Effective Strategies That Successful Mathematics Majors Use to Read and Comprehend Proofs

Preliminary Report

Abstract. Proof is a dominant means of conveying mathematics to undergraduates in their advanced mathematics courses, yet research suggests that students learn little from the proofs they read and find proofs to be confusing and pointless. In this presentation, we examine the behavior of two successful mathematics majors as they studied six proofs to identify productive proof comprehensive strategies. Prior to reading a proof, these students would attempt to understand the theorem by rephrasing and trying to determine why it was true. While reading a proof, these students would partition the proof into sections, attend to the proof framework being employed, and illustrate confusing aspects of the proof with examples. Implications and limitations of this study will be discussed.

Keywords: Proof, proof reading, proof comprehension.

1. Introduction

In advanced mathematics courses, much of students' time is spent observing their professor present proofs of theorems during course lectures and reading proofs in their textbooks. The implicit assumption underlying this practice is that students can effectively learn mathematics by studying proofs. However, many researchers in mathematics education question this assumption, noting that undergraduate mathematics majors often find the process of reading proofs to be confusing and pointless (e.g., Harel, 1998; Rowland, 2001) and students often do not develop an adequate understanding of a proof after reading it (e.g., Conradie & Frith, 2000). One area that has received little attention in mathematics education research is *how* students should read and study a proof to foster comprehension. The present study seeks to address this void in the literature by describing the proof reading strategies of two successful mathematics majors.

2. Related literature

In the mathematics education research literature, there is a great deal of research on mathematical proof. In analyzing this research, Mejia-Ramos and Inglis (2009) observed that the large majority of empirical studies on proof in mathematics education concerned students' construction of proofs rather than their reading of proofs. Mejia-Ramos and Inglis further noted that most studies focusing on students' reading of proofs analyzed the way students evaluated mathematical arguments; these studies, for instance, asked students if they found an argument to be convincing or if they thought the argument would qualify as a proof. There were few studies that concerned students' comprehension of proofs. As a main goal of presenting proofs to students in their advanced mathematics courses is to increase their understanding of mathematics, the lack of research into this area represents an important void in the literature.

3. Theoretical perspective

In the reading comprehension literature, it is widely accepted that the meaning that an individual obtains from a text is based on three factors: the individual, the text, and the way the individual interacts with the text (e.g., Alexander & Fox, 2004). While reading comprehension can be improved by improving the background knowledge of the reader or the quality of the text, it is also worthwhile to improve the ways that individuals interact with the text. A common approach to conducting research in this area is to identify strategies that effective readers use to comprehend text and to then instruct less successful readers on how to use these strategies (e.g., Palinscar & Brown, 1984; Chi et al, 1994). The study reported here is consistent with this research paradigm.

4. Methods

Two students, with the pseudonyms Kevin and Tim, from a large state university agreed to participate in this study. Both students were mathematics majors in their senior year; these students were also both simultaneously enrolled in a secondary mathematics teacher preparation program. They were invited to participate in this study because they performed well in their mathematics education courses, they were articulate, and they had successfully participated in mathematics education research studies in the past.

The participants met as a pair with the first author of this paper for two 2-hour videotaped task-based interviews. The participants were initially given a proof. They were asked to "think out loud" as they read and studied the proof. They were told to study the proof until they felt they understood it and informed they would be asked questions to assess their comprehension after they read the proof and they would not have the proof to refer to while they answered these questions. This process was repeated for each of the six proofs. Each proof was chosen so that it was of moderate length (between 4 and 20 lines), was based on calculus or basic number theory (to insure Kevin and Tim had an adequate background knowledge to comprehend the proof), and employed a novel technique.

As noted in the introduction, there are few research articles on proof comprehension (Mejia-Ramos & Inglis, 2009) and we are not aware of any research on the strategies that students should use to read proofs for comprehension. Consequently, we did not have any pre-existing categories in mind when analyzing this data and opted to use an open coding scheme in the style of Strauss and Corbin (1990).

In a first pass through the data, we independently noted each attempt that Kevin and Tim made to make sense of the theorem statement or the proof and provided a summary of the students' behavior. (Here, "attempt" was construed broadly to mean anything beyond a literal reading of the text). After these summaries were produced, the authors met to discuss their findings.

