Analysis of Undergraduate Students' Cognitive Processes When Writing Proofs about Inequalities

Preliminary Research Report

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The purpose of this presentation is to discuss undergraduate students' cognitive processes when they attempt to write proofs about inequalities involving absolute values. We employ the theory of conceptual blending to analyze the cognitive process behind the students' final proof of inequalities. Two undergraduate students from transition-to-proof courses participated in the study. Although the instruction about inequalities was given graphically, the students recruited algebraic ideas mainly when they attempted to construct a proof for the inequality. We illustrate how students apply the algebraic ideas and proving structures for their mental activity in their proving activity.

Keywords: proof construction, inequalities, absolute values, conceptual blending,

Introduction and Research Questions

The purpose of this presentation is to focus on undergraduate students' cognitive processes when they attempt to write a proof about an inequality. An understanding of inequalities plays an important role in comparing two quantities and identifying quantitative relationships between them. Research in mathematics education has paid little attention to students' ways of thinking and their difficulties with inequalities although some research reports that students encounter difficulties understanding the meaning of inequalities and their solutions (Tsamir & Almog, 2001; Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006). In this presentation, we discuss the following research questions:

1. What were the students' key mathematical ideas and proving frames used when proving inequalities?
2. What are the students' cognitive processes behind their final proofs of inequalities involving absolute values?

The research literature indicates that undergraduate students struggle with proof writing (e.g. Selden & Selden, 2008). Students tend to structure their proofs in the chronological order of their thought process instead of reorganizing it with proper implications (Dreyfus, 1999). Also, students have difficulty with utilizing conceptual ideas strategically to generate their proof (Weber, 2001). Therefore, students' challenges are related to how to structure a proof, construct a key idea, and strategically use their key idea in their proof structure (Zandieh, Knapp, & Roh, 2008). This research adds to that literature by describing students' cognitive process when proving inequalities involving absolute values, which have not been much addressed in previous work.

Theoretical Framework

We employ Fauconnier and Turner's (2002) theory of conceptual blending to analyze our data. This theory postulates the existence of a subconscious process in which an individual combines elements of current knowledge in order to build new knowledge. An individual may use their knowledge to form one or more mental spaces (referred to as *input spaces*), each of which involves an array of elements and their relationships to one another. Some elements of one input space may be matched with similar elements of another (we refer this as *cross matching* in this study). Such a cognitive process entails the blending of two or more input spaces to form a new mental space (called the *blended space*) as follows (Zandieh et al., 2008): Once the individual considers elements in each input space as important, he or she is *mapping* them into the blended space. As he or she organizes information in the blended space, they are *completing the blend*. This may be done by the use of knowledge outside of the input spaces (called a *conceptual frame*) to organize the blended space. Following this is called *running the blend*, which is a simulation or manipulation of the information in order to make inferences. In this presentation we will illustrate how students are constructing input spaces, cross-matching elements between the two input spaces, mapping from the input spaces to the blended space, applying a conceptual frame to complete the blend, and running the blend. We extend the idea of using conceptual blending to understand students' cognitive process in proofs to inequalities involving the absolute value.

Research Methodology

Data for this study comes from a teaching experiment (Steffe & Thompson, 2000) conducted at a southwestern university in the USA. The teaching experiment involved two undergraduates who were enrolled in different sections of a transition to proof course at the time. Both were strong students in their transition-to-proof courses and neither had instruction in real analysis before this. As a research team, we (identified as Instructor and TA in this study) met to design tasks prior to the teaching sessions, and team-taught during the teaching sessions. The tasks were also served to gauge students' reasoning and their understanding of topics. The data include transcripts of videotapes from the teaching sessions, photo-copies of students' written proofs and scratch work, and student reflections. In their reflections, the students reported aspects of the task or topic they found most interesting or challenging.

In this presentation we focus on the first session with the students, in which they were asked to prove an inequality involving absolute values. The session began with Instructor introducing the definition of the absolute value function and its graph. Instructor then presented properties of absolute values including the Triangle Inequality: For any $a, b \in \mathbb{R}$, $|a+b| \leq |a|+|b|$. The students used several values of *a* and *b* to make sense of the properties of absolute values. Instructor left the definition and theorems out for the students and told them that they may refer to it while working on problems with TA. TA then led the student discussion about how to construct a proof for the exercise statement: "Let $a, b, c \in \mathbb{R}$, then $|a-b| \leq |a-c|+|c-b|$."

For our data analysis, we identified each student's key mathematical ideas and their proving frame when proving the inequality. In terms of the theory of conceptual blending, we then identified how each student formed inputs and cross-matched elements from one space to another. We examined how the student is mapping the cross-matched elements into to the blended space, and how the student uses proving frames and his key mathematics ideas as he is completing the blend and running the blend, respectively.

