Student Approaches and Difficulties in Understanding and Using of Vectors

(Preliminary Research Report)

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Abstract

A configuration of vector representations based on multiple representation, cognitive development, and mathematical conceptualization, to serve as a new unifying framework for studying undergraduate student approaches and difficulties in understanding and using of vectors is proposed. Using this configuration, the study will explore 5 important transitions, 'physics to mathematics', 'arithmetic to algebraic', 'analytic to synthetic', 'geometric to symbolic', 'concrete to abstract', and corresponding student difficulties along epistemological and ontological axes. As a part of validation of the framework, a study on undergraduate students' approaches and difficulties in understanding and using of vectors with both quantitative and qualitative methods will be introduced, and we will see how useful this new framework is to analyze student approaches and difficulties in understanding and using of vectors.

Keywords: Vector, Representation, Vector Representation, Undergraduate Mathematics Education

Introduction

Undergraduate students usually experienced vectors in school physics and school mathematics. When students study undergraduate mathematics, they see vectors again in multivariable calculus, linear algebra, abstract algebra and geometry courses. (Figure 1) Some students see vectors in introductory physics or engineering courses while they are studying vectors in mathematics. Although undergraduate students' experiences with the concept of a vector varied, students still have difficulties in understanding and using vectors in various situations. In this research, we are going to explore the following: (1) constructing a framework to analyze student approaches and difficulties in understanding and using of vectors, (2) classifying approaches and difficulties, (3) seeing how much one approach prevails over the others in student thinking and in school and undergraduate mathematics curricula, and (4) locating the sources of student difficulties.

Root of a Theoretical Framework

Most of the studies about multiple representations are centered on the concept of a function (Janvier, 1987; NCTM, 2000). Unlike the representations of a function, vector representations have a hierarchy and are strongly dependent on the contexts of given questions. To grasp what student approaches and difficulties are in understanding and using of vector representations, many different contexts and different levels of sophistication should be considered. Many mathematics teachers and professors already knew student difficulties from their experience of teaching. However, those difficulties are not classified systematically and they are very scattered and isolated. As Tall (1992) mentioned, "the idea of looking for difficulties, then teaching to reduce or avoid them, is a somewhat negative metaphor for education. It is a physician metaphor - look for the illness and try to cure it. Far better is a positive attitude developing a theory of cognitive development aimed at an improved form of learning." To have more *positive attitude*, we need to have more deeper understanding of student approaches and difficulties on vector representations to the level of the theory of cognitive development.

Most studies about vectors are from physics point of views related with physical quantities and by physics educators. J. Aguirre and Erickson (1984); J. M. Aguirre (1988); Knight (1995); Nguyen and Meltzer (2003) are just a few of them. Some studies such as Watson and Tall (2002); Watson, Spyrou, and Tall (2003) attempted to analyze student approaches and difficulties on vector representations with more mathematical point of views. However, their studies cover only secondary level mathematics and the transition from physical thinking to mathematical thinking. This brings up a necessity of the new framework for investigating vector concepts that can cover vectors in more advanced and wider levels of undergraduate mathematics as well as in physics and secondary level mathematics.

Student approaches and difficulties in learning and using of vectors in undergraduate mathematics are very complex issues which have not yet definitely resolved. Dorier (2002) brought up these issues and analyzed them with a series of research. However this book placed the focus at linear algebra so that vectors in geometry were covered very briefly. Linear algebra courses are just one of the fields that requires the concept of vectors frequently, but most studies on the concept of a vector so far are regarded as parts of bigger topic research on linear algebra (Dorier, 2002; Harel, 1989; Dorier & Sierpinska, 2001).

Lesh, Post, and Behr (1987) pointed out five outer representations including real world object representation, concrete representation, arithmetic symbol representation, spoken-language representation and picture or graphic representation. Among them, the last three are more abstract and at a higher level of representations for mathematical problem solving (Johnson, 1998; Kaput, 1987). However, in most cases, picture representation is not geometric enough to show geometric structure, and graphical representation does not reflect synthetic geometry point of views but rather analytic.

