Redefining Integral: Preparing for a New Approach to Undergraduate Calculus

Preliminary Research Report

Abstract: This study is a pilot to a larger design research project that aims to explore an alternative approach to teaching a Calculus I course. Central to this approach is the introduction of the integral first, utilizing a non-standard definition, but which is equivalent to the standard definition. This is immediately followed by the introduction of derivative. This approach allows methods of derivation and integration, which are analogs of one another to be introduced in close succession, allowing the relationships between these methods to be a major theme of the course. The alternative definition of integral is the focus of this study. I present preliminary results of a teaching experiment that explores how students develop an understanding of this alternative definition of integral and how these understandings relate to prerequisite notions, such as area and arithmetic mean.

Keywords: Calculus, arithmetic mean, new methodology, teaching experiment.

Introduction

In the late 1980s and early 1990s the NSF responded to growing concerns regarding student success in Calculus by moving considerable resources into Calculus curricular reform efforts. The National Research Council (1991) followed suit issuing a challenge to "revitalize undergraduate mathematics." In spite of efforts which these initiatives sprouted student success remains much the same today as it did in the early 90's (Seymour, 2006). The larger research project – to which this study is a pilot – aims to work toward addressing this issue by developing and implementing a set of innovative course materials for a first course in Calculus and by studying the impact these have on student learning, disposition, and retention.

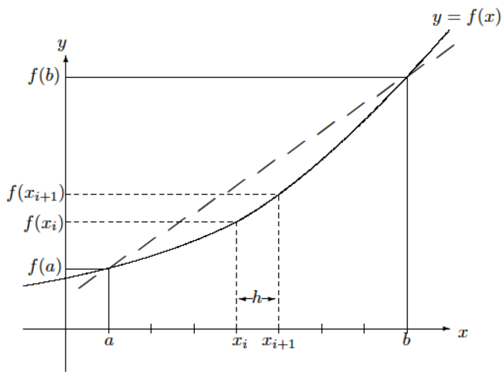
The Definition of Integral

The definition of integral utilized in the proposed reformed Calculus curriculum does not rely on the notion of Riemann sum. Instead, the definition relies on the mean of a uniform sample of *n* heights of a function:

Uniform Sampling: $x_{i-1} - x_i = h = \frac{b-a}{n}$, where $x_i = a + ih,\ i = 1,\ 2,\ \ldots,\ n$

Sample Data: $y_i = f\left(x_i\right),\ i = 1,\ 2,\ \ldots,\ n$

Statistical Mean: $\bar{y}(n) = \frac{1}{n}\sum_{i=1}^{n} y_i$



Definition of integral**:** $I \equiv \lim_{n\to\infty} \bar{y}(n)(b-a)$ is the integral of $f(x)$ over the interval $[a, b]$. This is written as $I \equiv \int_a^b f(x)dx$.

The Teaching Experiment

In order to explore how students come to interpret and understand this definition, a teaching experiment (Steffe & Thompson, 2000) was conducted with a group of students who typically do not take Calculus, pre-service elementary school teachers. This choice of participants exaggerates the potential struggles, misinterpretations and conceptual hurdles, providing a richer picture of what Calculus students are likely to encounter. Since pre-service elementary teachers likely have no prior exposure to Calculus, interference from previous instruction is limited.

The teaching experiment took place over four 50-minute sessions, conducted biweekly with a group of four students. The instructional sequence used student concepts and reasoning as the starting point, from which more complex and formal reasoning developed. Developing and understanding of the relationship between mean and area was the goal of the first session. Students were prompted to develop a standardized technique for finding the area of a display of uniform width columns. This served as a catalyst for gaining an understanding of why the method of multiplying the mean height of the columns by the width of the figure yields the area. The next session focused on estimating the area of more rounded figures utilizing a sample of their heights, which was obtained manually using a ruler. These figures were then replaced by functions in the subsequent session, where instead of finding heights manually the function values are utilized. Formal notation was then introduced. The final session explored students' conceptions of what happens as the sample size is increased. This culminated in the definition of integration. These sessions were video-taped and preliminary analysis will be presented.

The Questionnaire on Understanding Mean

An understanding of the alternative definition of integral relies heavily on a non-procedural understanding of mean. In the definition, the mean of a sample of heights of a function is abstract—not tied to any specific function. Hence, in order to understand the definition, students must be able to consider the abstract properties of mean. To explore understandings of mean amongst potential Calculus students, a questionnaire was designed and administered with pre-Calculus students. It included questions that could be approached both using procedural methods and non-procedural understandings of mean. For example, one such question shows two triangles and asks about the mean measure of their six angles. Another question involves a circle that is divided into several sectors. Students were asked to determine the average measure of the area of these sectors. In both problems the specific sizes of the angles are given, but are not necessary in order to find the solution. To account for students that possess conceptual understanding but prefer procedural solutions to demonstrate their method, as well as for students that develop conceptual insight after completing a procedural solution, students are prompted for an additional solution after each of the tasks. The results of this questionnaire will be presented.

Synopsis

This pilot study focuses on how students' develop an understanding of the alternative definition of integral and the hurdles they encounter along the way. Additionally, students' conceptions of mean, as well as how those conceptions relate to area, are explored. This additional focus is included in order to paint a better picture—one which includes an exploration of the kind of relevant tools students bring into the new Calculus curriculum. In the session participants will have the opportunity to reflect on and discuss the following questions related to the study:

- What student difficulties have you observed/experienced with the conventional definition of integral? What difficulties do you foresee (or have experienced) with the alternative definition of integral?
- What lens/framework do you suggest for the in-depth analysis of the teaching experiment?

References

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