Construct Analysis of Complex Variables:
Hypotheses and Historical Perspectives

Preliminary Research Report

ABSTRACT

Quantitative reasoning combined with gestures, visual representations, or mental images has been at the center of much research in the field of mathematics education. In this report we extend these studies to include complex numbers and complex variables. We provide a construct analysis for the teaching and learning of complex variables, which includes a description of existing frameworks that hypothesize about how students can best comprehend the arithmetic operations of complex numbers. In order to test these conjectures, we interviewed mathematicians, physicists, and electrical engineers to explore how they perceive complex variables content. Through phenomenolgogical and microethnography analysis methods we found how these experts integrate perceptuo-motor activity and metaphors into their descriptions.

Keywords: Complex variables, Operational components, Perceptuo-motor activity, Structural components

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The study of learning about numbers and their arithmetic operations is one of the best-developed fields in mathematics education research. The literature goes beyond the four basic operations to include composing and decomposing of whole numbers (Kilpatrick, Swafford, & Findell, 2001), verbal number competencies (Baroody, Benson, & Lai, 2003), ordering and comparing (Brannon, 2002), modeling and visual representations of the operations of whole numbers (Sowder, 1992). The research is not limited to whole numbers; rational numbers are part of the extensive literature related to number sense (Steffe & Olive, 2010). The studies on rational numbers entail investigating students' ability to create word problems that require division or multiplication of two fractions as well as students' visual representations of multiplication and division of two fractions. Studies of quantitative reasoning have elaborated the role of forming a mental image of the measurable attributes in a situation and conceiving of the relevant operations and relationships among these quantities (Thompson, 1994; Moore, Carlson, & Oehrtman, 2009). A natural extension to these studies is to investigate similar characteristics in the teaching and learning of complex numbers.

The main purpose of this preliminary report is to share a construct analysis for understanding complex numbers and variables. A secondary purpose is to describe how experts such as mathematicians, engineers, and physicists conceive of complex variables in both contextual and purely mathematical problems. Methodologically, we explore their use of geometric representations, gestures, verbiage, and symbolism to support their reasoning and convey their understanding. Our construct analysis is based on the few existing pieces of literature that hypothesize about students' understanding of complex numbers and a couple of studies that begin to provide empirical evidence about students' perspective for adding and multiplying complex numbers. We also include a description on how historical perspectives may influence the teaching and learning of complex variables.

Sfard (1999) argued for the need for students to become more flexible in moving between operational and structural conceptions of complex numbers. She encouraged viewing the operational and structural components of complex numbers as complementary pieces rather than as dichotomous. Researchers and instructors could support this perspective by integrating the two representations, for example through geometric illustrations of the operations since "visualization, … makes abstract ideas more tangible, and encourages treating them almost as if they were material entities" (Sfard p. 6). In order to transition from an operational to a structural perspective of complex numbers, Sfard posed three stages that students must navigate in order to develop their understanding of complex numbers.

The first stage is *interiorization*, which occurs when a process is performed on a familiar object. For the case of complex numbers Sfard claimed students who are just becoming proficient in using square roots, would be at the interiorization stage. *Condensation* is the second stage and it occurs when the learner is able to view a process as a whole without the tedious details. For example, students may continue to view 5+2i as a shorthand for certain procedures, but they would still be able to use this symbol in multi-step algorithms. The third stage, *reification*, is achieved when the learner has the ability to view a novel entity as an object-like whole. Learners who are at this stage would recognize 5+2i as a legitimate object that is an element of a well-defined set. According to Sfard (1999) this stage occurs as an "instantaneous leap" much like an "aha moment." Although the stages presented by Sfard are insightful, they do not provide empirical evidence that students actually follow these stages in learning complex variables. Furthermore, Sfard's analysis is restricted to introductory-level conceptions about complex numbers. We intend to elaborate the interplay between more advanced applications and learners' evolving conception of complex numbers/variables.