Results and Discussions

When the students attempted to construct a proof for the inequality, their key mathematical ideas were mainly algebraic although the instruction about inequalities was given graphically. Accordingly, we identified three algebraic ideas used by the students: The first one was observed when a student considered an element of one input space to be identical to the corresponding element of the other input space. The corresponding elements were therefore considered as equal so he could "replace" one with another. The second algebraic idea was "substitution", in which a student introduced a new object and substituted it with an element in an input space (e.g., substitute a variable *x* for the variable *a* in the triangle inequality $|a+b| \le |a|+|b|$). The third one was referred to "zero-trick" in which a student added in something equal to zero (e.g., adding "$-c+c$" to $a-b$).

We also found that students set up a conditional statement of form a conditional implies a conditional statement $(p \to q) \Rightarrow (r \to s)$, and manipulated premises and conclusions *p*, *q*, *r*, and *s*. They then framed their proof in terms of what is called a Conditional Implies a Conditional Frame (CICF) as Zandieh et al. (2008): a student assumes *r*, then induces *p*. Applying $p \to q$, he concludes *s* (Zandieh et al., 2008). However, there was some variation in recruiting CICF in proving the inequality. In particular, a student assumed all of all of *p*, *q*, and *r*, then induced *s*.

Example. We identified six episodes through our data coding procedure based on students' conceptual frames. Usually their conceptual frame consisted of one key algebraic idea and one proving frame. Here, we illustrate how conceptual blending can be used to describe the cognitive process in the second episode. In this episode, a student Jon stated: "What if we substitute this like: $a = a-c,\ b = c-b$? [...] Suppose we have $a = a-c$. Can we do this?" He wrote "$b = c-b$. $a-b = a-c\ -\ c-b$" then crossed out c's in "$a-c\ -\ c-b$" to induce $a-b$. One might note that Jon actually wrote $a-b = a-c\ -\ c-b$, and crossing out the *c*'s, he stated that he will have *a*-*b*. This calculation is incorrect. However, since he says that the *c*'s will cancel, it is probable that he meant $(a-c)+(c-b) = a-b$. The analysis below reflects this conjecture.

Analysis. We characterize Jon's conceptual frame in the second episode as a combination of "replacing" and CICF, and describe his conceptual blending as follows: To begin, he identified the exercise statement as one of his input spaces, say *Input A*. He used the statement of the Triangular Inequality as his strategic knowledge (Weber, 2001) to create another input space, say *Input B*. He then *cross-matched* |*a*|, |*b*|, and |*a*+*b*| in Input A with $|a-c|$, $|c-b|$, and $|a-b|$ in Input B, respectively. Identifying the elements he viewed as important ($a-c$, $c-b$, and $a-b$ from Input A, and *a*, *b*, and *a*+*b* from Input B), Jon was *mapping* the cross-matched elements into his blended space. Then Jon was *completing his blend* by recruiting his conceptual frame: he decided that he would begin with $a-c$ and $c-b$, and manipulate these elements by "replacing" to construct $a-b$. Finally, Jon was *running the blend* in four steps. First, he "replaced" $a-c$ and $c-b$ from Input A with *a* and *b*, respectively. Thus, he had constructed *a* and *b* in Input B. Second, he created *a*+*b* in Input B by adding these two elements. Third, by "replacing" again, he constructed $(a-c)+(c-b)$. Finally, his fourth step is to simplify this to eliminate the *c*'s and construct $a-b$ (See Figure 1).

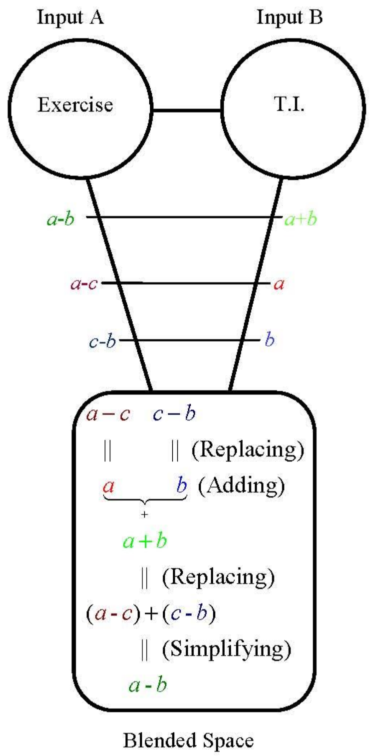

Figure 1: Jon's blending in the second episode

We found that the theory of conceptual blending accounts for students' cognitive processes behind their reasoning in proving inequalities involving absolute values. In particular, it sheds light on why and how students come to ignore the inequality when they prove or solve problems about inequalities. In fact, the students did not map the inequalities and the absolute value symbol into blended spaces, and hence they were not integrated in the blended spaces. In addition, logical structures in the input spaces were often dropped from the process of mapping to the blend, and as a consequence implication structures were obscured in the blended space. (e.g., conditionals $p \rightarrow q$ in input spaces were treated as p and q in the blended space.) Finally, the students also carried out algebraic ideas improperly while they recruited these ideas as their conceptual frames. (e.g., when running the blend, Jon recruited his key algebraic idea and hence identified the cross-matched elements into his blended space instead of using a proper substitution.)

Discussion Questions

1. What are the areas of research that are related to the proving of inequalities, but which are not considered in this study?
2. What are alternative frameworks for analyzing students' cognitive process while writing proofs and how are they going to be useful to explore our research question?

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