# PROCEEDINGS OF THE 15TH ANNUAL CONFERENCE ON RESEARCH IN UNDERGRADUATE MATHEMATICS EDUCATION

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PORTLAND, OREGON
FEBRUARY 23 – FEBRUARY 25, 2012

PRESENTED BY
THE SPECIAL INTEREST GROUP OF THE MATHEMATICS ASSOCIATION OF AMERICA (SIGMAA) FOR RESEARCH IN UNDERGRADUATE MATHEMATICS EDUCATION

CITATION: In (Eds.) S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman, *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education*, 2012, Portland, Oregon.

# FOREWARD

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its fifteenth annual Conference on Research in Undergraduate Mathematics Education in Portland, Oregon from February 23 - 25, 2012.

The conference is a forum for researchers in collegiate mathematics education to share results of research addressing issues pertinent to the learning and teaching of undergraduate mathematics. The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The program included plenary addresses by Dr. Alan Schoenfeld, Dr. Chris Rasmussen, Dr. Lara Alcock, and Dr. Cynthia Atman, a special session by Dr. Jacqueline Dewar, and the presentation of over 100 contributed, preliminary, and theoretical research reports. In addition to these activities, faculty, students and artists contributed to an inaugural display on Art and Undergraduate Mathematics Education.

The Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education are our record of the presentations given and it is our hope that they will serve both as a resource for future research, as our field continues to expand in its areas of interest, methodological approaches, theoretical frameworks, and analytical paradigms, and as a resource for faculty in mathematics departments, who wish to use research to inform mathematics instruction in the university classroom.

Last but not least, we wish to acknowledge the conference program committee and reviewers, for their substantial contributions to RUME and our institutions, for their support.

Sincerely,

Stacy Brown
RUME Organizational Director & Conference Chairperson

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**WITH MUCH APPRECIATION WE THANK THE FOLLOWING REVIEWERS**

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# Small Group Discussion and Student Evaluation of Presented Proofs

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Contributed Research Report

**Abstract:** Research has demonstrated that proof validation, the process of reading and reflecting on a proof to determine its correctness, is difficult for many mathematics majors. Rather than reading written proofs, however, students' classroom experiences often involve presentations of proofs either by their instructor or by other students. In this study, we explore mathematics majors success in recognizing the validity of presented proofs. Participants watched videotaped proofs either at the beginning or at the end of a transition-to-proof course. After completing an initial evaluation of the proof, students discussed the proof in small groups and then evaluated the proof a second time. The impact of the course and the effect of the small group interaction will be discussed.

**Key words:** proof validation, transition to proof, social norms

## 1 Introduction

Reading and writing mathematical proofs are tasks that play a significant role in many upper division undergraduate mathematics courses. Unfortunately, many students encounter significant difficulties with the notion of rigorous proof (see, for example, Sowder & Harel, 2003). In terms of reading proofs in particular, Selden and Selden (2003) recount the struggles undergraduate math majors had in validating proofs. They also argue that "constructing or producing proofs is inextricably linked to the ability to validate them correctly" (p. 9). While mathematical proofs are often viewed as either logically valid or logically invalid, Thurston (1994) points out that the social norms of the mathematical community for which the proof was written must also be considered. For instance, use of a particular result in a proof may be acceptable in a journal article but not in an undergraduate mathematics course. Taking this social aspect a step further, Yackel (2001) suggests that what counts as an acceptable mathematical argument is a sociomathematical norm that is formed the through interaction of participants in the classroom. Much of the interaction in proof-oriented math courses involves proof presentation (either by the instructor or by the students). Proof presentations go beyond the written text of a proof and may include verbal or nonverbal emphasis on certain key ideas or steps. This exploratory study investigates the impact of these social interactions on student evaluations of proofs. In particular, how successful are math majors in evaluating the correctness of presented proofs? How does completing a transition-to-proof course affect their success in evaluating presented proofs? Further, how does small group discussion of a presented proof impact their assessment of the correctness of the proof?

## 2 Previous related research

Focusing on proof as written argument establishing the truth of a theorem, Selden and Selden (2003) describe the process of reading and reflecting on a proof to determine its correctness as

*proof validation*. They comment that proof validation can include asking and answering questions, constructing subproofs, remembering other theorems or definitions, as well as general feelings of rightness or wrongness. Their data indicated that undergraduate mathematics majors were not particularly successful at proof validation, with students assessing the correctness of several written proofs at approximately chance levels. Weber (2010) replicated these results in a study of math majors that had recently completed a transition-to-proof course. Students in the study provided feedback on their level of understanding of the proof, the degree to which they were convinced by the proof, and whether they would classify it as a rigorous proof. While proof validation is a potentially complicated and time consuming task, Weber reported that almost none of the participants spent more than two minutes reading an argument. This suggests that students may be using other criteria in evaluating the correctness of a proof than line by line validation. This possibility is supported by Knuth's (2002) research where he described a variety of criteria used by secondary school mathematics teachers in judging proofs, such as familiarity, generality, shows why, and sufficient detail in the proof.

Toulmin's (1969) model of argumentation has also been used in investigating proof validation. In Toulmin's model, an argument consists of at least three parts: the data, the warrant, and the conclusion. The validity of the conclusion should follow from the data, while the warrant explains why the evidence presented in the data is sufficient to guarantee the conclusion. If the warrant is insufficient, additional backing for the warrant may be required. Alcock and Weber (2005) use this framework in investigating the efforts of students in an undergraduate analysis course in validating an incorrect proof that the sequence $\sqrt{n}$ tends to infinity as $n \to \infty$. The proof falls apart in the last line, where the conclusion that $\sqrt{n}$ tends to infinity supposedly follows from the data that $\sqrt{n}$ is an increasing sequence. Students worked in pairs in evaluating the proof, and while over half the students initially determined that the proof was correct, they observed that many students "seemed to focus on whether the assertions made were true, rather than considering whether they were substantiated" (p. 133) by a valid warrant. On the other hand, they also reported that, following leading prompts by the interviewer, over three quarters of the participants were able to identify the false warrant and to conclude that the proof was incorrect. This suggests that social interaction may have a positive impact in helping students to evaluate written proofs.

## 3 Methods

Eighteen students in an upper division transition-to-proof course taught at a public liberal arts university participated in the study. Participants watched six different videotaped proof presentations in groups of four to five students. The videotaped proofs were presented by two senior mathematics majors following an explicit script. The content was accessible to a student with a introductory calculus background, and each presentation lasted from two to four minutes. After watching each proof, participants completed an initial written assessment of the proof. The assessment had two parts: first, classifying the proof as mathematically correct, partially correct, or incorrect and, second, indicating the degree to which the argument was personally convincing using a Likert scale. Following this initial assessment, students were given the opportunity to discuss the proof as a group. During this discussion, the interviewer interjected several prompts to promote discussion (such as "who found this proof convincing and why?" or "were there any features of the proof that you found problematic?") However, the interviewer did not comment on the validity of any part of the discussion. Following the small group discussion, participants performed a second evaluation

of the proof in terms of correctness and the degree to which it was convincing. Half of the participants evaluated the proofs at the beginning of the semester; the other half assessed them at the end of the semester. The average course grades for the two groups of students were virtually identical. All participants received extra credit in the course.

A few additional comments regarding the study are necessary. First, in terms of student assessment of the videotaped proofs, no attempt was made to define the phrase "partially correct proof." However, participants were certainly familiar with having their work in previous mathematics courses graded as correct, partially correct, or incorrect. Because half the participants evaluated the videotaped proofs at the beginning the transition-to-proof course, these familiar assessment phrases were used in evaluating the proofs. Second, after watching the videotaped proof, students were not able to view the proof. This undoubtedly had a negative impact on their small group discussion of the proofs. However, if the written proof were left on the the screen, students would be able to watch the videotaped presentation and then read the written proof (or ignore the presentation and focus only on the text). This would make it difficult to determine student success at evaluating the presentation alone. Similarly, it would be difficult to separate the effect of small group discussion from the impact of additional time in reviewing the written proof. Third, one of the researchers was the instructor of the transition-to-proof course. The course was taught in a modified-lecture format with some student proof presentation of previously graded homework. There was no explicit instruction regarding proof validation or Toulmin's framework, and the instructor was not present during the students' assessment of the videotaped proofs.

## 4 Results

### 4.1 Success at evaluating presented proofs

In order to draw comparisons with previous research, we will focus on the initial proof assessments of students at the end of the transition-to-proof course. Nine participants watched six different proofs, of which two were valid and four were invalid proofs. Students identified the valid proofs as mathematically correct in 13 of 18 attempts, and students identified the invalid proofs as not correct (either partially correct or incorrect) in 31 of 36 attempts. This is a higher success rate than observed in previous research regarding student validation of written proofs. Students did perceive a difference between watching and reading a proof. For instance, one student remarked, "I was just thinking as he was doing it he is really good at explaining verbally, but if I was to receive that proof on paper, I would have a very hard time following what he was doing." On the other hand, it may be that the higher success rate may be explained by other factors, such as the level of difficulty of the proofs considered. In this regard, two of the invalid proofs in the current study appeared in Weber (2010). One of these involved the use of the converse in attempting to prove that $3|n^2$ implies $3|n$. In Weber's study, 12 of 28 students identified this as a rigorous proof, while 0 of 9 students in the current study categorized the proof as mathematically correct. The second invalid proof that appeared in both studies involved concluding that $\ln x \to \infty$ as $x \to \infty$ from that fact that $\ln x$ is an increasing function. From the written proof in Weber, 12 of 28 students identified this as a rigorous proof in spite of the invalid warrant, while 5 of 9 students in the current categorized the presented proof as mathematically correct. The results for these two invalid proofs are somewhat inconsistent; the error in the logarithm proof is more subtle and it may be more difficult to detect when watching the proof. Weber (2010) also observed that, contrary to previous research, students did not classify an empirical argument as a rigorous proof. Data from the current

study replicate that finding; none of the participants in this study categorized an empirical argument as a mathematically correct proof.

### 4.2 Impact of transition-to-proof course

To assess the impact of the transition-to-proof course, we will compare the initial proof evaluations of students watching the proofs at the beginning of the semester with those watching them at the end of the semester. At the beginning of the semester, nine students watched the six videotaped proofs. They correctly identified the proofs as mathematically correct or not correct in 39 of 54 attempts. At the end of the semester, the other nine participants correctly categorized the proofs in 44 of 54 attempts. Thus, the transition-to-proof course led to some small improvement in students' success in evaluating presented proofs. In fact, the improvement would have been significantly larger but for two particular proofs where students performance declined after completing the course. The first of these proofs made use a diagram with triangular arrays of dots to show that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ (see Knuth, 2002, p. 384). The argument is valid, but at the end of the semester almost half the participants classified it as not correct. It may be that some of these students had learned during the course to distrust proofs involving pictures (see Inglis and Mejia-Ramos, 2009). The second proof where students performed worse at the end of the semester was the invalid proof that $\ln x \to \infty$ as $x \to \infty$. In this case, it is not until the last line of the proof that things fall apart. In discussing this proof, Weber (2010) suggests that students may have had a false sense of confidence because the first part of the argument had been correct and made sense. This sense of confidence may be particularly influential when watching a proof. Rather than validating a proof line by line, students may simply be making an overall judgement regarding validity based on the proof having familiar form without any glaring errors.

After completing the transition-to-proof course, students appeared to be more confident in their assessments and were able to discuss the the proofs more effectively. For example, after watching an incorrect argument using the converse, students at the beginning of the semester often had some general distrust of the proof but had difficulty identifying the problem. One student remarked that "It seemed to go against ... it felt funny. He went about it the other way." In contrast, students at the end of the semester were quite confident in recognizing and discussing the error. One student remarked that "she started with what she wanted to prove and that's kind of a problem." Another student, referring the the person presenting the proof, commented that "I identified with him ... I've done that before ... I got slammed on my test. So when he did it I was like oh." Selden and Selden (2003) argue for the importance of recognizing proof frameworks, and the course clearly helped students in recognizing valid and invalid frameworks.

### 4.3 Effects of small group discussion

For the nine students who viewed the six proof presentations at the beginning of the semester, the group discussion led them to change their assessment of the correctness of a proof in 21 out of 54 times. At the end of the semester, the other nine participants changed their evaluation of proofs only 8 out of 54 times. Again, this suggests that students were more confident in their assessments after completing the transition-to-proof course. In terms of how convincing they found the arguments, in almost all cases participants' ratings either stayed the same or decreased after discussing the proof. In general, it would appear that group discussion either confirmed their own reaction or introduced additional doubts regarding the proof's validity. Much of the group discussion focused on form and style rather than logical details. For example, students comments included "I liked how she

wrote out a lot of the stuff, not just putting up the symbols," "I kind of felt it was too wordy," and, referring to a visual diagram, "for a visual learner, it would actually be easier to follow." On the other hand, there was discussion regarding particular details that led students to change their initial assessment. As an example, consider the following exchange regarding an invalid proof by cases:

> David: He failed to consider the case where either $x$ or $y$ is positive and the other one is negative.
> Julie: I didn't think about that.
> Claire: True.
> Emma: I didn't either. Wow!

Following this discussion, all group members correctly evaluated the presented proof. However, there were also situations where discussion of particular details of a proof did not resonate with the other members of the group. For instance, in the proof that $\ln x \to \infty$ as $x \to \infty$, one student pointed out that on three separate occasions that an increasing function need not tend to infinity as $x \to \infty$. Other members of the group appeared to completely ignore his comments, and instead talked about unrelated issues. Similar patterns were observed in other groups, and did not lead students to change their evaluation of the proof. This is quite different than the observations in Alcock and Weber (2005) where interviewer prompts led students to alter their assessment of a written proof.

## 5 Discussion

Looking at the overall data, students in the study were quite successful in evaluating presented proofs. When compared to previous research involving the validation of written proofs, students performed at approximately the same or higher levels. Similarly, the overall numbers indicate that students were more successful at recognizing correct mathematical proofs after completing a transition-to-proof course. Further, small group discussion had an overall positive impact on their ability to recognize correct proofs. On the other hand, this overall picture is an oversimplification that hides several contradictions in the data. There were several instances where proof presentation, completing the transitions course, and small group discussion had a negative impact on student success in evaluating proofs. The design of this exploratory study cannot explain these apparent inconsistencies. Further research using different methodologies will be needed. It is interesting, however, to conjecture at some possible explanations. It may be that students attend to different features when watching a presented proof than when reading a written proof. In this study, for instance, a familiar format and clarity of presentation seemed to play important roles in students' assessments of presented proofs. The format and "flow" of a proof were often discussed by students, and students were sometimes led to incorrect conclusions based on these features. These errors were occasionally worse at the end of the transitions course, when students' expectations regarding the proper form of a proof may have been refined throughout the semester. Given that much of students' classroom experience with proof involves proof presentations, a deeper understanding of the ways in which students evaluate presented proofs versus written proofs may have important implications in introducing students to the notion of rigorous mathematical proof.

