from Martin (2001) and Tucker (2002).

In the study reported here, eight students were interviewed individually in two 60-90 minute, videotaped sessions. The students were drawn from an upper-division mathematics courses at a large urban university and included mathematics majors, computer science majors, and post-baccalaureate students. In order to accomplish the goals above, the general interview protocol was as follows. In Interview 1, the subjects were given five to seven counting problems and were instructed to solve them as they naturally would (some talked with the author during this time, others were silent). Then, they were asked to explain their thought process and were posed questions about their work. At no point in either interview were they told whether or not a given answer was correct. In Interview 2, students were given alternative answers to the same problems they had solved in Interview 1. They were asked to evaluate the new answers, explore how the new answer compared to their original answer, and determine which answer they thought was correct.

The videotape of each interview was viewed repeatedly and transcribed. The methodological framework of grounded theory (Strauss & Corbin, 1998; Auerbach & Silverstein, 2003) was implemented in order to code the data. Coding consisted of the initial identification of repeated ideas (Auerbach & Silverstein, Ibid) and phenomena, which were then consolidated into themes related to the set/process distinction. The authors also drew upon Carlson & Bloom's (2005) problem solving cycle (which consists of four major stages: orienting, planning, executing, and checking) for analysis purposes. Coding thus took place along two dimensions: based on phenomena that the authors observed and categorized, and according to the problem-solving stages put forth by Carlson & Bloom.

Results. Preliminary analysis of the data indicates promising themes about the occurrences of set- and process-oriented thinking as students solve counting problems. In some contexts, there does indeed seem to be some correlation between the types of counting activity students carry out as they draw upon certain kinds of thinking. The primary theme reported in this study is that students show a strong tendency toward set-oriented thinking during the checking problem-solving phase, particularly when they engage in the evaluation of alternative solutions. Specifically, there is evidence of students categorizing an answer as incorrect by identifying a particular object that was counted more than once. Additionally, there are cases in which students identified two different answers as the same when they evaluated the process – it was not until they adopted a set-oriented perspective that they could explain the different numerical results.

While not all of these findings may be explored in this proposal, an example of student work on one problem is discussed below. The Test Questions problem states *A student must answer five out of ten questions on a test, including at least two of the first five questions. How many subsets of five questions can be answered?* One solution to this problem utilizes a case breakdown, based on whether exactly two, three, four, or five of the first five questions are answered, yielding $\binom{5}{2}\cdot\binom{5}{3}+\binom{5}{3}\cdot\binom{5}{2}+\binom{5}{4}\cdot\binom{5}{1}+\binom{5}{5}\cdot\binom{5}{0}$. A common incorrect answer is $\binom{5}{2}\cdot\binom{8}{3}$, which is obtained by first choosing two of the first five questions to answer, and then choosing any three of the eight remaining questions to answer. The rationale behind such a solution is that the "at least two" constraint is satisfied in the first step, and thus any remaining choice of three problems will still satisfy the constraint of the problem. The trouble with this strategy, however, is that some of the possible outcomes can be counted more than once. For example, in utilizing this strategy, suppose problems 1 and 2 are chosen as the first step. Then, in the next step, problems 3, 7 and 8 are chosen as the second step. Thus, the subset of five questions to be answered is {1, 2, 3, 7, 8}. However, this subset could be found in a different way using the same counting strategy, namely, by first choosing problems 1 and 3, and then choosing problems 2, 7, 8. Thus, the expression $\binom{5}{2}\cdot\binom{8}{3}$ actually counts some solutions more than once and is therefore incorrect. If the students solved this problem correctly initially, they were given the common incorrect answer in Interview 2. If they first arrived at an incorrect answer, they were asked to examine the correct solution.

Don was a student who displayed both set and process-oriented thinking at various times. In the excerpt below, while working on the Test Questions problem Don decides that the incorrect expression is too big. In justifying this belief, he appeals to two sets of questions generated by the incorrect attempt that are in fact the same. That is, in order to show that $\binom{5}{2}\cdot\binom{8}{3}$ is incorrect, he identifies a particular set of questions (a1, a2, a3, a9, a10) that is counted more than once.

> Don: And so let's see I have, we'll call them a1 and a2 [*he writes down the numbers as he's talking*], and then I also have a3, a9, and a10. But then, add up all these combinations again, you know next time I might have a1, and a3, and then a2 and a9 and a10, and, so this is the same, and this [the incorrect attempt] um, perhaps doesn't account for that.

Don's response reflects that, on some level, he was able to view the counting process as the enumeration of objects – he is counting objects in a set, and he identified one object that was counted too many times. Set-oriented thinking was his chosen way of justifying to himself that the incorrect attempt was too big.

Conclusion. The study described here suggests that students draw upon set-oriented thinking during particular moments in their combinatorial problem solving. These findings stand to inform current understandings of student thinking about counting, offering a meaningful contribution to the field of mathematics education. Subsequent research will include an additional round of data collection based on the preliminary themes identified in this study. The authors will continue to examine the data and make connections among themes that emerge in the new data set, coordinating new and existing themes appropriately. The first author also hopes to conduct further studies that address similar issues, perhaps investigating ways in which combinatorial mathematicians view the domain specific set versus process dichotomy.

Questions:

1) Is the distinction between sets and processes, as it relates to combinatorics, a relationship that teachers of combinatorics have noticed?
2) In what ways can the process-oriented thinking, as specified here, be related to other themes in mathematics education (such as students' notions of functions)?
3) For what other areas of combinatorics and discrete mathematics might this distinction be relevant, and in what ways?
4) What would a domain-specific problem-solving framework for combinatorics look like?

References

Auerbach, C. & Silverstein, L. B. (2003). *Qualitative data: An introduction to coding and analysis.* New York: New York University Press.

Batanero, C., Godino, J., & Navarro-Pelayo, V. (1997). Combinatorial reasoning and its assessment. In I. Gal & J. B. Garfield (Eds.), *The Assessment Challenge in Statistics Education* (pp. 239-252): IOS Press.

Batanero, C., Navarro-Pelayo, V, & Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, *32*, 181-199.

Brualdi, R. A. (2004). *Introductory Combinatorics* (4th ed.). Upper Saddle River, New Jersey: Pearson Prentice Hall.

Carlson, M & Bloom, I. (2005) The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58,* 45-75.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics*, *22*, 451-474.

English, L.D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G.A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning*. Kluwer, Academic Publishers, 40, 121-141.

Hadar, N., & Hadass, R. (1981). The road to solve combinatorial problems is strewn with pitfalls. *Educational Studies in Mathematics*, *12*, 435-443.

Kenney, M. J., & Hirsch, C. R. (Eds.). (1991). *Discrete mathematics across the curriculum, K-12: 1991 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Kavousian, S. (2008) *Enquiries into undergraduate students' understanding of combinatorial structures.* Unpublished doctoral dissertation, Simon Fraser University – Vancouver, BC.

Martin, G. E. (2001). *The Art of Enumerative Combinatorics*. New York: Springer.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Rosen, K. H. (2007). *Discrete Mathematics and Its Applications* (6th ed.). New York: McGraw Hill.

Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (2nd ed.). Thousand Oaks, California: Sage Publications, Inc.

Tucker, A. (2002). *Applied Combinatorics* (4th ed.). New York: John Wiley & Sons.