Title: Function Composition and the Chain Rule in Calculus
Preliminary Report

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The chain rule is a calculus concept that causes difficulties for many students. While several studies focus on other aspects of calculus, there is little research that focuses specifically on the chain rule. To address this gap in the research, we are studying how students use and interpret the chain rule while working in an online homework environment. We are particularly interested in answering three questions: 1) What characterizes student's understanding of composition of functions? 2) What characterizes student's understanding of chain rule? and 3) To what extent do students' understanding of composition of functions play a role in their understanding and ability to use chain rule in calculus?

Keywords: Calculus, precalculus, procedural knowledge, conceptual knowledge, technology

Literature Review

Studies have indicated that success in calculus is likely linked to a robust understanding of the concept of function (Carlson, Oehrtman, & Engelke, 2010; Ferrini-Mundy & Gaudard, 1992). Unfortunately, many students enter calculus with a weak understanding of the concept of function. Carlson (1998) investigated and described what is required for students to gain a mature understanding of the concept of function and concluded that a mature concept of function is slow to develop, even in strong students. Studies have also show that function composition is particularly problematic for students (Engelke, Oehrtman, & Carlson, 2005).

There have been a number of studies that focus on what it means to understand the concept of derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Ferrini-Mundy & Gaudard, 1992; Orton, 1983; Zandieh, 2000). "It is known that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure." (Orton, 1983, p. 242) In fact, many first semester calculus students earn a passing grade without ever achieving a conceptual understanding of the derivative. Students are adept at using rules to find the derivative function and using this result to compute the desired answer. When asked about the chain rule, most students will simply provide an example of what it is rather than explain how it works (Clark et al., 1997; Cottrill, 1999). The literature related to studies in calculus provide evidence that students develop more procedural understanding than conceptual in differentiation. However, there is a gap in the studies investigating the characteristics of student's understanding of composition of functions and the chain rule. We aim to provide a description of the possible relationships among these understandings.

Theoretical Perspective

Star (2005) carefully examines the existing literature on procedural and conceptual understanding in mathematics education and points out the necessity to develop broader frameworks to investigate both procedural and conceptual knowledge and understanding. Since the publication of Hiebert's book (1986) on conceptual and procedural knowledge many studies have used the definitions and the framework (Rittle-Johnson, Siegler & Alibali, 2001). Hiebert and Lefevre state that conceptual knowledge is "characterized most clearly as knowledge that is rich in relationships" whereas procedural knowledge "consists of rules and procedures for solving mathematical problem" (1986, p.3, p.7). Star (2005) suggests broadening these definitions in order to provide more in depth analysis of both procedural and conceptual understanding. He criticizes earlier research studies not providing in depth analysis of these concepts but rather focusing on the order of them: "Which comes first: procedural or conceptual knowledge?"

This study employs Star's (2005) approach toward procedural and conceptual knowledge and understanding to map out the students' understanding of composition of functions and the chain rule. We aim to describe possible characteristics of students' surface and deep procedural knowledge and understanding of composition of functions and the chain rule by examining student work.

Methodology

The 41 students in this study are first semester calculus students enrolled at a large Midwestern University who regularly take online quizzes using tablet computers. Student work on several function composition and chain rule problems was collected using a modified online homework system and digital ink. The system records and replays, in real-time, the work each student did to complete the problem. Students had three opportunities to submit each problem. The system also collected how they modified the problem, enabling us to focus on students whose initial work was incorrect and to identify the steps they thought needed to be fixed in order to answer the problem correctly.

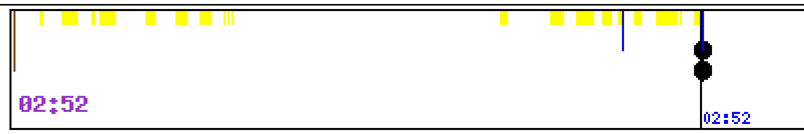
Student ability to complete precalculus tasks, including the chain rule, was measured during the first four weeks of the semester. Students were given pre-tests including both the Pre Calculus Concept Assessment (PCA) Instrument (Carlson et al., 2010) and an 84-item precalculus assessment focusing on procedural knowledge. Students were allowed to practice items from the procedural assessment during the subsequent two weeks, and completed a post-test involving the PCA and the procedural assessment during the fourth week. This data helps identify strengths and weaknesses in various students and as baseline data for our analysis of the student work gathered by the online homework system.

Results

Students completed function composition and chain rule problems during the first week and seventh week quizzes. For the purposes of this study, students are classified as strong, average, or weak according to their ability to work with function composition as measured by their initial and final scores on the PCA. The study is still collecting and analyzing data, some of which is shared below.