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# MAKING JUMPS: AN EXPLORATION OF STUDENTS' DIFFICULTIES INTERPRETING INDIRECT PROOFS

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*This paper reports findings from an exploratory study that examined undergraduate mathematics students' proof preferences, as they relate to indirect proof. While many have suggested that undergraduate students dislike and have metatheoretical difficulties with indirect proof, findings from 15 proof-preference surveys provide evidence that students' preferences are in fact more nuanced that previously anticipated and that in certain contexts students' prefer indirect proofs. Building on this work, 5 clinical one-hour interviews were conducted with a sample of survey participants, so as to better explain the discrepancy between these findings and those of other researchers. Data from the clinical interviews with advanced students are used to show: (1) aspects of indirect proofs, which others have argued are the root of students' dislike, were not salient to the students, and (2) students' difficulties recognizing secondary statements, rather than their metatheoretical difficulties, may account for their dislike of indirect proof in the form of proof by contradiction, but not of indirect proofs in the form of proof by contraposition.*

*Keywords*: Indirect Proof, Proof Preferences, Clinical Interviews

Over the past two decades, researchers have increasingly turned their attention to students' and teachers' production, understanding, and evaluation of proofs (Stylianou, Blanton, & Knuth, 2009). This increased interest is due to a growing understanding of the central role proof plays in the development of mathematics (Hanna & Barbeau, 2008; Rav, 1999) and of children's capacity to engage in acts of argumentation and justification (Valentine, Carpenter, & Pligge, 2005; Ball & Bass, 2000). In the area of undergraduate mathematics, researchers have begun to move beyond establishing the existence of students' difficulties (Moore, 1995) to characterizing students' proof schemes (Harel & Sowder, 1998), creating a richer picture of how students comprehend and evaluate mathematical proofs (Weber, 2009) and learn proof practices within mathematics classrooms (Fukawa-Connelly, 2010). One issue with research on undergraduate students is that few proof studies take into account the possibility that the specific practices students engage in when generating, comprehending, or evaluating proofs may be content or proof-type specific. There are two exceptions to this trend in the research: (1) Harel and Sowder's (1998) proof scheme studies and (2) research on mathematical induction. In the case of proof schemes, Harel and Sowder argue that learners can hold multiple proof schemes and that one's use of a particular proof scheme is often context dependent. For instance, a learner may employ a perceptual proof scheme within a geometric context and then use a symbolic proof scheme when asked to evaluate an algebraic proof. One implication of this work is that studies seeking to produce general descriptions of students' ways of producing, understanding, and evaluating proofs that do not carefully attend to the particular proof-types or content involved may, in fact, be presenting a narrow or exaggerated picture of students' approaches.

In addition to Harel and Sowder's proof scheme work, there is a large body of research focused solely on a particular proof-type, namely, mathematical induction (Avital & Hansen, 1976; Baker, 1995; Brown, 2003; Ernest, 1984; Dubinsky, 1986, 1989, 1991; Harel & Sowder, 1998; Harel 2001; Harel & Brown, 2008; Leron & Zazkis, 1986; Maher & Martino, 1996a;

1996b; Malcom, 1974; Movshovitz-Hadar, 1993a, 1993b; Reid, 1992). This research includes explicit lists of the difficulties students encounter (Dubinsky, 1986, 1989, 1991), analyses of students' understanding of recursion within MI proofs (Reid, 1992), powerful examples of children's generation of primitive MI-proofs (Maher & Martino, 1996a; 1996b), and trajectories for the emergence of MI-proofs with collegiate-level instructional innovations (Harel, 2002; Brown, 2003; Harel & Brown, 2008). Mathematical induction, however, may have functioned as a special case for researchers, for it can be viewed as a key piece of mathematical content (an axiom of the natural number system) and as a highly prevalent proof-type within undergraduate mathematics. Nevertheless, it is noteworthy that even though much is known in this area, these researchers have not suggested that the findings extend beyond MI-proofs.

One reason that researchers may have chosen not to focus on difficulties related to specific proof-types is that there are contexts where such an approach is likely to be far from fruitful. For instance, students' difficulties with direct proofs are likely to be tied to a variety of complex issues; such as, students' difficulties unpacking logical statements (Selden & Selden, 1995), reading proofs as texts (Selden & Selden, 2003) and using sematic and syntactic approaches (Weber & Alcock, 2004). Nonetheless, there are some forms of proof, which are ubiquitous in mathematics and have the potential to pose unique difficulties for students. Consider, for instance, how one's understandings of the pigeon-hole principle or diagonalization arguments might influence students' understanding of content developed with these techniques. Though common, these proof-types are not as ubiquitous as another form of proof that has received very little attention by researchers – indirect proof. Indeed, a survey of the proceeding volumes for the ICMI Study 19 Conference: *Proof and Proving in School Mathematics*, shows that of the 94 research papers presented, only 9 mention indirect proofs and only 1 of those 9 explicitly investigated indirect proofs (cf. Mariotti & Antonini, 2009). Similarly, a survey of the SIGMAA on RUME annual conference proceedings indicates that since the inception of its conference proceedings in 2008, there were no research reports that explicitly investigated students' understandings of indirect proof in 2008 ($n$ = 79), 2009 ($n$ = 88), or 2010 ($n$ = 74). Taken together, the proceedings of the ICMI and RUME conferences, indicate that students' difficulties with indirect proof have remained largely ignored. This is not say that research on indirect proof is non-existent. Indeed, researchers have offered explanations for and some evidence of students' dislike (Harel & Sowder, 1998) and discomfort (Antonini & Mariotti, 2008) with such proofs. Nevertheless, given the widespread presence of this form of proof within tertiary mathematics, one could still argue that there is a dearth of research in this area.

## Research on Indirect Proof

Research on students' production, understanding, and evaluation of indirect proofs has primarily focused on students' lack of preference for this form of proof. Harel and Sowder (1998) reported that students in their teaching experiments dislike and have difficulty with indirect proofs – as illustrated by the remarks of Dean, a university student, "I really don't like proofs by contradiction. I have never understood proofs by contradiction, they never made sense" (p. 272). Harel and Sowder argue that one reason for students' dislike is that students prefer constructive proofs – proofs which construct mathematical objects – over proofs the merely establish the logical necessity of a mathematical relation or object, as is often the case with indirect proofs. Similarly, Leron (1985), argued that students' difficulties are rooted in the "destructive nature" of indirect proofs: "Most non-trivial proofs pivot around an act of construction – a construction of a new mathematical object … In indirect proofs, however, …

We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world" (p.323).

Antonini and Mariotti (2008) have also suggested that students' experience difficulties with indirect proofs and have provided a rationale for these difficulties that is not solely rooted in issues of constructiveness (Leron, 1985; Harel and Sowder, 1998) and causality (Harel, 2007). In particular, Antonini and Mariotti argue that when engaging in such proofs, students must move from a principal statement (*e.g.*, $P \rightarrow Q$) to a secondary statement (*e.g.,* in a proof by contraposition, $\sim Q \rightarrow \sim P$) and then either interpret or produce a direct proof of the secondary statement. They argue that it is the lack of acceptance of this jump between principal statements (S) and secondary statements (S*) that is the source of students' difficulties and refer to such difficulties as *metatheoretical*. "This theorem is not part of the theory in which the principal and secondary statements are formulated, but it is part of the logical theory. Referring to their meta-theoretical status, we call the statement $S^* \rightarrow S$ meta-statement, the proof of $S^* \rightarrow S$ meta-proof, and the logical theory, in which the meta-proof makes sense, meta-theory" (p. 405).

To illustrate students' metatheoretical difficulties, Antonini and Mariotti asked students to evaluate indirect proofs, including a proof by contraposition of the statement, "If $n^2$ is even then $n$ is even," and showed that students struggle to accept the validity of the principal statement, given the proof of the secondary statement. For instance, consider the following remarks by Fabio, a university student: "… The problem is that in this way we proved that $n$ is odd implies $n^2$ is odd, and I accept this; but I do not feel satisfied with the other one." (p. 407). In other words, Fabio has accepted that a claim has been made and a proof given of the secondary statement, but is not "satisfied" with regard to the jump to accepting the principal statement.

While it may seem that Antonini and Mariotti's (2008) model of students' difficulties with indirect proofs offers an alternative to the characterizations provided by Leron (1985) and Harel and Sowder (1998), it is possible that this is not the case. Both Leron's and Harel and Sowder's characterizations provide a plausible basis for the proposed metatheoretical difficulties. Indeed, it may be the case that students do not view the transition to a secondary statement as constructive or fail to see how S* *causes* S to be true. Alternatively, it may be the case that while the Italian university students in Antonini and Mariotti's case studies recognized the transition to and proof of secondary statements, American students struggle to do so. If the latter is true, then American students' difficulties would be related to *recognizing*, as well as *accepting*, the equivalence of the principal and secondary statements. The purpose of this paper is to illustrate, using data from surveys and interviews, that American students' evaluations of proofs by contradiction (a form of indirect proof) support the latter hypothesis: American students' experience difficulties recognizing the equivalence of secondary statements in proofs by contradiction. However, this hypothesis is not supported in the context of proofs by contraposition.

## The Study

The research reported in this paper is drawn from a first-stage study of a multi-stage research program that seeks to: (i) document students' proof preferences, as they relate to indirect proof; (ii) verify current hypotheses about the existence and source of students' difficulties; and (iii) develop instructional innovations for indirect proof, if warranted. In particular, in this paper we report results from a small-scale exploratory study focused on students proof preferences. The study involved the administration of an 8-item proof preference survey to a diverse sample of 15 mathematics majors enrolled in one of four advanced mathematics courses and 5 one-hour follow-up clinical interviews. The survey instrument included three types of proof or 'proof-related' comparison tasks. Proof comparison tasks provided students with two proofs and asked

the students to rate the extent to which they were confident in their understanding of each proof and to indicate which proof they found more convincing. Three forms of proof comparisons were included. Type I tasks as participants to compare a direct proof to an indirect proof. Type II tasks asked participants to compare a *Proof by Construction* to an *Existence Proof*. Type III tasks asked participants to compare a proof by contraposition to a proof by contradiction. Type III tasks were used to explore the idea that there might be psychological distinctions to be made between the two forms of indirect proof. Type IV tasks were the 'proof-related' comparison tasks and asked participants to select a statement to prove out of three statements. Choices for the three statements include a principal statement and two secondary statements. Secondary statements were either of the logical form $\sim Q \Rightarrow \sim P$, which we will refer to as the Contra-P form, or the logical form "there exists no *n* such that, $P \wedge \sim Q$," which we will refer to as the Contra-D form. Results from the survey are shown in Table 1, with Type IV statement selection results reported as (Principal: Contra-P: Contra-D).

| Type I (Thm 1) PMI vs Indirect | Type I (Thm 2) PMI vs Indirect | Type I (Thm 3) Direct vs. Contra-P | Type II (Thm 4) Construction vs. Existence | Type III (Thm 5) Contra-P vs. Contra-D | Type III (Thm 6) Contra-P vs. Contra-D | Type IV (Thm 7) Statement Selection | Type IV (Thm 8) Statement Selection |
|---|---|---|---|---|---|---|---|
| 11:4 | 12:3 | 7:7 | 5:10 | 6:9 | 9:6 | 7:8:0 | 13:1:1 |

Table 1. Survey Results

As can be seen in Table 1, students' proof preferences were not consistent across comparison task type. Moreover, as indicated by participants' responses to Type I tasks, and as noted by others (Knuth, 2002; Healy & Hoyles, 2000), familiarity appears to play an important role in students' selection of an argument. In response to Type II tasks, participants tended to prefer the existence argument when contrasted with a constructive proof. No trends were observed in survey responses to Type III comparison tasks. Finally, in response to Type IV tasks, students tended to selected the principal statement, when the Contra-P form introduced negations and showed no preference when the use of negations did not increase in the Contra-P form. Of particular interest is the result that in only 1 of the 30 instances did a student select as a 'statement to prove' a secondary statement of the Contra-D form.

Given the small sample size, these findings should be considered with caution. A large-scale administration of the survey is needed to determine if the trends will hold. Nevertheless, it is surprising that students' overwhelmingly selected the existence argument in the Type II comparisons. This finding stands in contrast to current hypotheses about students' preferences for constructive proofs. In response to Type IV comparisons, the finding that students avoided Alternative 2, the Contra-D form secondary statement, is also interesting since no preference was indicated by the Type III comparison tasks responses.

In an effort to better understand the survey responses, 5 one-hour clinical interviews were conducted. Participants were asked to review their responses to 4 of the comparison tasks and to: (a) explain each of the given proofs, (b) describe any differences or similarities between the two proofs, and (c) explain their selection of the 'most convincing' proof. Participants were also asked to explain their response to a Type IV task.

## Interview Results

Due to space limitations, we will limit the discussion to two findings from the clinical interviews. First, after being asked to complete steps (a) – (c) for a Type II task (see Appendix A), students were asked if Argument B, the existence argument, provided a set of numbers that satisfied the given theorem. This prompted 4 of the 5 students to review the proof, in some cases multiple times, before responding to the question. This reaction suggests that the lack of

construction of a mathematical object that satisfied the theorem was not salient to the students during their evaluation of the argument, for if it had been they would have immediately responded to the question. The one student who immediately replied, however, did state that this was part of their reason for selecting Argument A, the constructive proof.

Why did the students avoid selecting the constructive argument, as research would predict? It is possible that the two arguments used in the Type II task may have clouded the constructive / non-constructive distinction. Argument A, the "constructive" proof, requires a subproof of the irrationality of $\log_2 9$ and the given subproof is modeled after the standard proof of the irrationality of $\sqrt{2}$. Thus, one possibility is that students' selection of Argument B can be explained in terms of an avoidance of a proof that relies on a subproof by contradiction. This hypothesis is supported by students' efforts to explain Argument A. Students either became confused when moving to the subproof or suggested that the proof could be improved by initially presenting the proof of the irrationality of $\log_2 9$ as a separate proof and then referencing this proof in the proof of Theorem 4. Even though students' selection of Argument B can be accounted for in terms of an avoidance of Argument A, and possibly of Contra-D form statements, it is still noteworthy that the students had not attended to the lack of constructiveness in Argument A when evaluating the given proof. The lack of attention to this aspect of the argument calls into question the extent to which students attend to constructiveness. Future research should examine: (i) if, in other instances, students attend to the existence or lack thereof of constructed objects in existence arguments; and (ii) if existence arguments are preferred when compared to purely constructive arguments.

