Do Leron's structured proofs improve proof comprehension?
Preliminary Research Report

Abstract
In undergraduate mathematics courses, proofs are regularly employed to convey mathematics to students. However, research has shown that students find proofs to be difficult to comprehend. Some mathematicians and mathematics educators attribute this confusion to the formal and linear style in which proofs are generally written. To address this difficulty, Leron (1983) suggested an alternative format for presenting proofs, named *structured proofs*, designed to enable students to perceive the main ideas of the proof without getting lost in its logical details. However, we are not aware of any empirical evidence that such format actually helps students comprehend proofs. In this presentation we report preliminary results of a study that employs a recent model of proof comprehension to assess the extent to which Leron's format help students comprehend proofs.

1. Introduction

In advanced mathematics courses, proofs are a primary way that teachers and textbooks convey mathematics to students. However, researchers note that students find proofs to be confusing or pointless (e.g., Harel, 1998; Porteous, 1986; Rowland, 2001) and undergraduates cannot distinguish a valid proof from an invalid argument (Selden & Selden, 2003; Weber, 2009). Some mathematicians and mathematics educators attribute students' difficulties in understanding proofs to the formal and linear style in which proofs are written (e.g., Thurston, 1994; Rowland, 2001).

To address this difficulty, several mathematics educators have suggested alternative formats for presenting proofs, such as using generic proofs (e.g., Rowland, 2001), e-proofs (Alcock, 2009), explanatory proofs emphasizing informal argumentation (e.g., Hanna, 1990; Hersh, 1993), and structured proofs (Leron, 1983). These suggestions have an obvious appeal; if changing the format of a proof can increase students' understanding of its content, then these alternative proof formats provide a practical way to improve the effectiveness of lectures and textbooks in advanced mathematics courses. However, we are not aware of any empirical evidence that suggests any of the proposed formats above actually increase students' understandings of the proofs they read or observe. In fact, at least one study suggests the opposite. When Roy, Alcock, and Inglis (2010) attempted to see if Alcock's (2009) e-proofs improve students' comprehension of proofs in a pilot study, they found that students who studied an e-proof surprisingly performed significantly *worse* on a post-test than students who studied the same proof from a lecture or textbook.

The goal of this study is to examine the extent to which Leron's (1983) structured proofs will improve student understanding. Leron (1983) suggests that linear proofs limit students' understanding because this format masks the overarching structure of the proof and the methods and variables introduced in a linear proof appear to come out of thin air. He suggests instead organizing a proof into levels with Level 1 providing a summary of the main ideas of the proof (without going into detail at his level into how they will be implemented), Level 2 giving a summary of how each of the main ideas is implemented, and successively lower levels filling in more of the details of the proof. In some cases, an "elevator" between levels provides an informal rationale for why the proof is proceeding the way that it is. This format enables the reader to perceive the main ideas of the proof without getting lost in its logical details, but still allows the reader to read about or verify these logical details if he or she desires to do so.

Although several mathematics education researchers cite Leron's structured proofs as a possible way to improve proof presentation (e.g., Alibert & Thomas, 1991; Hersh, 1993; Movshovits-Hadar, 1988), we are not aware of any empirical evidence that such proofs will help students. Indeed, in an exploratory study, Cairns and Gow (2003) present theoretical difficulties that students may encounter with a structure proof and illustrate how some students experience these difficulties based on interviews with three students. They concluded a structured proof "is not *a fortiori* the best presentation for proofs" (p. 186).

2. Theoretical perspective

Our model and means of assessing proof comprehension is based on Mejia-Ramos et al's (2010) presentation at last year's RUME conference. This model posits that students' proof comprehension can be measured along six dimensions: (a) understanding of terms and statements in the proof, (b) ability to cite justifications for statements in the proof, (c) the logical structure of the proof, (d) the high-level ideas of the proof, (e) the method used in the proof, and (f) how the proof relates to examples or informal images. Our assessment of students' proof comprehension was based on this model.

