Student Understanding of Integration in the Context and Notation of Thermodynamics: Concepts, Representations, and Transfer
Preliminary Research Report

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Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Little empirical research has investigated how readily students are able to "transfer" their mathematical knowledge and skills from their mathematics classes to other courses. In physics education research (PER), few studies have distinguished between difficulties students have with physics concepts and those with either the mathematics concepts, application of those concepts, or the representations used to connect the math and the physics. We report on empirical studies of student conceptual difficulties with (single-variable) integration on mathematics questions that are analogous to canonical questions in thermodynamics. We interpret our results considering the representations used as well as the lens of knowledge transfer, with attention to how students solve problems involving the same mathematical principles in the differing contexts of their physics and mathematics classes.

Keywords: Physics, integrals, conceptual understanding, representations, transfer

Mathematics is a vital part of how physics concepts are represented (e.g. equations, graphs and diagrams), manipulated and how problems are solved, from the introductory to the upper level. It allows students to simplify the analysis of complex problems by representing complicated conceptual physics problems as a relatively simple relationship between variables. Appropriate interpretation of these representations requires recognition of the connections between the physics and the mathematics built into the representation and subsequent application of the related mathematical concepts (Redish, 2005). Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Despite the fact that students are expected to carry out such interdisciplinary study as a matter of course, little empirical research has investigated how readily students are able to "transfer" their mathematical knowledge and skills from their mathematics classes to other courses.

With many physics areas, specific mathematical concepts are required for a complete understanding and appreciation of the physics. Meltzer (2002) has shown a link between mathematical acumen and success in an algebra-based physics class. Tuminaro and Redish (2007) have combined the frameworks of resources (Hammer, 1996), epistemic games, and framing to analyze student use of mathematics in physics. To date, however, there have only been a few PER studies exploring physics students' difficulties with calculus concepts (Pollock et al., 2007; Rebello et al., 2007, Black and Wittmann, 2009).

Our work aims to identify the extent to which mathematical knowledge and understanding affects physics conceptual knowledge, specifically in the context of upper-level thermal physics. We have two main research questions: What difficulties do advanced-level undergraduate students have when learning thermal physics concepts? To what extent does students' mathematical knowledge and understanding influence their responses to physics questions?

The empirical framework that guides descriptions of student reasoning in our research is that of *specific difficulties* (Heron, 2003). We start with targeted, context-dependent results and then generalize across contexts, seeking larger patterns of student responses in our data. Our emphasis is on gathering and interpreting empirical data that can act as a foundation for future studies on reasoning in physics and for curriculum development to address specific difficulties. The more cognitive framework, ideally suited for the study of knowledge transfer and the context-sensitivity of mathematical knowledge, is that of *transfer in pieces* (Wagner, 2006).

A key feature of thermal physics is the reliance on many ways of thinking about integration. In physics in general, the idea of an integral is tied closely with a graphical interpretation as the area under the curve.

Previous findings on student understanding of integral calculus concepts in mathematics education research indicate that students do not possess the necessary knowledge to allow them to successfully complete problems involving concepts of integration, especially with regard to considering the integral as the area under the curve. The literature in mathematics education repeatedly documents the lack of student understanding of the relationship between a definite integral and the area under the curve (Orton, 1983; Vinner, 1989; Thompson, 1994; Grundmeier, 2006). These include student difficulties with recognition of integrals as limits of (Riemann) sums (Orton 1983, Sealey, 2006); student confusion about the concept of "negative area" for integrals of curves that fall below the x-axis either conceptually (Bezuidenhout and Olivier, 2000), computationally (Orton, 1983; Rasslan & Tall, 2002) or both (Hall, 2010). Thompson and Silverman (2008) showed that the reliance on *area under curve* reasoning may limit applicability of the conception of integrals.

In our current research, we seek to isolate mathematical difficulties that may underlie observed physics difficulties by asking physics questions that are completely stripped of their content, and focus on the calculus concepts under investigation (e.g., integration and partial differentiation). We call these *physicsless physics questions,* since they typically use notation that is more consistent with representations used in physics rather than following the conventions of mathematics (Christensen & Thompson, 2010).