The second finding concerns students' responses to the Type IV "proof related" comparison task. This task did not call on students to evaluate or select a proof but rather provided students with three statements: a principal statement, a Contra-P secondary statement, and a Contra-D secondary statement (see Appendix B). Survey responses were split between the two forms (principal and Contra-P), with no students selecting the third option, the Contra-D form.

To gather addition data, participants were asked to review their responses and to explain why one could or could not prove the given theorem by proving either of the alternative statements. Across all interviews, students' responses to this interview question indicated that students immediately recognized the Contra-P form of a statement and knew that it was an acceptable alternative to the theorem. Students were also asked to label the theorem's premise (P) and conclusion (Q) and then to demonstrate that the selected statement was the contrapositive of the theorem. In all cases, the interview participants were able to quickly and correctly complete this task. In contrast, in all five interviews students struggled to describe the logical structure of Alternative 2, the Contra-D form statement, and in 3 of the 5 interviews, participants became uncertain as to whether or not Alternative 2 could be used to prove the original statement. For example, after having identified P and Q in the principal statement, Ivana (a senior mathematics major) examined Alternative 2 and determined that the statement's form was, "there exists no *n* such that P and ~ Q." She was then asked if this statement was equivalent to the theorem.

> I: Are these equivalent?
>
> Ivana: No [*picks up pencil*] … yes … umm [*pauses*] … I want to say yes but my gut is saying no.

Grace (also a senior mathematics major) had a harder time identifying the logical form of Alternative 2 and spent several minutes of the interview unpacking the statement's structure. Furthermore, after having identified the structure of the Contra-D statement, she was uncertain as to whether or not Alternative 2 was an acceptable alternative to the principal statement.

I: And, what about this one?

Grace: This one, I'm not even sure, right now, if it's true or false. I think my gut instinct was false but then, I don't know why, I changed it to true.

Finally, Stephen (a sophomore mathematics major enrolled in advanced mathematics courses) appeared to have the most difficulty explaining the structure of the Contra-D statement.

Stephen: I think if we just negate theorem 7 we get alternative 2.

I: Okay, so maybe this [*Alternative 2*] is a negation, actually?

Stephen: [*Alternative*] 2 is a negation of [*theorem*] 7.

Students' responses to the Type IV tasks clearly indicate that students' struggled to recognize the Contra-D secondary statement as logical equivalent to the principal statement. In fact, students repeatedly attributed their lack of selection of the Contra-D statement to finding the form of the statement confusing. This finding supports the hypothesis that American students experience difficulties *recognizing*, as well as *accepting*, the logical equivalence of secondary statements, when the statements are of the Contra-D form. The same hypothesis, however, did not hold for the Contra-P form. However, the findings of this contributed report should be viewed with caution. Further research with a larger sample is needed before implications can be derived. Subsequent research should seek to determine the extent to which the reported difficulties are typically among students enrolled in advanced mathematics courses.

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## Appendix A

Theorem 4: There exists irrational numbers $a$ and $b$ such that $a^b$ is rational.

| Argument A | Argument B |
|---|---|
| Let $a = \sqrt{2}$ and $b = \log_2 9$<br>It is well known that $\sqrt{2}$ is irrational.<br><br>Claim: $\log_2 9$ is also irrational<br>Recall, that if $y = \log_z x$ then $x = y^z$.<br>Thus, if $y = \log_2 9$, then $2^y = 9$.<br><br>Suppose $\log_2 9$ is rational, then $\log_2 9 = m/n$, for integers $m$ and $n$. This implies $2^{\frac{m}{n}} = 9 \Rightarrow 2^m = 9^n$. Since $9^n$ is odd, for all integers $n$, and $2^m$ is even, for all integers $m$, it follows that no such integers, $m$ and $n$ exists. Therefore, $\log_2 9$ is irrational.<br>If $a = \sqrt{2}$ and $b = \log_2 9$, then $a$ and $b$ are irrational, and<br>$a^b = \sqrt{2}^{\log_2 9} = (2^{\frac{1}{2}})^{\log_2 9} = (2^{\log_2 9})^{\frac{1}{2}} = (9)^{\frac{1}{2}} = 3$<br><br>Thus, $a^b$ is rational. | Let $a = \sqrt{2}$ and $b = \sqrt{2}$.<br>It is well known that $\sqrt{2}$ is irrational.<br><br>$a^b = \sqrt{2}^{\sqrt{2}}$<br><br>If $\sqrt{2}^{\sqrt{2}}$ is rational, then the theorem is true.<br><br>If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, then<br><br>$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2.$<br><br>Thus, $a^b$ is rational. |
| 1. I am confident about my understanding of Argument A. (Please mark one)<br><br>☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree | 2. I am confident about my understanding of Argument B. (Please mark one)<br><br>☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree |
| 3. Which argument, in your opinion, is the most convincing? ☐ Argument A ☐ Argument B<br>*Please explain your selection. (If you need additional space please use the back of this page.)* | |

## Appendix B

| Theorem 7: If $n$ is a positive integer such that $n$ mod(3) = 2, then $n$ is not a perfect square. | |
|---|---|
| Alternative 1 for Theorem 7: | Alternative 2 for Theorem 7: |
| If $n$ is a perfect square, then $n$ is a positive integer such that $n$ mod(3) $\neq$ 2. | There exists no positive integer $n$ such that $n$ mod(3) = 2 and $n$ is a perfect square. |
| 1. You can prove the statement by proving alternative statement 1. Please check one box.<br>☐ True ☐ False | 2. You can prove the statement by proving alternative statement 2. Please check one box.<br>☐ True ☐ False |
| 3. If you were asked to prove Theorem 7, which formulation would you pursue first? Please check one box.<br>☐ The original statement ☐ Alternative statement 1 ☐ Alternative statement 2 | |
| *Please explain your explain your response to question 3*: | |

# Does a Statement of Whether Order Matters in Counting Problems Affect Students' Strategies?

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**Abstract**

Counting problems ask students to compute the number of ways a certain set of requirements can be satisfied, and they are important in such mathematical subjects as probability, combinatorics, and abstract algebra, among others. Students are often taught to solve counting problems by looking for specific clues to help categorize the problems and identify solution strategies. In this study, we investigate how the wording of certain counting problems, specifically whether or not "order matters", affects students' solution strategies. In particular, we gave students questions involving explicit statements as to whether or not order matters, some of which were intentionally misleading, and questions that do not contain such an explicit statement. Data was collected in the form of written responses and student interviews. The results show that many students do, in fact, rely heavily on such explicit statements about whether order matters, even when such statements are misleading.

*Keywords:* combinatorics, permutations, combinations, problem solving

## Introduction

Counting problems are a type of combinatorial problem which ask the solver to determine the number of ways a certain set of requirements can be satisfied in a given situation. For example, the question might ask, "How many 5-card poker hands contain cards all of the same suit?" Such questions arise in elementary probability questions in high school classes, in more advanced probability classes at the undergraduate level, as well as in abstract algebra, combinatorics, and other areas of the undergraduate curriculum.

Students are given several tools to solve counting problems. The two most basic tools are the *multiplication principle* (also known as the fundamental counting principle) and the *addition principle*. Students are also introduced to some useful formulas: The combination formula $C(n,k)$ counts the number of unordered subsets of size $k$ that can be made from a set of size $n$; the permutation formula $P(n,k)$ counts the number of ordered subsets[1]. Both of these formulas are derived from the multiplication principle, and can be viewed as "shortcuts" for specific applications of the multiplication principle. In almost every textbook used in the United States, these formulas are defined (as above) in terms of a *selection model*, in which a sample of elements is drawn from a set of objects. In some problems, repetition of selected objects is allowed. Therefore, four basic combinatorial operations can be defined as in Table 1 (Godino et al., 2005; Batanero et al., 1997; Rosen, 2011). We use the notation $P(n,k)$ for permutations without repetition, $C(n,k)$ for combinations without repetition, $PR(n,k)$ for permutations with repetition, and $CR(n,k)$ for combinations with repetition.

[1]Some authors (Eizenberg and Zaslavsky, 2004; Godino et al., 2005, e.g.) refer to ordered subsets as *arrangements* and view permutations as a special case of arrangement, in which all $n$ elements of the set are ordered. We will use the term *permutations* to refer to both cases.

While other combinatorial models (distribution, partition) can appear in counting problems (Dubois, 1984, cited in Batanero, et. al. 1997), the selection model is the most familiar to most students, and solving problems using other models often involves "translating" the problem into a selection model (when possible) and applying one of the basic combinatorial operations (Godino et al., 2005). Several student difficulties with counting problems have been identified in the literature, and students may be more or less prone to make errors depending on several factors: the type of combinatorial operation (permutation or combination, with or without repetition); the nature of elements to be combined (letters, numbers, people, or objects); the implicit combinatorial model (selection, distribution, or partition); and the values given to $n$ and $k$ (Fischbein and Gazit, 1988; Batanero et al., 1997; Eizenberg and Zaslavsky, 2004).

Batanero et al. (1997) also catalogue several types of student error. In particular, one type of error is the "error of order," which Batanero et al. (1997) describe as, "confusing the criteria of combinations and arrangements, that is, distinguishing the order of the elements when it is irrelevant or, on the contrary, not considering the order when it is essential." This issue will be the focus of this study.

As noted earlier, most students are familiar with counting problems based on a selection model. Students are often taught to solve such problems by identifying the sampling conditions of the problem, recognizing the appropriate combinatorial operation (as in Table 1), and applying the required formula. While it is well-known that students often have difficulty recognizing the appropriate combinatorial operation (Batanero et al., 1997; Eizenberg and Zaslavsky, 2004; Godino et al., 2005), there have not been, to our knowledge, any studies examining the strategies students use to identify the combinatorial operation. Students are often taught to focus on whether or not order is allowed, and whether or not repetition is allowed. However, even in simple counting problems, these factors may not be obvious, and in fact, can be somewhat misleading. For example, consider the problem:

> A club has five members. In how many ways can a president, vice-president, and treasurer be elected?

The standard solution to this problem interprets this as "permutations without repetition", $P(5,3)$, assuming that a club member cannot simultaneously hold more than one office. However, it is not immediately clear to many students exactly how "order matters" in this problem. One explanation is that the selection of three officers can be *mapped* to an ordered subset of the club members by making the first selected member to be the president, the second to be the vice-president, and the third to be the treasurer. However, there are other ways in which the order does *not* matter: for example, the order in which the elections are held does not matter. Thus, in problems of this type, the question of whether or not "order matters" may not be the right question, and perhaps a different strategy might be more successful for more students.

## Methods

We claim that the burden for successfully answering questions about combinations and permutations often falls upon the solver's careful reading and interpretation of the questions. In particular, we believe that a student's interpretation of whether order matters, and what it means for order to matter, greatly impacts that student's thinking. We claim that the wording of questions in this area has a crucial impact on how students view them.

To investigate this claim, we prepared a written quiz of six combinatorics problems; see Table 2. Questions 1, 3, 4 and 5 make statements concerning whether or not "order matters", with question 3 and 5 written intentionally to present the question of "order" in a non-standard way. As our results will show, these statements may have influenced students to solve those problems incorrectly.

This quiz was given to students enrolled in combinatorics courses[2] at a large state university during the Fall 2011 semester. This quiz was given twice, once before the students received direct instruction about combinations and permutations and again a few weeks after.

A group of ten graduate students were also interviewed following both rounds of the quiz regarding their thinking process on the quiz. The interviews were video recorded and analyzed, and pseudonyms were assigned to each student.

## Results

Data was collected both before direct instruction and afterwards, in both classes. At the time of this writing, only the data collected prior to direct instruction is available, but we expect both sets of data to be collected and analyzed well before the date of presentation.

Data analysis was conducted with attention to how students interpreted the questions particularly in regard to phrasing about whether or not order matters. We hypothesized that the phrasing "order does not matter" and "order matters," particularly in questions three and five, would result in students identifying those key phrases and use a combination or permutation formula accordingly. Several students did just that. In her interview, Jane summarized her strategies with, "I remember in high school learning about if order matters, it is a permutation, and if order does not matter, it is a combination." Questions three and five were written specifically to "mislead" students by including a statement about order that would not fit this principle. We discuss the results of these questions in particular below.

## Question Three: Hockey Players

The question stated: A youth hockey team has twelve members. How many ways are there to choose a starting lineup of center, left wing, right wing, left defense, right defense, and goalie, if the order in which these positions are filled does not matter?

Problem three included the statement that "the order in which these positions are filled does not matter," which is true: the starting lineup is not changed if the position of goalie is filled before that of center, or vice-versa. However, this is not the standard meaning of "order does not matter." In fact, since the order in which the positions are filled does not matter, the usual approach to this problem is to define an arbitrary order of positions (center first, left wing second, etc.) and map ordered subsets of players to starting lineups using this order of positions. By focusing the attention on the order of the positions, the problem mislead students who relied on the "order matters" principle.

For some students, the phrase "the order in which these positions are filled does not matter" was an indication to use the choose function, $\binom{n}{k}$. In the written results, eighteen of the participants used some form of a combination formula in their response, while only seven participants correctly solved the problem.

[2]There were thirteen graduate students and twenty-one undergraduate students who participated in this study.

In her interview, Fiona stated, "When I saw the words 'does not matter,' I said OK that's a choosing problem, I remember 12 choose 6 is the formula where order wouldn't matter." Similarly, Ruth's response was, "This problem is almost the same as the first problem, except order does not matter. Order does not matter is combinations and order matters is permutations." Another student, Emily stated, "It said the order does not matter, and so I know that we are choosing positions and we are going to divide by the number of positions factorial because order does not matter, so we are going to take away the redundant orders." When prompted by the interviewer about the phrasing of the question, she elaborated that without the words "order does not matter" she would have used a different strategy.

Other students were not swayed by the phrasing of the question. Lowell stated, "It says that the order that the positions are chosen does not matter, which made it sound like it was going to be a combination problem instead of permutations. But each of the positions is different so it matters which person gets chosen in which position, so the order that you pick them does still matter, which put it back in a permutations question."

This question was functionally identical to question six, which states that all colors on the main, trim, accent and siding must be different, but makes no explicit statement of whether or not order matters. This change in wording made it much clearer to the students that a combination formula was unnecessary here, and more students were successful in solving problem six than problem three.

