Where is the Logic in Proofs?
Preliminary Research Report

Milos Savic
New Mexico State University
milos@nmsu.edu

Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs. Logic, in some form, does seem to play a crucial role in constructing proofs. Yet, this study of 43 student-constructed proofs of theorems about sets, functions, real analysis, abstract algebra, and topology, found that only 1.7% of proof lines involved logic beyond common sense reasoning. Where is the logic? How much of it is just common sense? Does proving involve forms of deductive reasoning that are logic-like, but are not immediately derivable from predicate or propositional calculus? Also, can the needed logic be taught in context while teaching proof-construction instead of first teaching it in an abstract, disembodied way? Through a theoretical framework emerging from a line-by-line analysis of proofs and task-based interviews with students, I try to shed light on these questions.

Keywords: Logic, transition-to-proof courses, analysis of proofs, task-based interviews

To obtain a Masters or Ph.D. in Mathematics, one must be able to construct original proofs. This process of proof construction is usually explicitly taught, if at all, to undergraduates in a transition-to-proof or "bridge" course. At the beginning of such courses, teachers often include some formal logic, but how it should be taught is not so clear. Epp (2003) stated that, "I believe in presenting logic in a manner that continually links it to language and to both real world and mathematical subject matter" (p. 895). However, some mathematics education researchers maintain that there is a danger in relating logic too closely to the real world: "The example of 'mother and sweets'[1] episode, for instance, which is 'logically wrong' but, on the other hand, compatible with norms of argumentation in everyday discourse, expresses the sizeable discrepancy between formal thinking and natural thinking…" (Ayalon & Even, 2008, p. 245).

There are also those who do not think that logic needs to be explicitly introduced. For example, Hanna and de Villiers (2008) stated, "It remains unclear what benefit comes from teaching formal logic to students or to prospective teachers, particularly because mathematicians have readily admitted that they seldom use formal logic in their research" (p. 311). Selden and Selden (2009) claimed that "Logic does not occur within proofs as often as one might expect … [but] [w]here logic does occur within proofs, it plays an important role" (p. 347). Taken together,

[1] The scenario is that the mother says to the child, "If you don't eat, you won't get any sweets" and the child responds by saying, "I ate, so I deserve some sweets." (Ayalon & Even, 2008)

these views suggest that it would be useful for mathematics education researchers to further examine the role of logic and logic-like reasoning within proofs.

In this paper, I begin to answer the question, “Where is the logic in students’ proofs?” by first searching for uses of logic in a line-by-line analysis of 43 student-constructed proofs in various areas of mathematics, and then examining the actions of the proving process in search of additional uses of logic. This research was done in conjunction with a course, “Understanding and Constructing Proofs”, at a large Southwestern state university, giving Masters and Ph.D.’s in mathematics. Students in the course were first-year mathematics graduate students along with a few undergraduates. Topics covered included sets, functions, real analysis, algebra, and topology. The 43 proofs analyzed were all of the student-constructed proofs in the course. The professors verified all of these as correct. For example, some theorems that were proved by the students included: “The product of two continuous functions is continuous”; “Every semigroup has at most one minimal ideal”; and “Every compact, Hausdorff topological space is regular”.

In the process of coding the lines of the proofs, a theoretical framework emerged. Twenty-three categories were developed and used to code the lines. Here I will describe just four categories: informal inference, formal logic, interior reference, and use of definition; and the others will be in the research report. *Informal inference* is a category that refers to a line of a proof that depends on common sense reasoning. I view informal inference as being logic-like, as it seems that when one uses common sense, one does so automatically and does not consciously bring to mind formal logic. For example, given $a \in A$ and $A \subseteq B$, one gets $a \in B$ as a common sense conclusion, which need not call on formal logic such as Modus Ponens. By *formal logic*, in this report I mean conscious use of predicate and propositional calculus beyond common sense. *Interior reference* is the category for a line in the proof that uses a previous line as a warrant for a conclusion. For example, if there were a line indicating $x \in A$ earlier in the proof, then subsequently stating “Since $x \in A$…” later in the proof would be an interior reference. Lastly, *use of definition* or *definition of* refers to when a line in the proof calls on the definition of a mathematical term. For example, consider the line “Since $x \in A$ or $x \in B$, then $x \in A \cup B$.” The conclusion “then $x \in A \cup B$” is implicitly calling on the definition of union.