Lakoff and Núñez (2000) also offered a framework for the conceptual development of complex numbers. Their framework entails a conceptual blend of the real number line, the Cartesian plane, and rotations combined with the use of metaphor for number and number operations. Similar to historical descriptions, they portrayed multiplication of a real number $x$ by $-1$ as a rotation of 180° to obtain $-x$. Thus, multiplying a number by $i$ is equivalent to rotating by 90° counterclockwise. The beauty of this description is that it works mathematically, but empirical evidence suggests students do not view multiplying a number by $-1$ as a rotation of 180°, rather they perceive it as a reflection (Conner, Rasmussen, Zandieh, & Smith, 2007). This might be explained by the fact that students are focused on the real number line rather than the Cartesian plane.

In a more recent study, Nemirovsky, Rasmussen, Sweeney, and Wawro (in press) described the results of a teaching experiment with prospective secondary teachers enrolled in a capstone course. The goal of the teaching experiment was to create an instructional sequence that allowed students to create and discover the conceptual meaning behind adding and multiplying complex numbers. In this phenomenological study, the researchers incorporated microethnography to portray students' body activities over short time periods. These depictions included language use, gaze, gestures, posture, facial expressions, tone of voice, etc. As a result of their study, the researchers found:

1. mathematical conceptualization of adding and multiplying complex numbers was communicated through and comprised of perceptuo-motor activity, and

2. perceptuo-motor activity situated by the learning environment and the setting influenced the learning about the structural components behind adding and multiplying complex numbers.

This study is the first to provide hypotheses about how students make sense of arithmetic operations of complex numbers with supporting empirical data. As such it may provide insight into how best to introduce complex numbers to students besides as a mechanism for solving $x^2 + 1 = 0$. Incorporating reconstructed historical pieces of the development of complex numbers may also engender structural understanding of this area of mathematics (Glas, 1998).

Historically, the introduction and initial development of complex numbers was purely algebraic to resolve the issue of finding the real solution to certain cubic equations. Even after the square root of negative numbers was introduced, mathematicians such as Cardan found such numbers to be *sophistic* because they could not attach a physical meaning to these numbers (Nahin, 1998). These mathematicians tended to ignore the conceptual difficulties of these numbers and proceeded to apply the procedures "mechanically" (Glas, 1998, p. 368). It was Wallis who first dedicated much of his career attempting to represent the square root of a negative number through geometric constructions. Although, Wallis made progress his work was not convincing to other mathematicians or himself. It was more than a hundred years later that Wessel introduced the interpretation of placing $i = \sqrt{-1}$ at a unit distance from the origin on an axis perpendicular to the real number line to form the complex plane and that multiplying by $i$ geometrically represents a rotation of 90º counterclockwise. This representation allowed mathematicians to begin to think about complex numbers as vectors, which in turn led to geometric representations of the arithmetic operations of complex numbers. These models were essential for mathematicians to prove that extended theories of complex numbers (i.e., quaternions, Cauchy-Riemann equations) are consistent and preserve the structure of the complex number system. Such historical developments may provide insights into how "concepts and theories can be best brought to light" for students (Glas, 1998, p. 377).

From the literature and personal reflection we hypothesize a framework in which learners may gain a better understanding of complex numbers and complex valued functions if they have opportunities to visualize arithmetic operations of two complex numbers, complex valued solutions to a quadratic equation, mappings of complex-valued function, poles, geometrical illustrations of theorems, etc. In order to better prepare ourselves to conduct teaching experiments that corroborate this hypothesis, we began our investigation by interviewing "experts." We used phenomenological methods with microethnography to synthesize their responses and to describe how they integrate perceptuo-motor activity and metaphors. We have chosen to interview experts since a goal of this research program is to eventually build a theory that describes how students understand the structural components of complex variables beyond the arithmetic operations of two complex numbers. Our hope is that this framework informed by experts' perceptions will help inform our future research.

As part of our presentation, we will show video clips of our interviewees so that the audience has an opportunity to confirm or argue against our interpretations. Questions for our audience are:

1. How can the interview questions be improved?
2. Is there another framework for data interpretation besides microethnography that might be more appropriate?
3. What impact might social constructivism have on student responses that are not evident in expert responses?

4. How might recognizing the structural component of complex numbers and/ or complex variables contribute to understanding the abstract facets of this mathematical domain?

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