The students were given the following problem during the first week of the semester: *The graph of y=f(x) and y=g(x) are shown below. Calculate f(f(-2) and f(g(2)).* Figure 1 shows a *work map* of Student 1 and Student 2, selected from the strong and weak groups. Vertical bars on the work map indicate when the student was drawing, graphing, erasing, adding images, navigating between problems, and submitting correct or incorrect answers for a problem. Figure 2 shows part of the work done by each student. From our initial observation of Student 1 and Student 2 work, we noticed the procedurally more proficient student (Student 1) was capable of using a graphical representation of a function to find the composition of function. This could be a possible characteristic needed for a deep procedural knowledge of a composition of function.

Student 1:



Student 2:

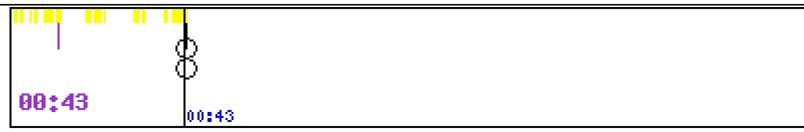
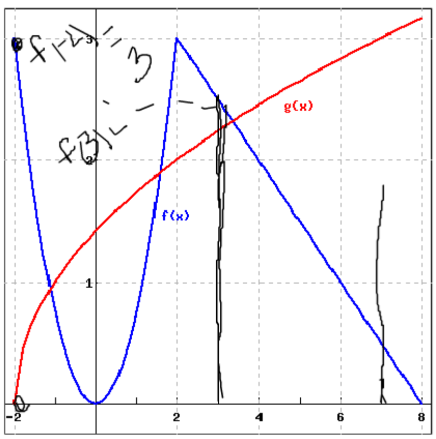


Figure 1: Work maps of Student 1 and Student 2 on Function Composition Problem.



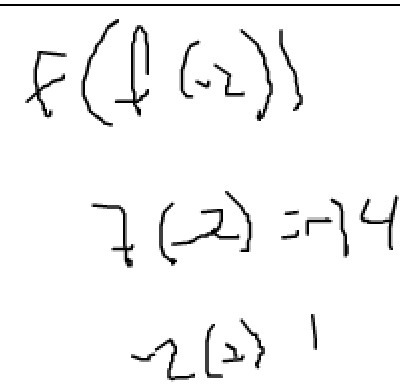
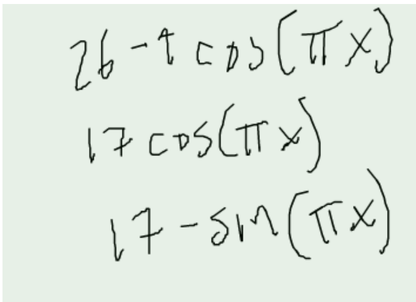
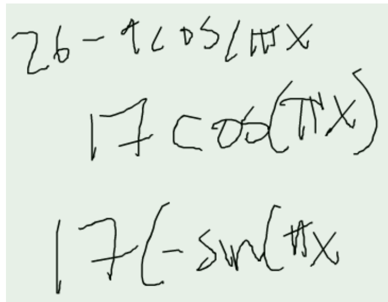
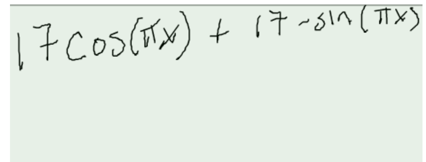


Figure 2: (Partial) work of Student 1 and Student 2 on Function Composition Problem.

During the seventh week of the semester, students were asked: *Find the derivative of R(x) = 26 – 6 cos(πx)*. Initial observations show that students who consistently scored high on the function composition problems on the PCA correctly apply the chain rule in this simple case, while students from the moderate and weak groups typically failed to recognize that differentiating *cos(πx)* required the use of the chain rule. Replay of student work shows many students struggling with function manipulations involving signs, addition, and multiplication. For instance, Figure 3 shows three separate attempts by Student 2 to solve the problem. In each case, the student struggled with the application of an incorrect procedure but failed to address the use of the chain rule.

In this study we would like to investigate these cases further in depth to elicit more features of surface and deep procedural knowledge and understanding. Also, we plan to include contextual problems to investigate student's conceptual understanding further.

		
Work before first attempt: *17(-sin(pi x))*	Work before second attempt: *17 - sin(pi x)*	Work before third attempt: *17cos(pi x) + 26 - 9 - sin(pi x)*
Figure 3: Student 2 work on solving a simple chain rule problem		

Questions for the audience:

- What would you like the technology to be able to do?
- How would you envision using the work map?
- What does a work map tell us about student's procedural and conceptual knowledge?
- We are currently looking at this as further refining some of the procedural/conceptual frameworks that have come before now. Is there a better theoretical perspective that we could be working with?

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