In the line-by-line analysis of the proofs, 14% of the 630 lines were informal inference, and less than 2% of the lines were formal logic, such as Modus Tollens and DeMorgan’s laws. In fact, collecting all the logic-like categories together, I found that only 18% of the lines were logic-like. These *logic-like* categories included induction cases, induction hypothesis, induction conclusion, contradiction hypothesis, contradiction conclusion, informal inference, and formal logic. If only 18% of the lines were logic-like, what were the rest of the lines of the proofs like? I found that 21% of the lines were use of definition, 15% were interior reference, and 13% were categorized as *assumption*, meaning that the proof-writer introduced a new object into the proof. Thus, use of definition, interior reference, and assumption accounted for 49% of the lines in the analyzed proofs. While most of the lines of a proof may aid reasoning, they are not themselves logic-like. Also, in a randomly selected line, there is about a 98% chance that there is no formal logic.

Is logic that might not appear in the finished proofs called on by the actions of the proving process? To begin to answer this question, five proofs were selected and the possible actions a student might take in the proving process were hypothesized and analyzed. There were also task-based interviews with three students who had taken the “Understanding and Constructing Proofs” course one year earlier to observe their actions while proving one of the theorems. A one-page set of notes was given to the students (excerpted from the course notes they had used), starting with the definition of a semigroup, and ending with the theorem to prove, “Every semigroup has at most one minimal ideal.” The students were videoed while they thought aloud and attempted to prove the theorem at the blackboard. An interesting result was that these students took three different approaches to the proof, including voicing different concept images for concept definitions. For example, in the notes there was a definition of a “minimal ideal of a semigroup”, and one student considered Venn diagrams while reflecting on the definition, while the other two students stated in a subsequent debriefing that they had not thought of using a diagram.

Another result was that the actions hypothesized for the proof construction did not match the actual actions of the interviewed students. For example, I had hypothesized that the students would write the first line or assumptions, leave a space, and then would write the last line of what was to be proved (as they had been encouraged to do in the earlier “Understanding and Constructing Proofs” course). This is a proving technique (Downs & Mamona-Downs, 2005) that is not often taught. While all three interviewed students wrote “Let S be a semigroup” almost immediately at the beginning of their proofs, only one student wrote the conclusion after playing a bit with the algebra of a semigroup. An analysis of the proof actions in another student’s interview revealed that she wanted to understand and write definitions on scratch work before attempting the proof. She then attempted to comprehend what a minimal ideal is, because she had previously assumed A and B were minimal ideals and intended to arrive at the conclusion $A = B$. She then used the definition of minimal ideal to claim (without justification) that either $A = B$ or $A \cap B = \emptyset$. After using a theorem listed in the notes, she concluded $A = B$, which in her mind finished the proof. Most of the above mentioned actions (e.g., assuming two minimal ideals, deriving a conclusion, and using modus ponens with a theorem) are examples of logic-like actions in the proving process.

An implication for teaching that arises out of this study is that it might be useful for teachers to explicitly attend to students’ logic-like actions in the proving process. Also, because formal logic occurs fairly rarely, one could teach it in context as the need arises. In addition, it would be good to explicitly help students to learn how to read and understand definitions, and when to introduce mathematical objects into a proof, because these together with interior reference constituted 49% of the lines analyzed. Some interesting questions arise from this study:

How many beginning graduate students need a course specifically devoted to improving their proving skills? Can one identify a range of logic-like actions that students most often need to use in constructing proofs? Would a structural analysis of proofs, in contrast to a line-by-line analysis, yield different results? In particular, is it reasonable to regard certain structures in a proof as logic-like? For example, knowing one can prove P *or* Q by supposing *not* P and arriving at Q has the effect of using logic. So is it reasonable to regard *not* P*…then* Q as a logic-like structure in a proof of P *or* Q?

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