3. Methods

For this study, we recruited two groups of six students. Each participant met individually with one of the co-authors of this paper. The participants were asked to study a proof and were told they would be asked a series of questions about the proof. After they studied the proof to their satisfaction, they returned the proof to the interviewer. The participants were asked on a scale of 1 through 5 how well they understood the proof, with a 5 indicating they understood the proof completely. They were then asked an open-ended question about the proof (e.g., "How was the fact that $f'(x)>0$ used in the proof?") followed by a multiple-choice question of the same item. After they answered all the questions, the proof was returned and they were permitted to change their answers. This process was repeated with a second proof. The assessment questions were based on the model of Mejia-Ramos et al (2010).

Participants in the first group (Group A) first studied a linear presentation of a proof of the assertion "The only solution to the equation $x^3 + 5x = x^2 + \sin x$" (from here on Proof 1). They then studied a structured proof of the statement "There are infinitely many primes of the form 4k+3". Participants in the second group (Group B) studied a structured version of Proof 1 and a linear version of Proof 2. The structured and linear versions of Proof 2 were taken with minor modifications from Leron (1983). If participants read a structured proof, they were also asked about their opinions of the proof, what (if anything) they found positive about it, and what (if anything) they found negative about it.

Our analysis focuses on: (a) how well participants felt they understood the proofs they read, (b) participants' performance on the open-ended questions that they answered immediately after reading the proof (without having the proof to refer to), and (c) participants' comments on the benefits and drawbacks of structured proofs.

4. Results

For Proof 1, Group A (who received the linear version of the proof) appeared to perform better than Group B (who received the structured version) on the assessment items. On average, they answered 5 of the 8 assessment questions correctly (63%) while the students in Group B answered only 2.33 questions correctly (29%). Group A and Group B reported nearly equal levels of understanding Proof 1 (4.17 vs. 4.00).

For Proof 2, Group A (who received the structured version of this proof) performed slightly better than Group B (who received the linear version). They answered 2.5 of the 7 assessment questions correctly (36%) while Group B answered 2 questions correctly (29%). Group B reported a higher level of understanding than Group A for Proof 2 (3.83 vs. 2.33).

Combining across proofs, participants studying the linear proofs reported a mean understanding of 4.00 and answered an average of 7 of the 15 assessment questions correctly (47%), while students studying structured proofs reported a mean understanding of 3.13 and answered 4.83 out of 15 assessment questions correctly (32%).

Among the 12 participants, two reacted positively to the structured proof format, citing that it made explicit the goals of proof and the relationships between its different parts. The remaining 10 participants cited drawbacks with the approach, with some claiming they found it generally confusing.

5. Discussion

In summary, this study did not find evidence that structured proofs improved students' comprehension of proofs. When the participants read a structured proof as opposed to a linear proof, they reported less understanding and performed worse on the assessment questions. Only two participants cited more benefits of structured proofs than drawbacks, with the remaining participants citing that the difficulties in following the structured proofs hindered their understanding.

Of course, it is imperative to note our study does **not** demonstrate that structured proofs are ineffective as the design of our study could be criticized on several grounds. Most importantly, our sample size was limited and we cannot infer that the results of our study would not change if we expanded our sample. We also note that there are threats to the construct validity of our study. In mathematics classes, students are not given a short period of time to read a proof and then are given a test on it; they are often given a proof and expected to study it for a longer time over several days. Finally students' difficulty may have been due to the novelty of the structured format. Perhaps giving students more exposure to structured proofs, or instruction on reading them, may have improved their performance.

On the other hand, Roy, Alcock, and Inglis (2010) illustrate how a theoretically motivated alternative proof presentation format can, in some cases, decrease students' understanding of the proof. We note that our results about students' difficulties with structured proofs are consistent with the findings of Cairns and Gow (2003). Finally, we also note there are no empirical studies that offer any evidence that structured proofs do improve understanding.

We are not arguing such studies cannot be done, but we believe they would take careful thought to design, and would likely include instruction for students on how a structured proof should be read. We contend such studies are necessary if structured proofs are to continue to be proposed as a means of increasing students' proof comprehension, both because claims of this type in mathematics education should require empirical support and because a study of this type can offer practical pedagogical direction for teachers who wish to incorporate pedagogical proofs in their own classrooms.

6. Questions for audience

Under what conditions might we see the benefits of structured proofs? What type of evidence would be required to convince the community that structured proofs (or, more generally, any pedagogical suggestion) might not be effective?

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