### Question Five: Block Stacking

This question stated: A toddler has an essentially unlimited supply of red and blue blocks, and is building stacks of these blocks. If the toddler makes a stack of eight blocks, how many ways are there to stack the blocks so that exactly three blocks are red? (The order in which the blocks are stacked matters.)

Problem five stated that "the order in which the blocks are stacked matters." This statement is not misleading on the surface: outcomes of stacked blocks are different if the same blocks are rearranged, and this problem can be considered to be a "permutations with repetition allowed" problem. However, a simple permutation formula (without repetition) cannot be applied. In fact, one way to solve the problem is to choose an (unordered) subset of the eight positions to be filled with red blocks, leaving the remaining five positions to be filled with blue blocks. That is, even though "order matters", a combination formula can be appropriately used to solve this problem.

Fewer students behaved according to our hypothesis on this question: Five students clearly indicated the use of a permutation formula in their written work. However, this was also the most difficult problem: only six participants gave a completely correct answer. None of the interviewed students approached the problem as a "permutations with repetitions allowed" problem, and if they did attempt to use a permutation formula, it was for permutations without repetition. Jane, using her primary strategy (stated above) said, "Now since the order does matter, because we are thinking about lining things up, that why I use a permutation."

Emily, on the other hand, wrote $\frac{8!}{3!5!} + \frac{8!}{5!3!}$, and described her solution by "choosing" the three blocks to be red and "choosing" the five blocks to be blue. Here, she discarded her strategy of using the key phrase "order matters" and used combination formulas (in conjunction with the addition principle, in the mistaken belief that both terms were needed to account for the two colors). Emily indicated that the fact that the blocks were indistinguishable prompted her to modify her strategy

of focusing on whether or not order mattered. During her interview, she used the term "choose" to refer to any kind of selection process, with or without order (or in terms of the multiplication principle), but also distinguished (with less than total confidence) between "n choose r" and "n factorial over r factorial" (a misremembered version of $\frac{n!}{(n-r)!}$).

Lowell, again undisturbed by the statement about order, described his strategy as looking at the positions of the blocks and "choosing" three of them to be red. While Lowell was very successful in avoiding this pitfall, it should be noted that Lowell was simultaneously enrolled in both classes involved in this study, and therefore had much more recent experience with counting problems.

This question also produced an unanticipated phrasing difficulty for students: At least three students interpreted the requirement that "exactly three blocks are red" to mean that the three red blocks were to be stacked adjacently at some point in their interviews.

## Conclusion

Counting problems can be quite difficult, and many different types of error are possible. Our study takes a closer look at one dimension of the error types identified by Batanero et al. (1997); namely, that of the "error of order". In order to help students avoid this error, instructors and textbooks have adopted a single organizing principle for dealing with combinations and permutations: "If order matters, use permutations; if not, use combinations." However, such a principle belies the difficulty of such problems, and in fact, can be misleading.

In our two "misleading" problems, we give statements about order that do not conform to the usual meaning of "order matters". The usual interpretation of "order matters" is that, when a subset is selected from a set, a difference in the order in which the elements of the subsets are selected constitutes a different outcome. In other words, ordered subsets are counted. In our study, we found that an over-reliance on the "order matters" principle can lead students to misinterpret counting problems. We believe that students may need a much more nuanced view of permutations and combinations in order to consistently solve counting problems successfully.

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| | Ordered sample | Unordered sample |
|---|---|---|
| No repetition | $P(n,k) = \dfrac{n!}{(n-k)!}$ | $C(n,k) = \dfrac{n!}{k!(n-k)!}$ |
| Repetition | $PR(n,k) = n^k$ | $CR(n,k) = \dfrac{(n+k-1)!}{k!(n-1)!}$ |

Table 1: Four basic combinatorial operations based on selection model

(1) A bag contains 26 marbles, labeled A through Z. In how many ways can six marbles be chosen, where each of the six chosen marbles is different and the order in which they are chosen matters?

(2) A chess club has 9 members. If the club puts on a friendly tournament in which each member plays every other member exactly once, how many games will be played?

(3) A youth hockey team has twelve members. How many ways are there to choose a starting lineup of center, left wing, right wing, left defense, right defense, and goalie, if the order in which these positions are filled does not matter?

(4) A child has 8 different stuffed animals. When leaving to visit her grandmother, the child is allowed to select three animals to take along. How many ways are there for the child to select the three animals? The order in which the animals are selected does not matter.

(5) A toddler has an essentially unlimited supply of red and blue blocks, and is building stacks of these blocks. If the toddler makes a stack of eight blocks, how many ways are there to stack the blocks so that exactly three blocks are red? (The order in which the blocks are stacked matters.)

(6) A painter has fourteen colors of paint available. When painting a house, she needs to choose a main color, trim color, accent color, and siding color, and all of these colors must be different from one another. How many ways are there for the painter to pick colors for the house?

Table 2: Counting problems presented to student participants

# THE EFFECT OF STRUCTURE-BASED INSTRUCTION ON THE TRANSFER OF LEARNING TO SOLVE ALGEBRA WORD PROBLEMS

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## ABSTRACT

A problem in learning to solve mathematics word problems students have been facing is to transfer the learned problem-solving knowledge from one story context to another story context. Some studies have provided evidence that structure facilitates transfer of learning to solve word problems. In this study we examine the effect of teaching structures (structure-based instruction) on the transfer of learning to solve algebra word problems. Sixty-one college students participated in a 2-hour controlled experiment. The results showed that students who received structure-based instruction had better performance in some types of transfer of solving algebra word problems.

Keywords: Structure-Based Instruction, Algebra Word Problems, Linear Equations, Transfer

## Introduction

A vision of school mathematics education is to help students understand and be able to use mathematics in everyday life and in the workplace (NCTM, 2000). Word problems play an essential role for achieving the vision to help students appreciate mathematics in daily life and learn to solve real-life problems using mathematics. However, it has been suggested students have difficulty in the transfer of solving algebra word problems (Bassok & Holyoak, 1989; Fuchs et al., 2003, 2004; Gick & Holyoak, 1983; Hayes & Simon, 1977; Holyoak & Koh, 1987; Nickerson, Perkins, & Smith, 1985; Reed, Ernst, & Banerji, 1974; Reed, 1999). Transfer means to solve problems that are situated in new contexts. Specifically, if students couldn't solve a word problem when its context is different (or a new context) at school, how do we expect students to be able to transfer their word problem solving skills to solve real life problems?

Structure has been suggested helpful in the transfer of learning to solve word problems (Bassok & Holyoak, 1989; Catrambone & Holyoak, 1989; Cooper & Sweller, 1987; Gick & Holyoak, 1983; Holyoak and Koh, 1987; Kaminski, Sloutsky and Heckler, 2008). Particularly, structure can help subjects organize and discriminate information of word problems and problem-solving skills, and help subjects recall types of problems and their associated solution methods when the subjects encounter similar/novel problems (Bassok & Holyoak, 1989; Blessing & Ross, 1996; Fuchs et al., 2004; Gick & Holyoak, 1987). Since studies concerning algebra word problem structure and its effect in transfer are under development, the purpose of the study is to explore if teaching structures help in the transfer of learning to solve algebra word problems.

This study defines structure of a word problem as the particular components (e.g., objects, events) and the relationships between the components (Fuchs et al., 2004; Mayer, 1981). It is inherent in the word problem.

The topic of one-variable linear equation word problem, as one kind of algebra word problems, was chosen for this study for the following three reasons. First, one-variable linear equations represent an important initial topic in algebra. Mastery of this topic plays a critical role in almost all subsequent mathematical courses and topics. Second, international assessments show that U.S. students have been facing difficulties and don't do well in algebra, which includes solving algebra word problems (Blume & Heckman, 1997; Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999). Third, a literature search (e.g., ERIC), shows that little research has been conducted on the transfer of learning to solve linear equation word problems, although there are many studies about the comprehension and translation of linear equation word problems, for example, how to translate word statements to algebraic expressions and how to translate them correctly (Clement, 1982; Herscovics & Kieran, 1980; MacGregor & Stacey, 1996).

## Framework

The framework included the designs of structure-based instruction and traditional instruction, and the method for assessing the degree of transfer.

For the design of structure-based instruction, this study chose "rate" structure (e.g., miles per hour, dollars per item) and two-car model (e.g., two cars running toward or away from each other) as the to-be-taught structure. Rate was chosen because it has been the central focus of many studies about the transfer of algebra word problems (Bassok & Holyoak, 1989; Blessing & Ross, 1996; Reed, 1989; Reed, Dempster, and Ettinger, 1985), and it has been suggested successful in facilitating transfer in the domain of elementary mathematics word problems (Fuchs et al., 2004; Jitendra et al., 2000). This study used one principle and three steps for structure-based instruction

based on several schema-based instruction methods (e.g., Fuchs et al., 2004; Jitendra et al., 2002; Xin, 2005), which are teacher-directed instruction principle and the three steps of teaching the definition of the structure, helping students generate the schema, and application of the schema.

Schema is the mental representation of structure. Since it is "mental representation", different subjects may have different mental representations for the same structure, for example, a diagram, a list of components, or images of objects, events, and situations (Armbruster, 1996; Dansereau, 1995). The same word problem structure could have various schemas created and exist in the mind of different individuals. Individuals may conceptualize a word problem structure mentally in various ways (e.g., a diagram or different diagrams or a component list)

The traditional instruction for learning to solve algebra word problems was derived from textbooks (e.g., Bellman et al., 2004), which was based on that several studies of evaluating the effectiveness of schema-based instruction designed their control group treatment by following the teaching plan (or script) on textbooks (e.g., Fuchs et al., 2004; Xin, 2005). The traditional instruction consisted of the following four elements: identify quantities, identify relationships between quantities, identify the unknown quantity and its relationships with other quantities, and connect piecewise relationships together and translate to an equation.

Three transfer measures were constructed to measure the effect degree of transfer, which were SS (similar context and similar equation), SD (similar context and different equation), and DS (different contexts and similar equation), based on Reed's (1999) definitions of *equivalent*, *similar*, and *isomorphic* respectively. The following four examples illustrate the three transfer types (SS, SD, and DS) according to an original word problem:

The original word problem:

Suppose you are helping to prepare a large banquet. You can peel 2 carrots per minute. You need 60 peeled carrots. How long will it take you to peel 60 carrots? (Modified from *Algebra 1*, p. 84 (Bellman et al., 2004))

Equation translated from the word problem: $2x = 60$

1. The SS type (similar story context and similar translated equation):

   Suppose you are helping to prepare a large banquet. You can set up plates for 3 tables per minute. You need to set up 60 tables. How long will it take you to set up 60 tables?

   Equation translated from the word problem: $3x = 60$

2. The SD type (similar story context and different translated equations)

   Suppose you are helping to prepare a large banquet. You can set up plates on 3 tables per minute. You need to set up 60 tables. How long will it take you to finish if you have already set up 24 tables?

   Equation translated from the word problem: $3x = 60 - 24$

3. The DS type: (different story contexts and similar translated equation)

   Suppose you are attending a marathon. You can run 3 miles per hour. You need to run 15 miles in the marathon. How long will it take you to finish the 15-mile marathon?

   Equation translated from the word problem: $3x = 15$

**Method**

To explore the effect of structure-based instruction on learning to solve transfer algebra word problems, an experiment with the pretest-posttest control group design was conducted. Sixty-one undergraduates (Male: 18, Female: 43) from Michigan State University who were taking intermediate algebra or college algebra in fall, 2009 participated in the study. The 61 students were randomly assigned to two groups, the experimental (or structure-based instruction) group

and the control (or traditional instruction) group. Three instructors (two females, one male) with college mathematics teaching experience were recruited to implement the teaching and interview tasks. Prior to the experiment, the three instructors were given five sessions of professional development.

Each student was only in one treatment session, using either the structure-based or traditional approach. These treatment sessions were held 12 times within 7 weeks (6 sessions for the control group and 6 sessions for the experimental group). Each treatment session lasted about 2 hours containing a pretest (20 minutes), a treatment (1 hour), a posttest (20 minutes), and an interview (10-20 minutes). The treatment contained a 15-minute lecture and a 45-minute problem-solving exercise. In the pretest and posttest, students were given 12 word problems to solve. The 12 posttest word problems were generated based on four types of equation, which were (1) $ax+b(cx+d)=2$ (2) $ax+bx=c$ (3) $ax=b(cx+d)$, and (4) $ax+b(c-dx)=e*f$ . Three problems were generated for each equation type associated with either travel or mixture or other context (three rate types of word problems). To allow for assessment of transfer, each posttest problem was matched with a pretest problem in one of the following three ways, Similar context and Similar equation (SS), Similar context and Different equation (SD), or Different context and Similar Equation (DS).

For the treatment of the structure-based instruction, the teacher first asked students what rate was, and showed students examples of rate in daily life and in equations. Second, the teacher taught students the definition of rate (a ratio relationship between two quantities with different units). Third, the teacher helped students form and consolidate the rate structure by giving students several rates and asking students what real-life contexts were related to these rates. For example, given the rate "pages per minute" and "dollars per month", what are the related contexts? (Example answers: "copy machine" and "salary" respectively). Fourth, the teacher helped students apply the rate structure by (1) asking students to translate several real-life rate relationships to symbolic expressions (e.g., "If you have the information $x$ miles per hour and 3 hours, what does $3x$ mean?"), and (2) asking students translating several monomials to real-life situations (e.g., "Can you find a real-life story for the expression "60x"? Try to identify a rate first, for example, can 60 be a rate?"). For the instruction of the two-car model, the teacher taught students three scenarios of two cars for comprehending travel type problems. The two-car model consisted of three scenarios that were (1) Two cars travel in opposite directions (2) Two cars travel toward each other, and (3) Two cars travel in the same direction.

The treatment for the traditional instruction group consisted of four-step strategy and an application of the strategy. The teacher taught the first step "Identify quantities" by first describing the definition of quantity (a number associated with a unit), and illustrated with examples (e.g., 3 persons, 10 gallons, 70 miles per hour). The teacher taught the second step "Identify relationships between quantities" by illustrating with examples how two quantities could be related together. For example, "We know the speed of a car is 70 miles per hour, and we know the car has run 3 hours. The two quantities imply a distance relationship, e.g., 70 MPH x 3 hours = 240 miles." The teacher taught the third step "Identify the unknown quantities and find relationships related to the quantities" by illustrating that some quantities might be implicit, which meant there were no numbers associated, and illustrating with examples how an unknown quantity could be related to other quantities in a word problems (similar to step 2). The teacher taught the last step "Find the integrated relationship" by illustrating with examples how to connect piecewise relationships found in the previous three steps to make an integrated relationship. For example, the integrated relationship "Cell phone monthly payment = charged minutes x 0.75 dollars per minute + monthly program fee" is composed of the two relationships that are "charged minutes x 0.75 dollars per

minute is part of the total charge" and "monthly program fee is part of the total charge". After teaching the four-step strategy, the teacher explained with an example how to apply the strategy.