The problem of vector representations lies not only on the multiple representations but also on the translations. Sfard and Thompson (1994); Yerushalmy (1997) are based on the assumption that students ability to understand mathematical concepts depends on their ability to make translations among several modes of representations. Tall, Thomas, Davis, Gray, and Simpson (1999) analyzed several theories of these. These transitions are referred to as "encapsulation" by Dubinsky (1991) and "reification" by Sfard (1991). The proposed framework tries to reflects this idea of encapsulation or reification not just in symbolic modes of representation but also in geometric modes of representation that has not been studied much along with algebra view point (Meissner, 2001b, 2001a; Meissner, Tall, et al., 2006).

Construction of a Framework to Analyze

In this research, a configuration of vector representations based on multiple representations, cognitive development, and mathematical conceptualization, to serve as a new unifying framework for studying student approaches and difficulties in understanding and using of vectors is proposed. Using this configuration, the study will explore five important transitions, 'physics to mathematics', 'arithmetic to algebraic', 'analytic to synthetic', 'geometric to symbolic', 'concrete to abstract', and corresponding student difficulties along epistemological and ontological axes. (See figure 2.) As Zandieh (2000) stated in her study on the framework for the concept of a function, "The framework is not meant to explain how or why students learn as they do, nor to predict a learning trajectory. Rather the framework is a 'map of the territory,' a tool of a certain grain size that we, as teachers, researchers and curriculum developers, can yield as we organize our thinking about teaching and learning the concept...", this new framework serves as a 'map of the territory'. Therefore, with this new framework, we will classify approaches and difficulties, see how much one approach prevails over the others in student thinking and in mathematics curricula, and locate the sources of student difficulties.

Comparison with Other Frameworks

This new framework has some important features. First, it suggests that the interplay between ontological aspect and epistemological aspect is critical in understanding and using of vectors and the key transitions between representations require both ontological and epistemological aspects of understanding simultaneously. Second, it can distinguish and put greater emphasis on difference between analytic geometric representations of vectors and synthetic geometric representations of vectors. It can also distinguish and put greater emphasis on difference between physical representations of vectors and mathematical representations of vectors. Furthermore, it distinguishes/shows/embeds/connects parallel developments of symbolic representations and geometric representations along with cognitive development theories such as reification, or APOS theory. And finally it systematizes the transitions between various representations of vectors.

Research Questions

This research focuses on specific issues arising when representations for vectors are utilized in undergraduate mathematics instruction: (1) What student approaches and difficulties can be identified in understanding and using of vectors?, and (2) How is the students understanding and using of vectors similar to and different from vectors as seen in the written curricula? By proposing the configuration of vector representations, related with the above issues of vector representations, we hypothesize that students difficulties lies on ontological and epistemological jumps in the configuration of vector representations, and students have more difficulties in geometric representations of vectors than symbolic representations at some levels. Hence, the following will be the research questions that we will investigate in this study:(1) What is the theory that explains the process of undergraduate students understanding and using of vectors?, (2) Do students tend to use particular vector representations more?, (3) Do students tend to use vector representations in particular developmental order?, (4) Do different representations of vectors constitute different entities that may not convey the expected vector concepts?

Questions for Discussion

(1) What are the better ways of validating this framework both qualitatively and quantitatively?
(2) Can we think of any philosophical considerations on the framework? (wording issues such as ontological, epistemological, etc.)
(3) What are the views from mathematicians, mathematics educators, physicists?