We administered the physics questions as well as the analogous physicsless questions to the students in our thermodynamics course. Data have been obtained from written responses to ungraded free-response questions, and interviews have been conducted on physics students at various levels. Student responses to the physics questions were compared to reported categories in the literature (Loverude et al. 2002, Meltzer 2004). Responses to the physicsless questions were analyzed for patterns and categorized; the categories were then compared to those from the (paired) physics questions. The results from the paired physics and math questions among physics students show that some of the difficulties that arise when comparing thermodynamic work based on a pressure-volume (*P-V*) diagram may be attributed to difficulties with the *mathematical* aspect of the diagram, in particular with the correct application of an understanding of integrals, rather than physics conceptual difficulties (e.g., treating work as an equilibrium state function). These results suggest that some students aren't necessarily attributing state function properties to work so much as failing to recognize the same variable as two different functions during integration.

To further explore the question of transfer closer to the source of the concept of integration (and area under the curve), we asked the physicsless integral questions near the end of a multivariable calculus class to over 150 calculus students from 3 different semesters. These results from the multivariable calculus course suggest that the observed mathematical difficulties are not just with transfer of math knowledge to physics contexts. Some of these difficulties seem to have origins in the understanding of the math concepts themselves.

We have recently extended this work to vary the features of the representation(s) used in the math-based question, with the goal of exploring the extent to which students are using features of the representation – either tacitly or explicitly – to interpret the question being asked.

In one case, different formats of the basic question discussed (Figures 1a and b) were administered near the end of a few recitation sections of calculus classes, both introductory integral calculus (Calculus II) and multivariable calculus (Calculus III). The questions asked students to compare the absolute value of the definite integrals for two functions, *f(y)* and *g(y)* on

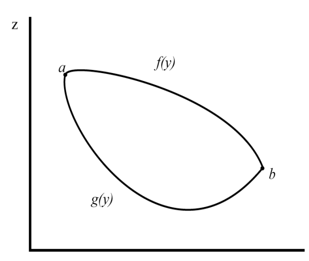

Figure 1(a)

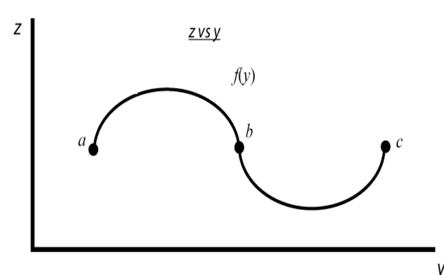

Figure 1(b)

a qualitative graph.

The reasoning given with answers was not always clear, but they could roughly be put into a number of categories: area under the curve, position of the function, curvature of the curve, slope of the line. Just over 50% explicitly used the word "area" in their reasoning. While there is evidence of these categories of reasoning, it does not provide definitive proof that students have a concept image that is consistent with their explanations. One interesting result is that calculus 2 students outperformed calculus 3 students on the question, with nearly all of calculus 2 students giving the correct answer and 60% of calculus 3 students successfully able to compare the absolute values for the integrals. Building on these results, clinical interviews are being carried out to explore the nature of students' concept image (Vinner, 1989) of the integral in variations of the same question.

Furthermore, we are interpreting many of the above results through the lens of knowledge transfer, with attention to how students solve problems involving the same mathematical principles in the differing contexts of their physics and mathematics classes. Existing data are being examined through a *transfer-in-pieces* framework, initially to consider how student performance may be influenced by the different representational forms used in the domains of mathematics and physics. This analysis complements existing analyses in ways that suggest further research tasks. This research angle focuses on how undergraduate students come to see contextually different problems or situations as "alike," in that they demonstrate instances of the same mathematical principle. This perspective complements and expands the initial research questions about how students recognize and understand the relationship between the concepts of physics and the mathematics that underlies them.

Questions for audience

What are the implications for the teaching of these topics, both in mathematics and in physics?

We intend to continue research along this vein of the mathematics-physics connections. We recognize that our questions aren't presented in rigorous mathematical notation; how reasonable are our questions in the opinion of mathematicians, given their relevance to the way physicists use and apply the mathematics?

Even if students come out of their mathematics classes with a good *mathematical* understanding of the principles at stake, it is reasonable to expect that transferring their mathematical knowledge into a physics context should be unproblematic?

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