**Results**

The treatment for the SB (structure-based) instruction group was significantly effective (t=6.392, df=28, $p<0.001$; See Table 14), and the treatment for the TB (traditional-based) instruction group was also significantly effective (t=6.952, df=31, $p<0.001$; See Table 15). However, students' performance between the two groups was not significantly different (t=0.949, df=59, $p>0.1$).

Two MANOVA tests were conducted to evaluate if the two groups of students had different performance in the three types of transfer (SS, SD, and DS) and the three real-life problem types (travel, mixture, and other). The first MANOVA test was conducted comparing the mean scores of the structure-based (SB) instruction group (N=29) and the traditional-based (TB) instruction group (N=32) regarding the three transfer types, SS, SD and DS. The test showed the two groups performed significantly different on SD type word problems ($F$=4.3, $p$=0.042). The effect size (Partial Eta Squared = 0.68) of the difference is medium (Cohen, 1992). The structure-based instruction group performed significantly better than the traditional-based instruction group on SD type word problems. However, there was no difference between the two groups on SS and DS type word problems. The second MANOVA test was conducted comparing the mean scores of the structure-based (SB) instruction group (N=29) and the traditional-based (TB) instruction group (N=32) regarding the three real-life problem types, Travel, Mixture and Other. The test showed the two groups performed significantly different on Travel type word problems ($F$=4.1, $p$=0.048). The effect size (Partial Eta Squared = 0.065) of the difference is medium (Cohen, 1992). The structure-based instruction group performed significantly better than the traditional instruction group on Travel type word problems. However, there was no performance difference between the two groups on the Mixture or Other type word problems.

**Discussion**

In general, the structure-based instruction does not make significant difference in helping students solve algebra word problems compared to the traditional-based instruction. However, this study showed that teaching rate structure and two-car model (the structure-based instruction group) helped students achieve significant immediate transfer when contexts were similar and equations embedded in problems were altered, or when problems were travel type problems (e.g., distance, speed, and time). Teaching structures did not make significant difference on the transfer types of SS and DS, and on the mixture or other type problems, compared to the traditional instruction.

There are two possible explanations for the failure of the structure-based instruction on the SS and DS types of transfer. First, traditional instruction group students could easily recall what they just learned (the stories and their solution methods) when encountered problems with similar context and similar equation (SS type). Second, since the treatment lasted only one hour, the dosage of treatment for the structure-based instruction group might not be enough to result in significant transfer on DS type (different contexts and similar equation) problems, compared to the traditional instruction group. DS type of transfer, compared to the other two types of transfer, is far transfer (or far from what students had just learned), which typically takes more practice time and still is difficult to achieve.

Teaching rate structure and two-car model also helped students achieve significant immediate transfer when problems were about travel (speed, time, and distance). However, teaching the

structure and model did not make significant transfer difference on mixture and other types of problems compared to the traditional instruction. There are two possible explanations. First, the two-car model was directly related to the travel type problems. It could help students comprehend travel type stories. However, the model was not directly related to the other types of word problems, although isomorphic relationship could exist (e.g., two pumps with different rates filling a tank can be understood as two cars running toward each other), and therefore the two-car model might not help students comprehend the other types of word problems. Second, since the treatment lasted only one hour, the dosage of treatment for the structure-based instruction group might not be enough to result in significant transfer in solving problems with different contexts/stories, that is, different from the two-car model.

The structure adopted by the structure-based instruction was restricted to rate structure and the two-car model. It is still unknown whether the effectiveness of structure-based instruction on SD type of transfer proved in this study is true for other kinds of structure (e.g., multiplicative structures like multiple or part/whole, or additive structures like change or compare). More generally, it is unknown whether teaching other kinds of structure will have different effect on the three types of transfer (SS, SD, and DS). Therefore, future explorations to test the effectiveness of other kinds of structure by means of structure-based instruction are needed for the two purposes (1) to generalize the effectiveness of structure-based instruction in the transfer of learning to solve algebra word problems, and (2) to explore the effective relationships in facilitating transfer between kinds of structure and types of word problems (e.g., two-car model for travel type).

The intervention time of this study was restricted to one hour due to limited budget. It would be too early to conclude that structure-based instruction is not effective in solving SS and DS types of transfer word problems. Studies with longer intervention time may provide more insights on the effect of structure-based instruction on the SD and DS types of transfer.

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# PROVIDING ANSWERS TO A QUESTION THAT WAS NOT ASKED

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Abstract. *The purpose of this article is to contribute to research on teachers' conceptions of probability. To meet this objective, prospective mathematics teachers were presented two different answer keys to a 10 question multiple-choice quiz and were asked to determine and justify which of the two answer keys was least likely to occur. This article utilizes the attribute substitution model (Kahneman & Frederick, 2002) to account for certain normatively incorrect responses from prospective teachers. This new perspective provides evidence that certain individuals, when presented a particular question, answer a different question instead. Results demonstrate that participants substitute a variety of heuristic attributes instead of making the intended relative likelihood comparison of the answer keys presented. Through recognizing that there is more than one particular candidate for the role of heuristic attribute, results further demonstrate that certain participants substitute more than one heuristic attribute in their response justifications.*

Keywords: *probability; representativeness; attribute substitution; relative likelihood comparisons; answer key*

The purpose of this article is to contribute to the dearth of research on teachers' conceptions of probability (Stohl, 2005). In addition, the purpose of this article is to recognize, embrace and influence the current, minimal coordination between mathematics education research on probabilistic thinking and the teaching and learning of probability and dual-process theories from cognitive psychology (for exceptions see, for example, Leron & Hazan, 2006, 2009; Tzur, 2011). To meet these objectives, prospective mathematics teachers were presented two different answer keys to a 10 question multiple-choice quiz and were asked to determine and justify which of the two answer keys was least likely to occur. This research utilizes the attribute substitution model (Kahneman & Frederick, 2002) – which emerged from a revisitation of Kahneman and Tversky's (1972) representativeness heuristic – to account for certain incorrect responses. This new perspective provides evidence that certain individuals, when presented a particular question, answer a different question instead. Results demonstrate that participants substitute a variety of heuristic attributes instead of making the intended relative likelihood comparison of the answer keys presented. Through recognizing that there is more than one particular candidate for the role of heuristic attribute, results further demonstrate that certain participants substitute more than one heuristic attribute in their response justifications.

## The representativeness heuristic

Tversky and Kahneman (1974) found that "people rely on a limited number of heuristic principles which reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations" (p. 1124). In other words, people evaluate probabilities based on a comparison to a perceived ideal. For example, Kahneman and Tversky (1972) presented individuals with birth sequences (using B for boy and G for girl) that, while equally likely, might not be interpreted by the participants as "equally representative" (p. 432). Of the three sequences presented – GBGBBG, BGBBBB and BBBGGG – the sequence BGBBBB was considered less likely than GBGBBG because BGBBBB does not reflect the ratio of boys to girls found in the larger population. Further, BBBGGG was deemed less likely than GBGBBG because BBBGGG did not reflect the random nature associated with the birthing of boys and girls.

As evidenced from subsequent research in mathematics education (e.g., Abrahamson, 2009a, 2009b; Borovcnik & Bentz, 1991; Chernoff, 2009, 2011; Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Konold et al., 1993; Rubel, 2007; Shaughnessy, 1977, 1981), the resettlement of the representativeness heuristic has provided a foundation for researchers in mathematics education to develop their own theories, models, critiques and developments. The nineties and the aughts, however, were a quiet period for research on comparisons of relative likelihood and the representativeness heuristic. However, as the end of the aughts approached, a resurgence of research focused on relative likelihood comparisons, based to varying degrees on the representativeness heuristic, occurred (e.g., Abrahamson, 2008, 2009a, 2009b; Chernoff, 2008, 2009a, 2009b, Rubel, 2007).

**Revisiting the representativeness heuristic**

In his 2002 Nobel Prize Lecture, Daniel Kahneman detailed the (earlier) results of his close collaboration with Amos Tversky and the two system view of cognitive operations: "the ancient idea that cognitive processes can be partitioned into two main families – traditionally called intuition and reason – [and which] is now widely embraced under the general label of dual-process theories" (Kahneman & Frederick, 2002, p. 51). In addition, Kahneman's lecture detailed how "Shane Frederick and [he] recently revisited the conception of heuristics and biases, in the light of developments in the study of judgment and in the broader field of cognitive psychology in the intervening three decades" (Kahneman, 2002, p. 465).

Kahneman and Frederick (2002) define attribute substitution as follows: "We will say that judgment is mediated by a heuristic when an individual assesses a specified target attribute of a judgment object by substituting another property of that object – the heuristic attribute – which comes more readily to mind. Many judgments are made by this process of attribute substitution" (p. 53). As Kahneman (2002) would note, "This definition elaborates a theme in early research, that people who are confronted with a difficult question sometimes answer an easier one instead (p. 466). A consequence of the new model, that is, attribute substitution, which differs from the earlier work on heuristics (e.g., a common process that explains how judgment heuristics work), is that "The word 'heuristic' is used in two senses in the new definition. The noun refers to the cognitive process, and the adjective in 'heuristic attribute' specifies the substitution that occurs in a particular judgment. For example, the representativeness heuristic is defined by the use of representativeness as a heuristic attribute to judge probability" (Kahneman, 2002, p. 466).

In order to get a better sense of the new model and new uses of the term heuristic, an example is now discussed in detail. Consider the example that has been discussed previously: which of the following sequences of births of boys and girls is least likely to occur BBBGGG or GBGBBG or BGBBBB or BGBGGB (Kahneman & Tversky, 1972). Framed within the model of attribute substitution, the assessment of the target attribute, that is, comparing the relative likelihood of birth order sequences, is substituted with particular heuristic attributes, that is, the random nature associated with the birthing of girls and boys and population ratio of boys to girls. "The target attribute does not come to mind immediately, but the search for it evokes activates the value of other attributes that are conceptually and associatively related" (Kahneman & Frederick, 2002, p. 54). Alternatively stated, when confronted with making a relative likelihood comparison certain individuals answer, instead, the easier question of how regular is the pattern or what is the ratio of boys to girls. As seen with the example presented, the notion of heuristic is now used in two distinct ways. First, heuristic is used to describe the cognitive process that takes place during the process of attribute substitution and, second, the representativeness heuristic is defined by the use of representativeness as a heuristic attribute to judge probability.

## Task and participants

The task given to participants, denoted the answer key task, is presented in Figure 2 below. The two sequences presented to participants are, theoretically, equally likely to occur. The two sequences, however, are not considered (according to established literature on relative likelihood comparisons) equally representative.

> Which of the following, answer key 1 or answer key 2, is least likely to be the answer key for a 10 question multiple choice math quiz? Explain your answer
> Answer key 1: A C C B D C A A D B
> Answer key 2: C C C B B B B B B B

Figure 1. The answer key task

Research investigating comparisons of relative likelihood has been conducted with a wide range of individuals including: elementary and high school students (Abrahamson, 2008, 2009a, 2009b; Rubel, 2007); college students (Hirsch & O'Donnell, 2001; Shaughnessy, 1977, 1981), which includes prospective mathematics teachers (Chernoff, 2009a); graduate students (Cox & Mouw, 1992; Hirsch & O'Donnell, 2001); and mathematical psychologists (Tversky & Kahneman, 1971). In this research, participants were prospective mathematics teachers and were chosen for two specific reasons. First, as documented (e.g., Stohl 2005), there is a limited amount of research investigating teachers' knowledge and beliefs about probability. Second, prospective mathematics teachers, it was assumed, would represent a group of individuals that would provide unique perspectives to answer keys, which might not be a part of, for example, mathematical psychologists' perspectives.

Given the former and latter points, data for this research was gathered from 59 prospective elementary and middle school teachers – teachers of students from 4 to 13 years of age. Participants were enrolled in a methodology course, which introduces them to content, strategies and approaches from research and practice related to the teaching and learning of mathematics. The 59 prospective teachers were from two different courses of 31 and 28 students taught by two different teachers. The topic of probability had yet to be addressed in either of the two classes. Participants were asked, and given as much time as required, to determine which of the sequences were least likely to occur and, further, to justify their choice via written response.

## Results

A total of 48 out of a possible 59 participants, approximately 81%, chose AK2, that is, CCCBBBBBBB, as least likely to be the answer key for a 10 question multiple choice quiz. Alternatively, seven participants, approximately 12%, chose ACCBDCAADB, that is, AK1, as least likely. Worthy of note, four participants, approximately 7%, determined that each AK1 and AK2 were equally likely to occur, even though the equally likely option was not explicitly presented as a choice for participants.

## Analysis of results

As presented in the results, responses fall into three distinct categories: AK1 least likely, AK2 least likely and AK1 and AK2 as equally likely to occur. Given the objectives of this research, the analysis of results will focus on certain response justifications from the 48 (out of 59) participants that chose AK2 as least likely to occur. As will be demonstrated throughout the analysis of results, their judgments of relative likelihood were mediated with a heuristic. Utilizing the attribute substitution model for the analysis of results further reveals that judgments were made through the process of attribute substitution. That is, certain participants, when asked

to make a comparison of relative likelihood, answer a different question. The relative likelihood of the answer keys, that is, the target attribute, is substituted with other properties of that object, that is, heuristic attributes. As expected, the representativeness heuristic was one of the heuristic attributes that emerged from the analysis. However, "there is sometimes more than one candidate for the role of heuristic attribute" (Kahneman & Frederick, 2002, p. 55) and a number of other heuristic attributes – entitled: prototypical answer key, answer key encounters, test maker tendencies and pattern recognition heuristics – arose from the analysis of results. Of the heuristics mentioned, the analysis of results details the answer key encounters and pattern recognition heuristics.

**The answer key encounters heuristic**

The responses from Raymond, Sue, Tara and Uma, as was the case with Mary and Oliver, make reference to the use of only C's and B's (no A's and D's) and, further, that there are too many of the answers are in a row in AK2.