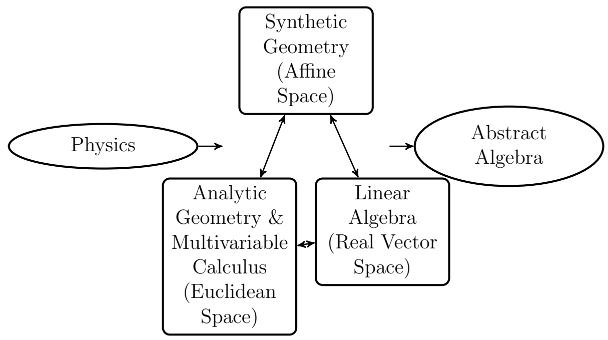

Figure 1. Vectors in Undergraduate Mathematics Curricula

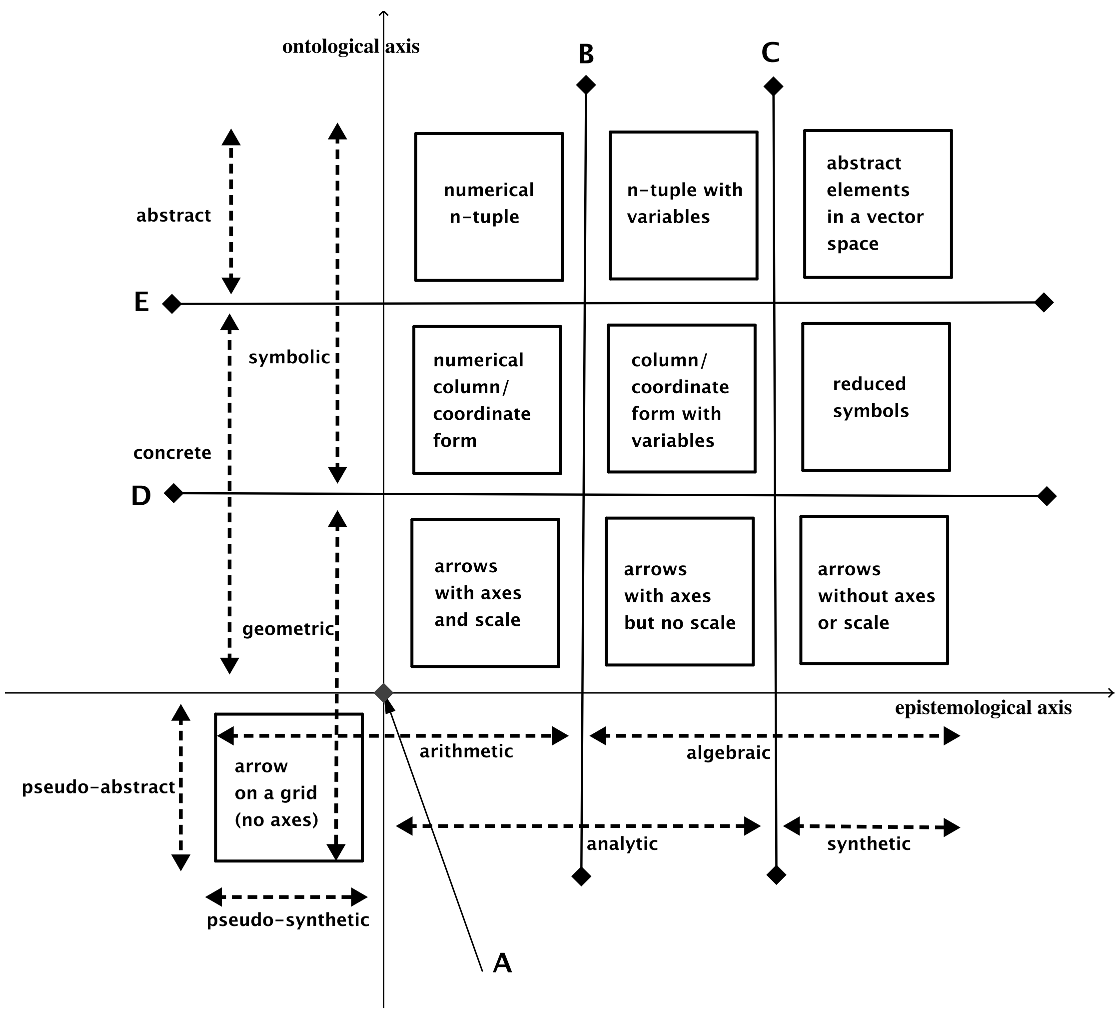

Figure 2. The Configuration of Vector Representations

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