From here, it was noted that Kevin and Tim's proof reading could be divided into four phases: (a) studying the theorem, (b) reading the proof, (c) re-reading and summarizing the proof, and (d) critically evaluating the proof.Within each phase, similar proof-reading attempts were grouped together to form categories of the proof reading strategies that Kevin and Tim employed. After categories were named and defined, we again independently viewed the videotape, coding for each instance of the proof reading strategies. We then compared notes and discussed disagreements until they were resolved. Most disagreements were the result of oversight on one of our parts. After our coding, Kevin and Tim were again interviewed about whether the strategies we observed were commonly used and why they engaged in those proof-reading strategies.

4. Results

Kevin and Tim spent considerable time studying the proofs, with the time spent on each proof ranging between 3 minutes and 16 minutes, with an average of 7 minutes and 20 seconds per proof. We note this is significantly longer than other studies on undergraduates' proof reading (e.g., Selden & Selden, 2003; Weber, 2009). It is not clear if Kevin and Tim spent more time reading the proof was due to them being unusually thoughtful and deliberate or because of the task design (they were given an assessment test after reading each proof).

Kevin and Tim averaged nearly three minutes studying the theorem *prior* to reading its proof, in one case spending nearly six minutes studying the theorem. This finding suggests that in examining that strategies for proof comprehension should not focus only on how students interact with proofs, but also the things they do to understand theorems.

Kevin and Tim would attempt to understand the theorem by rephrasing the theorem and by attempting to see why the theorem was true for themselves before reading the proof. The latter was done for each of the six theorem-proof pairs that Kevin and Tim read. They cited numerous benefits to trying to see why a theorem was true, both in terms of understanding the proof and motivating the need to read the proof.

When reading the proof, Kevin and Tim would explicitly attend to the proof framework (in the sense of Selden and Selden, 1995) employed (for example, by once engaging in a lengthy process where they verified that the assumptions and conclusions of the proof actually satisfied the framework for proof by contraposition), partition the proof into sections to verify it, and check problematic assertions in the proof with examples.

After reading the proof, Kevin and Tim would sometimes re-read the proof, summarizing the proof based on its high-level ideas. In other cases, Tim would point out assertions within the proof that appeared to be inconsistent (either with other assertions or his own mathematical understanding), at which point he and Kevin would resolve these apparent inconsistencies together.

5. Significance

This presentation outlines strategies that the two successful mathematics majors used to effectively comprehend six proofs. Clearly, due to limitations of the study (in particular, only using two students and six proofs), no definitive claims can be made. The purpose of this presentation is to make a contribution to the literature by suggesting strategies that other students can be taught to use to improve proof comprehension.

6. Questions for the audience

+ This study will be replicated with another pair of successful students. How can this study be modified to elicit more proof reading strategies?
+ What other types of methodologies can be used to investigate the successful proof reading strategies of these students?
+ What types of classroom environments might foster the use of these strategies?

References

Alexander, P.A. & Fox, E. (2004). A historical perspective on reading research and practice. In R.B. Ruddell & N.J. Unrau (eds.) *Theoretical models and processes of reading*. Newark, DE: International Reading Association.

Chi, M. T. H., de Leeuw, N., Chiu M-H., & Lavancher C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science, 18,* 439-477.

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics, 42*, 225-235.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly*, 105, 497-507.

Mejia-Ramos, J. P. & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88-93), Taipei, Taiwan.

Palinscar, A.S. & Brown, A.L. (1984). Reciprocal teaching of comprehension-fostering and monitoring activities. *Cognition and Instruction*, 1(2), 117-175.

Rowland, T. (2001). Generic proofs in number theory. In S. Campbell & R. Zazkis (eds.) *Learning and teaching number theory: Research in cognition and instruction*. Westport, CT: Ablex Publishing.

Selden, J. and Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123-151.

Strauss, A. and Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. London: SAGE Productions.

Weber, K. (2009). Mathematics majors' evaluation of mathematical arguments and their conception of proof. In *Proceedings of the 12th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2009/proceedings.html