> *Raymond*: Every multiple choice answer key over a twenty-year career in academics has looked more like AK1. I am using probability to make an educated guess that AK2 is less likely.
>
> *Sue*: I say that AK2 is least likely mostly for the fact that I have personally never had an exam (like this) where only 2 answers are correct.
>
> *Tara*: From my experience with multiple choice exams, the answers never line up one after the other, like in AK2. The multiple choice exams I studied for such as math, have always looked more like AK1, where there is a variety of answers such as ACCBD instead of CCCBBB.
>
> *Uma*: there are too many answers that are the same ex)cccbbb. This (as a student) always made me confused. If the answers are all in a line like that, it makes the student feel like they did something wrong.

The responses from the four participants further describe that their expectation of seeing all the possible answers, not just C and B, and more "variety" instead of answer "in a line" is based upon their personal encounters with answer keys. Based on their different yet similar experiences with multiple-choice answer keys, they expect frequent switches and short runs between answers. The alternative property associate with answer keys, not the intended relative likelihood comparison of the answer key, they describe in their responses is a description or notion of personal answer key encounters.

Further analyzing their responses from the notion of personal answer key encounters and within the attribute substitution framework, the responses of Raymond, Sue, Tara and Uma demonstrate that their judgment was also mediated by a heuristic. More specifically, the individuals are assessing the target attribute of the judgment object (i.e., the relative likelihood of the answer keys), by substituting an alternative property associated with answer keys (i.e., a heuristic attribute), which, in this instance, is their personal encounters with answer keys. As such, the respondents are also answering and providing a reasonable response, instead, to a question that was not asked of them. Based on their justifications, perhaps Raymond, Sue, Tara and Uma are responding to some variation of the following question: "Which of the following two answer keys have you previously encountered the least?"

**The pattern recognition heuristic**

Unlike the majority of the previous responses analyzed, Aaron and Doug do not, at least explicitly, discuss the long run of one answer and the sole use of answer C and B. Instead, Aaron and Doug reference the presence of a pattern in AK2

*Aaron*: Usually there isn't a pattern to the answer key. When students recognize that the answer has been B for the last few questions then they will tend to just pick B for the next ones without reading/fully answering the questions.

*Doug*: I think AK2 is least likely to be the answer key because there is only 2 lines going straight down. AK1 has a zig-zag and it just seems better to have the answers all over rather than a boring pattern. Everyone knows the answers don't follow a pattern, if they did, everyone would get the answers right.

Aaron and Doug further discuss that patterns are not typically found in answer keys. Answer keys, for Aaron and Doug, cannot have a pattern because once a savvy student taking the multiple choice quiz picks up on the pattern the integrity of the quiz is compromised, which is also an issue for Uma. If, however, the answer key "zig-zags" all over the place, students will not be able to pick up on any discernable pattern. As such, the alternative property (to relative likelihood) associated with answer keys being expressed by Aaron and Doug can be captured as a notion or description of pattern recognition. One cannot run the risk of having a pattern in an answer key lest a student finds the pattern.

The notion of pattern recognition found in the responses of Aaron and Doug demonstrates that their judgment, too, was mediated by a heuristic. Aaron and Doug are assessing the relative likelihood of AK1 and AK2 by substituting an alternative property associated with answer keys, which, in this case, is pattern recognition by the test taker. In other words, they assess the target attribute by substituting the pattern recognition heuristic and, in doing, are, instead, answering a different question that one they were asked. Recognizing the pattern recognition heuristic and based on their responses, perhaps Aaron and Doug are responding to some variation of the following question: "Which of the following answer keys patterns does not follow a pattern?"

## Discussion and Conclusion

Recent developments from the field of cognitive psychology, such as attribute substitution, have largely been ignored by those investigating probabilistic thinking and the teaching and learning of probability in the field of mathematics education. Through a combination of the new task and the use of attribute substitution as a theoretical framework, it was established, in general, that participants' judgments of relative likelihood were mediated by a heuristic (cognitive process sense of the word) because participants assessed the relative likelihood of the answer keys by substituting other properties associated with answer keys. For example, and according to prior research, as expected, representativeness was used as a heuristic (substitution sense of the word) attribute to judge relative likelihood. Beyond the representativeness heuristic, a number of other heuristic attributes were revealed. For example, prototypical answer key, answer key encounters, test maker tendencies and pattern recognition were all revealed as alternative properties of answer keys (i.e., heuristic attributes), which were substituted instead of making a relative likelihood comparison. In some cases, certain individuals substituted more than one heuristic attribute during the process of attribute substitution. As a result, it was established that, although asked to make a relative likelihood comparison, many participants provided reasonable answers to questions that were not asked, which is the essence of attribute substitution. This research demonstrates "that there are important opportunities for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard, Dooren, Schaeken & Verschaffel, 2009, p. 103) and bolsters the current, minimal coordination between mathematics education research on probabilistic thinking and the teaching and learning of probability and dual-process theories from cognitive psychology.

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# PREPARING MATHEMATICS TEACHERS TO TEACH DEFINITIONS

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*Mathematics teachers, at all levels, must help their students become thoughtful users of mathematical definitions. This paper examines pre-service secondary mathematics teachers, at the end of their undergraduate training, interacting with mathematical definitions. They were tasked with choosing and using definitions; evaluating the equivalence of definitions; and interpreting a definition from a high school mathematics textbook. Their performances indicated that many of these future mathematics teachers have difficulty reasoning with and about mathematical definitions. These deficiencies have implications for undergraduate teacher preparation.*

*Keywords:* definitions, functions, pre-service secondary teachers

Pre-service secondary mathematics teachers (PSMTs) learn many mathematical definitions in their undergraduate training, but what should they learn *about the role of* mathematical definitions? This paper explores that question by examining data from an undergraduate capstone course for mathematics majors who intend to become secondary mathematics teachers. The relevance of this study is underscored by the call for teachers to prepare mathematically proficient students as envisioned in the Common Core State Standards for Mathematics (CCSSM); these are students who use definitions to construct arguments, to reason, and to communicate (Common Core State Standards Initiative, 2010). The essential role of definitions in mathematics is widely noted (e.g., Tall & Vinner, 1981; Zaslavsky & Shir, 2005) and, accordingly, some mathematics educators have called for explicit instruction about definitions in teacher training programs (e.g., Winicki-Landman & Leikin, 2000). However, there has been little research on the learning or teaching of definitions (deVilliers, 1998; Moore-Russo, 2008; Vinner, 1991; Zaslavsky & Shir, 2005), and the research which does exist indicates that many PSMTs have a deficient understanding of the roles of definitions (Leikin & Winicki-Landman, 2001; Linchevsky, Vinner, & Karsenty, 1992; Moore-Russo, 2008).

The qualitative study presented herein is an analysis of the work produced by 23 PSMTs on three tasks in which they were required to choose, use, compare, and interpret definitions of functions. These tasks are, at least in part, aligned with how the participants, in their careers as teachers, will need to interact with mathematical definitions. The research was guided by the question: What do the PSMTs' performances on these tasks indicate about their meta-mathematical knowledge *about* mathematical definitions? This specialized knowledge includes awareness of the qualities of good mathematical definitions; for instance, they must not be self-contradicting or ambiguous and they must be invariant under choice of representation (Zaslavsky & Shir, 2005). Additionally, mathematics teachers should have a sense of the arbitrariness of mathematical definitions (there are many equivalent ways of defining a mathematical object). The PSMTs' performances on these tasks offer insights about their training both as undergraduate mathematics majors and as secondary mathematics teachers.

## Literature Review

There is limited research on college student and on pre- or in-service teacher understanding of the use, nature, and roles of mathematical definitions. However, in one study of particular relevance, Vinner and Dreyfus (1989) found, in a survey of college students and junior high teachers, that of the 82 respondents who supplied a Dirichlet-Bourbaki definition of functions (a correspondence between sets), 56% of them did *not* use this conception of functions to answer other questions about functions. They described this inconsistency as potentially a result of conflicting cognitive schemes for which concept images and concept definitions were not mutually supportive. That is, the way that the respondents think about functions was incompatible with the words used to define function. This phenomenon has elsewhere been described as compartmentalization (Vinner, Hershkowitz, & Bruckheimer, 1981). Vinner (1991) advised that, in negotiating these conflicts, educational goals should dictate the roles of definitions in a mathematics class.

Research with pre- and in-service has generally indicated that many struggle with constructing, comparing, and using definitions. Linchevsky et al. (1992) found that, out of a group of 82 pre-service mathematics teachers, only 21 were "aware of the arbitrariness aspect of definition" (p. 53). Moore-Russo (2008), in a study of pre- and in-service secondary mathematics teachers, found that none of the 14 participants had any prior experience with definition construction. Both Moore-Russo and Leikin & Winicki-Landman (2001) described explicit work with definition construction as a means to deepen teachers' subject matter and meta-mathematical knowledge. Leikin & Winicki-Landman noted that many teachers in a professional development workshop were unaware of the arbitrariness of definitions and of the consequences of particular definition choices. Shir and Zaslavsky (2001) described inconsistencies amongst mathematics teachers evaluating the equivalence of definitions of squares and observed that the 24 teachers in their study considered both mathematical and pedagogical concerns in determining equivalence.

Despite the limited attention given to definitions in mathematics education research, the importance and value of definitions throughout mathematics education has been widely acknowledged both by researchers (e.g., deVilliers, 1998; Harel, Selden, & Selden, 2006; Ouvrier-Buffet, 2006; Vinner, 1991; Winicki-Landman & Leikin, 2000; Zaslavsky & Shir, 2005) and in standards documents (Common Core State Standards Initiative, 2010). Indeed, secondary mathematics teachers must interact with definitions as they evaluate, interpret, and model the use of definitions from a variety of sources. This task is often complicated by the inadequate definitions which teachers may encounter in curricular materials; often, these definitions do not foster conceptual understanding or support a logical foundation for future mathematics studies (Harel & Wilson, 2011). Vinner (1991) advises teachers and textbook writers to be cognizant of the "cognitive power that definition has on the student's mathematical thinking"; something which he warns is often neglected (p. 80).

## Methodology

The participants in this qualitative study were 23 students enrolled in a capstone course in the mathematics department at a large masters-granting university in the western United States. The course is required for all mathematics majors intending to teach high school mathematics. Nineteen of the students were in the last semester of their BS in Mathematics degrees; the others had just completed their BS and were enrolled in a credential or graduate program. The author of this paper was also the instructor. The data are comprised of student work on three tasks intended to assess (1) knowledge of the roles of definitions, (2) the arbitrariness of definitions, and (3) the

pedagogic dimension of definition choice. The tasks were assigned either as homework or as test problems (they were not specifically designed as research instruments). The analysis process was an iterative search for patterns through coding of student responses (Coffey & Atkinson, 1996). The initial rounds of coding were driven by the three concerns listed above. Subsequent rounds were driven by emergent themes.

## Results

### *Task 1: Choosing & Using Definitions*

Task 1 was a two part question; students were asked (a) to choose a definition for functions (from any source) and then (b) to use that definition in order to explain why sequences are functions. Nineteen of the 21 students who answered this question chose definitions which described functions as mappings, rules, relations, correspondences, or relationships between sets, variables, or inputs/outputs. The other two described functions as types of equations; this was a problematic choice since not all functions can be described by equations. The analytical focus was on whether the PSMTs used their chosen definitions in the second part of the task. An imperfect but demonstrative way of determining this was to see if the student either directly or indirectly referenced the object of their chosen definition. For example, a direct reference would be if a student defined functions as mappings and then describe sequences as a type of mapping or as objects which map (i.e., they used the verb-form of the noun). An indirect reference would, for example, be if they defined functions as relations and then described sequences as sets of ordered pairs.

Table 1. *Summary of Responses to Task 1*

| | Referenced Object | No Reference to Object | Total |
|---|---|---|---|
| Correct | 5 | 7 | 12 |
| Incorrect | 6 | 3 | 9 |
| Total | 11 | 10 | |

Out of 21 students, only eleven used their definition of function to attempt to justify why sequences are functions. Of these eleven, only five correctly justified why a sequence is a function. Seven students provided a correct justification without referencing their definition. For example, one student chose a definition for functions which restricted the domain to open intervals; this definition was too restrictive to support her otherwise correct answer for the second part of the task. The students who described functions as equations also suffered from choosing overly restrictive definitions; neither correctly answered part (b). Of the five students who described functions as rules, only one even attempted to describe sequences as types of rules. Six students used their definitions yet did not adequately explain why sequences are functions; for example, one student who described functions as sets of ordered pairs wrote, "Since in [part] (a) we get a set of ordered pairs, a sequence $\{a_n\}$ is also a set of numbers written in a definite order." On this task,, student work revealed that content knowledge (about functions and sequences) and knowledge about using definitions were not mutually supportive for most students.

### *Task 2. Interpreting and Comparing Definitions*

Students were asked to determine if the following two textbook definitions of functions were equivalent.

I. "A function is a rule that takes certain numbers as inputs and assigns to each a definite output number." (Hughes-Hallett et al., 2004, p. 2)

II. "A function is a special type of relation in which each element of the domain is paired with *exactly one* element of the range." Relation had been previously defined as "a set of ordered pairs... The domain of a relation is the set of all first coordinates from the ordered pairs, and the range is the set of all second coordinates of the ordered pairs." (Holliday et al., 2005, p. 57)

Of the 22 PSMTs who did this task, only five noted that these definitions are not equivalent because definition (I) requires that the domain and range be sets of *numbers*. Four others said that they are not equivalent because definition (I) does not require that each input be assigned a unique output. That is, they did not recognize that the phrase "definite output number" satisfied this requirement. Twelve students believed these definitions to be equivalent and one said "yes and no". This ambivalent student was perhaps distracted by the fact that Definition (I) came from a calculus textbook and concluded that this definition, and not definition (II), from and Algebra 2 textbook, allowed for multivariate functions. Some PSMTs were also struck with confusion when asked what the word "special" meant in definition (II); five of the students erroneously noted that it meant that each input has a *unique* output.

***Task 3. Interpreting Definitions***

Task 3 required the PSMTs to interpret the following definition from a high school algebra textbook:

> A **rational function** is an equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. (Holliday et al., 2005, p. 485)

This definition was chosen because it is problematically dependent on a choice of representation and because the condition that $q(x) \neq 0$ may not be clear to many PSMTs. Students were given choices of multiple questions to answer about this definition. Of the eight who responded to the request to change the definition so that it is more clear that $y = 3x^{-3} + 2x^{-1} - 5x^2$ is a rational function, six of them improved the definition by changing it to, "A rational function is an equation which *can be written* in the form $f(x) = \frac{p(x)}{q(x)}$, …". The other two produced a new definition which was also dependent on representation. Only six out of 22 PSMTs correctly noted that the condition that $q(x) \neq 0$ means that $q(x)$ cannot be the zero polynomial. The others believed it to be a domain restriction. This result may be more indicative of the communicative shortcomings of this definition. Surely, much of this confusion would be pre-empted by a more precisely-worded definition such as the following:

> A rational function is a function that can be put in the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomials, and $b(x)$ is not the zero polynomial (McCallum et al., 2010, p. 407)

## Discussion

In an undergraduate mathematics capstone, pre-service teachers of secondary mathematics were required to choose, use, compare, and interpret definitions. In the context of a secondary mathematics classroom, these constitute authentic teacher interactions with definitions; that is, these tasks are relevant to the PSMTs' future careers. High school teachers of mathematics will need to evaluate the (sometimes flawed) definitions they encounter in a multitude of curricular sources and, moreover, they will need to train their students to be thoughtful users of mathematical definitions. However, the three tasks described above

demonstrated that many pre-service teachers, even those near completion of an undergraduate mathematics degree, may not be prepared for these mathematical requirements of their chosen careers.

The PSMTs' performances on these tasks also revealed that the relationship between mathematical content knowledge and knowledge about definitions may, indeed, be complicated and context-dependent. In Task 1, about half of the students demonstrated that they were aware of what it means to "use" a definition, yet this did not guarantee that they used the definition correctly. Among the students who correctly described sequences as functions, more than half did not reference their definitions. In many cases, this seems to have been caused by a poor choice of definition, as was the case in the study by Vinner & Dreyfus (1989). Indeed, these results may be due to conflicting cognitive schemes for the PSMTs' concept images and the concept definitions. Yet they may also be related to a lack of knowledge about the role of mathematical definitions. Task 2 further illustrated that many PSMTs had trouble comparing definitions; most did not notice an important detail (one definition defined functions only on sets of numbers). Some misinterpreted other details either by not recognizing the equivalence of the phrases "definite output" and "unique output", or by misunderstanding the meaning of "special" in the phrase "a special type of relation". Task 3 presented the PSMTs with a problematic definition from a high school textbook. The data did not reveal the extent to which they recognized that this definition was based on a choice of representation (i.e., a rational functions is "an equation of the form $f(x) = \frac{p(x)}{q(x)}$ …"), however, two out of eight of the PSMTs who attempted to fix the definition did not address this issue. The work on these tasks highlights the importance of thoughtful choice of mathematical definition.

Indeed, these undergraduate mathematics majors were often confused or hindered by their definition choices or by the definitions which were supplied for them. Yet, as these PSMTs transition to their careers as mathematics teachers, they will be tasked with helping their own students use definitions to construct arguments, to reason, and to communicate. Vinner (1991) noted that definitions create "a serious problem in mathematics learning" as they represent "the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (p. 65). The present study indicates that some pre-service secondary mathematics teachers may not have the knowledge about definitions needed to help their future students navigate this conflict.

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# TWO PARALLEL CALCULUSES: THE ONE TAUGHT AND THE ONE USED

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*Calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their courses. Unfortunately, due to the parallel nature of the calculus taught, many students leave their calculus course(s) with an understanding misaligned with what is needed in the discipline courses and are thus ill-prepared. By working with presumed experts (undergraduate mathematics and other discipline faculty members), this study developed a small number of prototype tasks that elicit, document, and measure students' understanding of a few calculus concepts the faculty participants believe to be essential to successful academic pursuits within their respective disciplines. This presentation details the data and analysis from the concluding rounds of research. Implications of this research for calculus instruction and curriculum are mentioned.*

*Key words:* Calculus, understanding, STEM preparation, design research

*Mathematics can and should play an important role in the education of undergraduate students. In fact, few educators would dispute that students who can think mathematically and reason through problems are better able to face the challenges of careers in other disciplines – including those in non-scientific areas. Add to these skills the appropriate use of technology, the ability to model complex situations, and an understanding and appreciation of the specific mathematics appropriate to their chosen fields, and students are then equipped with powerful tools for the future.*

*Unfortunately, many mathematics courses are not successful in achieving these goals. Students do not see the connections between mathematics and their chosen disciplines; instead, they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings and whose importance to their future careers is not appreciated. Indeed, the mathematics many students are taught often is not the most relevant to their chosen fields. … The challenge, therefore, is to provide mathematical experiences that are true to the spirit of mathematics yet also relevant to students' futures in other fields.*

(Ganter & Barker, 2004, p. 1)

These claims detail the rationale for The Mathematical Association of America's (MAA) Curriculum Foundations Project (CFP, http://www.maa.org/cupm/crafty/cf_project.html). Portions of the mathematics community and its partner disciplines, what I refer to as "client" disciplines (e.g., biology, business, chemistry, computer science, several areas of engineering), worked together to generate a set of recommendations that have assisted mathematics departments plan their programs to better serve the needs of client disciplines (Ferrini-Mundy & Gücler, 2009).

What does it mean for a mathematics course (e.g., calculus) to serve the needs of client disciplines? More often than not, client departments expect the pre-requisite calculus course(s) to provide the mathematical foundation needed for success in their calculus-based courses (Klingbeil, Mercer, Rattan, Raymer, & Reynolds, 2006). Are the calculus courses emphasizing the understanding needed for success in the client courses? Much research shows they are not and the graduates of the calculus course(s) leave with an "exceptionally primitive" understanding of fundamental calculus concepts (Ferrini-Mundy & Graham, 1991; Zerr, 2010) and are ill-prepared for client courses (Ganter & Barker, 2004; Kasten, 1988; Klingbeil, et al., 2006).

As Ganter and Barker (2004) implied, client department faculty often complain that students are unable to apply calculus in the client coursework. Sometimes this coursework asks students to use the calculus concepts in ways not familiar to them. At other times, even when the concept is used in a similar fashion, differences in notation or a lack of familiar cues derails students. Such difficulties in transferring knowledge between disciplines are stark indicators of a lack of understanding (Hughes Hallett, 2000). Muddying the waters further are the numerous characterizations offered by literature that do not clarify what it means to understand calculus (Hiebert et al., 1997), much less provide resources for measuring this understanding.

Ferrini-Mundy and Gücler's (2009) review of the education reform efforts put forth within the undergraduate STEM disciplines provided an indication of the nation's willingness and commitment to ensure students learn these disciplines to the levels needed for competitiveness and for literacy. Before students can compete nationally, they must be successful within the academic world. Success in this world requires an applicable understanding of calculus because "modern scientific thought has been formed from the concepts of calculus and is meaningless outside this context" (Bressoud, 1992, p. 615).

The changes during the reform years placed greater emphasis on conceptual understanding (Hughes Hallett, 2000), but as Ganter and Barker (2004) pointed out, it has not been enough; the disconnect between what the client disciplines need and what the calculus courses provide still exists. For this reason, this study sought to answer these questions:

1. What calculus is needed and in what context?
2. What does it mean to understand this calculus?
3. How will teachers know if their students understand calculus?

Following in the footsteps of the CFP, this study explored the potential disconnect between the calculus taught and the calculus used at a particular undergraduate engineering institution. Through exploring this disconnect, this study identified several fundamental calculus concepts students need for successful academic pursuits outside the calculus classroom. This study pushed beyond the CFP by describing what it means to understand these concepts and developing tasks that allow teachers to assess calculus understanding.

## Description of Study

Describing the fundamental calculus concepts and developing the prototype tasks constituted a design research study (Brown, 1992; Collins, 1992). As design research, each cycle of this study included divergent ways of thinking, selection criteria for the most useful ways of thinking, and sufficient means of carrying forward the ways of thinking so they could be tested during subsequent cycles.

Twenty-one faculty members (9 "teachers" and 12 "users") at an engineering undergraduate institution participated in an iterative series of interviews during which they expressed, tested, and revised the descriptions of fundamental calculus concepts, frameworks for understanding each concept, and draft tasks. Mathematics and client department faculty (referred to as "teachers" and "users" of calculus, respectfully) were selected based on their proximity to the calculus courses and the calculus-based client courses.

The framework for this study can be thought of as a multi-tier design experiment (Lesh & Kelly, 2000). As Table 1 outlines, there were three tiers in this research project: 1) students, 2) faculty members/researchers, and 3) researcher/facilitator. For the research described here, the goal was not to produce generalizations about students or faculty members. Instead the primary goal was to work with presumed "experts" (instructors that taught a course of interest two or more times) to develop a small number of prototype tasks to elicit, document, and measure students' understanding of a few calculus concepts the faculty participants believe to be essential to successful academic pursuits within their respective disciplines.

| | |
|---|---|
| Tier 3: The Researcher Level | Researcher develops models to make sense of faculty members' and students' calculus understanding. The researcher's interpretations are revealed through facilitation of the faculty interviews and student work sessions. Describing, explaining, and predicting faculty and student behaviors and responses further reveals the researcher's interpretations. |
| Tier 2: The Faculty Level | As faculty members develop shared tools (such as guidelines to assess student responses) and as they describe, explain, and predict students' responses, they construct and refine models to make sense of students' calculus understanding. |
| Tier 1: The Student Level | Individual students work on several tasks in which the goals include eliciting, documenting, and measuring the individual student's calculus understanding. |

*Table 1.* A three-tiered design experiment (adapted from Lesh & Kelly, 2000).

The calculus concepts (function, limit and continuity, rate of change, accumulation, and the fundamental theorem of calculus), together with frameworks and tasks, believed to be fundamental by mathematics and mathematics education researchers formed the basis of the interviews. After each cycle, a compare/contrast analysis was conducted on the emerging products. This consolidated set of products formed the basis of the next cycle. Subsequent cycles centered on analyzing and revising the tasks. Tasks were evaluated and analyzed first through the faculty lens[1] and then through the medium of student work. Task modification and writing completed each interview. The cycles, including data collected and results, are outlined in Table 2.

| Data Collection | Goals | Data Collected | Results |
|---|---|---|---|
| **Cycle 1: Describing Calculus and its Fundamental Concepts, Developing Draft Tasks** | ▪ Make explicit what calculus is and how students need to understand the necessary calculus concepts within respective disciplines<br>▪ Develop drafts of prototype tasks | ▪ Interview notes from each intradisciplinary group<br>▪ Draft tasks<br>▪ Audio and video recordings of each group interview session | ▪ Preliminary list of fundamental calculus concepts<br>▪ Preliminary version of understanding frameworks<br>▪ Drafts of 19 tasks |
| **Cycle 2: Analyzing and modifying tasks based on faculty testing and student work. (Implicit revisions of concept list and frameworks)** | ▪ Revision of tasks<br>▪ Revisions of concept list and frameworks | ▪ Student work for 11 selected tasks<br>▪ Interview notes from each interdisciplinary group<br>▪ Modified tasks<br>▪ Audio recordings of each faculty group interview session | ▪ Revised list of fundamental calculus concepts<br>▪ Revised version of understanding frameworks<br>▪ Revisions of 11 tasks |
| **Cycle 3: Clarifying Distinctions, Evaluating Tasks** | ▪ Clarify and make explicit the distinctions between teachers and users<br>▪ Revision of tasks | ▪ Student work for 12 selected tasks<br>▪ Interview notes from each interdisciplinary group<br>▪ Modified tasks<br>▪ Audio recordings of each faculty group interview session | ▪ Final list of fundamental calculus concepts<br>▪ Final version of understanding frameworks<br>▪ Final versions of 12 tasks |

*Table 2.* Description of Cycles, Data Collected, and Results.

[1] The "faculty lens" is comprised of any pre-existing beliefs and/or knowledge about calculus, any previous experience with the task themselves or with similar tasks, and any work done to complete the prototype tasks.

## Results

The results and analyses discussed here are not in their raw form. All data interpretations and follow-on analyses were reviewed by the faculty participants to ensure completeness and accuracy. The quality, usefulness, and effectiveness of the prototype tasks (see Figure 1 for examples of the tasks written) were tested through administration to single-variable calculus students and analysis by the interdisciplinary groups of faculty participants. A complete cycle of data collection, data analysis, and interpretation verification occurred for a given session prior to conducting subsequent sessions.

### Cycle 1: Describing Calculus and its Fundamental Concepts, Developing Draft Tasks

For a detailed description of Cycle 1, see Ferguson and Lesh (2011).

### Cycle 2: Analyzing and Modifying Tasks based on Faculty Testing and Student Work

Implicit in the task modification discussions lay opportunities for revisions of the faculty participants model(s) of calculus and the concepts that comprise the field, as well as revisions of what it means to understand calculus. The revised list of calculus concepts is (**bolded** concepts are considered fundamental): **function**; infinitesimals; infinity; limit; continuity; **derivative** ≈ rate of change; related rates and simultaneously changing rates; Riemann sums; **integral** ≈ summation, accumulation; concavity, **relationship between derivatives and integrals**; sequences and series; and differential equations. With the exception of adding concavity to the concept list and elevating the relationship between derivatives and integrals to a fundamental concept, the faculty participants' models of calculus and understanding did not seem to change from that discussed in Cycle 1. The unified view of calculus as a tool dominated the interviews: Calculus is a tool used to explain how a physical situation works, to make a prediction about some physical situation, or to solve a problem. While the teachers focus on students knowing how and why to use the tools in their toolbox (i.e., procedures), the users focus on students assessing physical situations and selecting the calculus tool which will make a sensible prediction about the situation. According to both teachers and users of calculus, understanding is assessing a given situation and intelligently selecting an existing description (i.e., model or procedure) of the expected concept and applying the associated procedures correctly to get a reasonable answer and/or prediction.

The method chosen to assess whether a student has this understanding illuminated another distinction. To assess understanding, teachers of calculus are content with explanations and descriptions (e.g., writing assignments that basically ask students to explain the idea of a derivative or integral based on some sort of scenario). The users of calculus questioned whether the understanding needed to write about a concept will also enable a student to be able to apply the concept. Application of calculus tools in the client discipline courses requires the students to recognize 1) that calculus is applicable to the given situation and 2) what calculus tool is appropriate and when to use it. Therefore, the users of calculus assess understanding with applications. This difference begins to explain the misalignment between the preparation the students receive in calculus courses and the preparation the client courses require. To satisfy both the teachers and users, determining whether a student has a flexible, durable, and transferrable understanding requires both explanation and application.

### Cycle 3: Clarifying Distinctions, Evaluating Tasks

Analysis of the previous interview sessions revealed further distinctions in the ways the faculty participants view calculus understanding and elicit it from students. These distinctions illuminated two "parallel" calculuses: calculuses that span the same concepts, but differ when it comes to objectives for understanding, application, awareness, extraction, guidance, and representation. The following statements articulate these distinctions.

1. *Understanding*: The distinction between how the teachers and users view understanding calculus (beyond computations) is best articulated by the level of understanding students are expected to demonstrate upon leaving their calculus course(s) or entering their client course(s). Using Bloom's taxonomy (Anderson & Krathwohl, 2001), these levels are:
   - Understanding: Can the student explain ideas and concepts? (teacher preference)
   - Applying: Can the student use the information in a new way? (user preference)

   Because the teachers of calculus are focused on developing explanatory abilities in their students, they have little to no time for applications. This reality does not meet the users' expectation (or desires) for students to be steeped in applications.
2. *Application*: For the teachers, understanding a concept and applying a concept are different and exist in hierarchy. Application without understanding is repetition of a teacher- or textbook-demonstrated procedure. Application with understanding is being able to "undress" the given situation, recognize the underlying concept(s), and select the appropriate tool which will solve the problem. The teachers believe computations are required to elicit application, while they are not needed to elicit understanding. For the users, application is the ability to apply a concept with understanding (i.e., recognizing the concept within a new situation, knowing what procedures then apply, and proficiently solving for the answer). Computational ability and versatility must combine with understanding to get the ability to solve novel problems.
3. *Awareness*: Several Cycle 2 comments opened a discussion of the necessity for students to *think* they are doing calculus when they are doing calculus. While all the faculty participants agreed it would be beneficial for students to recognize what was causing their difficulties when solving a problem (e.g., deficit in algebra not calculus when trying to optimize), they differed on whether it was important to label the tools that allowed the problem to be solved. For some, the ability to label the tools is completely unnecessary; while for others, the ability to label is synonymous with selecting the appropriate tool.
4. *Extraction*: The users expect the students to be steeped in applications – applications with understanding – when they leave the pre-requisite calculus courses. As a result, computations (i.e., numerical or graphical solutions) are mandatory. Meanwhile, the teachers of calculus focus on developing explanatory abilities in their students. While explanations do a good job of eliciting understanding, simultaneously eliciting understanding and mechanics is the agreed upon "best" method for elicitation.
5. *Guidance*: The amount of "leading" a task must do to either guide a student in the desired direction or determine where a student is having difficulties depends on the timing and purpose of the task. The teachers felt the students need more guidance because the purpose of their instruction is to help students develop the procedures and concepts; it is the users' job to help students develop the flexibility to apply the procedures and concepts. The users did not disagree, but stressed the use of guidance depended on the type of assessment: formative meant leading the student and summative meant "throwing the students out of the nest and seeing if they could fly."
6. *Representation, specifically tables versus graphs*: While the faculty participants viewed each representation as merely a different way to look at a situation, they felt working with a table requires more assumptions and thus requires a deeper understanding than working with a graph does. The teachers emphasized that because most students do not realize the assumptions necessary to work with tables, they (the teachers) prefer graphs. Also impacting this choice is the abstract nature of their favored situations (i.e., situations lending themselves easily to continuous, smooth graphs). The users prefer tables because most real-world data are collected, stored, and displayed using tables; real life does not typically have very nice and neat functions associated with it; and making the assumptions explicit is essential to the students' understanding of the situation.

**The Ball Task**

Consider the following problem: A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. How much time will it take for the area of the circular ripple to exceed 50,000 square centimeters?

Write up a solution to the above problem as if you were writing a textbook example (i.e., explain how any function, formula, and/or computation will be used in the next step(s)).
For example: *Find the second derivative and interpret its sign for* $g(x) = x^3$.

*Solution: If* $g(x) = x^3$, *then* $g'(x) = 3x^2$, $g''(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3 \cdot 2x = 6x$. *This is positive for* $x > 0$ *and negative for* $x < 0$, *which means* $g(x) = x^3$ *is concave up for* $x > 0$ *and concave down for* $x < 0$.

**The Hiking Task**

To celebrate their 40th wedding anniversary, Helen and Brendan O'Neill are planning a hike with their children and grandchildren. They are considering a nearby 5-kilometer hike. The local park provided a graph of the trail's grade at every point, but the O'Neills want to make sure it is suitable for them. Helen wants to know if there is a summit where they can have lunch and enjoy the view, while Brendan wants to know where the hiking gets difficult.

**The O'Neills need your help!**

Design a method that the O'Neills can use to sketch a distance-height graph of the original trail. You can assume the trail begins at sea level.

Write a letter to the O'Neills explaining your method, and use your method to describe what the hiking trail will be like. In particular, you must clearly show any summits and valleys on the trail, uphill and downhill portions of the trail, and the parts of the trail where the slopes are steepest and easiest.

Most importantly, your method needs to work not only for this hiking trail, but also for any other hiking trail the O'Neills might consider.

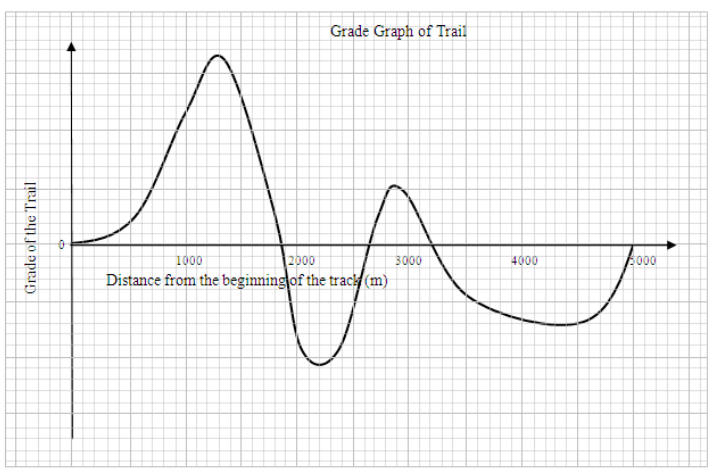


*Figure 1.* Two prototype tasks: the Ball Task (modified from a composition contextual problem discussed in Carlson, Oehrtman, & Engelke, 2010) and the Hiking task (modified from the Tramping Problem in Yoon, Dreyfus, & Thomas, 2010).

## Conclusion and Implications

Teachers of calculus push students to develop a deep understanding of the concepts such that they can explain it to another student well enough to make that student understand; whereas the users expect the students to walk into their classrooms with a deep understanding of the concepts such that they can recognize a concept within the given situation or context, select a calculus tool that will efficiently get them an answer, compute the answer, and make an accurate determination or prediction. As this study revealed, the end goal of the calculus courses and the beginning goal of the client disciple courses do not align. This misalignment has caused and continues to cause many students, instructors, and researchers much frustration. This misalignment also carries with it many implications: implications such as client discipline courses creating mathematics courses to be taught in-house which will remove students from mathematics classrooms, or implications that mathematics classrooms should teach only mechanics and leave the bridging of the procedures and context (i.e. the applications) to the client discipline courses.

When asked to rank the tasks according to which task they felt best assessed the necessary calculus understanding with respect to their respective disciplines, the faculty participants implicitly revealed a preference for tasks with personally familiar manners (e.g., extreme open-endedness and use of creativity in the Toy Train task) and/or styles (e.g., the Theater task table versus the Hiking task graph). This familiarity has implications for instruction. If the faculty participants are not as familiar or comfortable with the manner or style of a task, they will be less likely to engage their students in such activities (Cooney, Badger, & Wilson, 1993). The faculty participants gave a variety of reasons for their prioritization: lack of familiarity, difficulty, appropriateness for a testing environment, etc.. Other issues raised included concerns over the time required to complete the task and the perceived need to assign a number or letter grade to the task. The implications of these reasonings and issues could be that it is not just the students that need to be challenged, stretched, and pushed outside their comfort zones, but the faculty participants need to be too.

As stated before, calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. This study hopes to offer a collective vision to focus the content of beginning calculus courses on the meeting the needs of client disciplines. However, in the end, it is the mathematicians that have the responsibility to create courses and curricula that embrace the spirit of this vision while maintaining the intellectual integrity of mathematics. By explicitly knowing what and how students should be prepared for client courses, teachers and curriculum developers of both calculus and client disciplines can work together to prepare students for academic success in any discipline.

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# DYNAMIC GEOMETRIC REPRESENTATION OF EIGENVECTORS

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*This study summarizes the five participants' exploration of dynamic representations of an eigenvector of a 2×2 matrix associated with a negative eigenvalue. I drew on the complementary use of the different theoretical constructs to study the role of an instrument of semiotic mediation in learners' developmental process of mathematical understanding. My data analysis reveals that the participants' use of the dragging tool (which became an instrument of semiotic mediation) enabled them to understand properties of eigenvectors and eigenvalues. It also suggests an integration of analysis of participants' gesture and speech to provide an insight of the participant's modes of thinking.*

*Keywords*: transformation of vectors, dynamic geometric representation, instrumental genesis and shifts of attention

## Introduction

This study summarizes the five participants' exploration of dynamic representations of an eigenvector of a 2×2 matrix associated with a negative eigenvalue. I chose to focus on this particular task to examine the participants' cognitive development in a digital technology environment. This is in line with my previous contributed research report (see the author, 2011) which I suggested combining the theory of instrumental genesis with the theory of shifts of attention to enable an analysis of learners' cognitive development.

## Theoretical Constructs

Mason (2008) sees attention and awareness as two aspects of human psyche in the developmental process of mathematical being. Awareness refers to what enables us to act, calling upon our conscious and unconscious powers, and the sensitivity to detect changes and to choose proper actions in certain situations (Gattegno, 1987; Mason, 2008). To educate awareness is to draw attention to actions which are being carried out with lesser or greater awareness. Attention can be drawn not only to mathematical objects, relationships and properties, but also to manifestations of mathematical themes, and to heuristic forms of mathematical thinking (Mason, 2008). According to Mason, the structure of attention comprises macro and micro levels; what is being attended to is as important as how it is being attended to. At the macro level, Mason describes the nature of attention as follows: "attention can vary in multiplicity, locus, focus and sharpness" (p.5). At the micro level, he distinguishes five different states of attending: holding wholes, discerning details, recognizing relationships, perceiving properties and reasoning on the basis of agreed properties. Mason argues that different states of attention can be triggered more prominently than others by different cues. The flexibility of shifts among various forms of attention is a factor that influences one's awareness. This suggests providing learners with opportunities where their attention can be drawn to identifying the invariants of a mathematical

concept which would enable them to perceive properties of the concept. For example, finding eigenvalues of a 2×2 matrix by solving the characteristic equation could draw a learner's attention to algebraic calculations, whereas the dynamic geometric representation of eigenvectors could enable the learner to perceive an eigenvalue as a scalar factor. The use of dynamic interactive representations in mathematics education has been shown to provide learners with such opportunities (Mariotti, 2000).

The theory of instrumental genesis (Verillon & Rabardel, 1995) draws on actions and procedures taken by a student to use a tool. The tool can be transformed into an internally oriented tool (instrument of semiotic mediation) by the process of internalization (Vygotsky, 1978) that occurs through semiotic processes. In mathematics education research, the study of the use of dragging tool from a cognitive perspective suggests that dragging can mediate the relationships between perceptual and conceptual entities. For example, Arzarello et al. (2002) write that "dragging [...] allows users to discover their [objects] invariant properties" (p. 66). They identify different dragging modalities such as guided dragging and line dragging. Guided dragging involves dragging an object in order to find a particular configuration. Line dragging is dragging along a line in order to keep the regularity of the discovered configuration.

As discussed above, Mason's theory of shifts of attention appears potentially fruitful in terms of revealing the developmental process of mathematical being. Mason highlights the importance of accumulated experience of the different forms of attention and also the flexibility of shifts among various forms of attention. It seems that providing learners with opportunities where their attention can be drawn to identifying the invariants of a mathematical concept would enable them to perceive properties of the concept. The developmental processes of a learner's understanding of a concept can be described through analyzing shifts in her attention, and in her use of visualization, imagery and embodied cognition. Given the important role of the dragging tool in the DGE-based activity, I decided to explore the complementary use of the theory of instrumental genesis and the theory of shifts of attention in analyzing data.

## Methodology of Study

Data was collected using one-on-one interviews with five students. The three participants (Mike, Kate and Jack) had completed a Linear Algebra course and studied eigenvectors, whereas the two (Rose and Tom) were taking it at the time of interview. Tom was introduced into the concept of eigenvectors, but Rose had not studied it at the time of interview. I videotaped the interviews and analyzed their dragging strategies and modalities, and their speech and gesture. In this paper, I presented my analysis of their dragging strategies and modalities to discuss the effect of instrument of semiotic mediation on shifts of attention.

The participants were given the *eigen* sketch (see Figure 1) and a worksheet. The worksheet included a formal definition of eigenvectors and eigenvalues and a task that required using the *eigen* sketch to find eigenvector(s) and associated eigenvalues(s) of the four given 2×2 matrices.

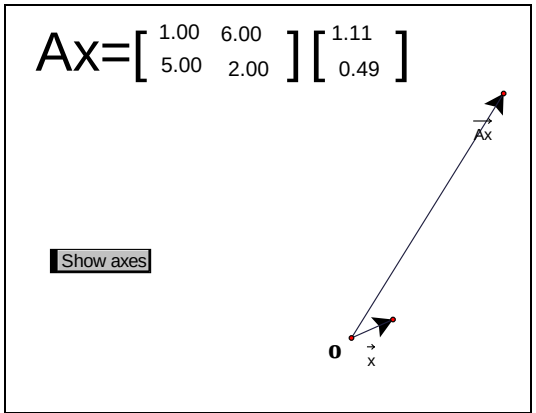


Figure 1. A snapshot of *eigen* sketch

Using *The Geometer's Sketchpad* (Jackiw, 1991), I designed the *eigen* sketch to enable exploration of eigenvectors and eigenvalues for given 2×2 matrices. As shown in Figure 1, the sketch includes a draggable vector *x*, as well as a non-draggable vector *Ax*. As, vector *x* is dragged about the screen the vector *Ax* moves accordingly. The sketch also includes numeric values of the matrix-vector multiplication (*Ax*). The user can change the values of matrix *A*. In this paper, I reported on the participants' interaction with the *eigen* sketch to find eigenvector(s) and associated eigenvalue(s) of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Given that this matrix was the third one on the task, they already developed some dragging strategies and modalities in finding eigenvector(s) and associated positive eigenvalue(s) of the two first matrices. They have all discovered that the geometric representation of eigenvectors associated with a positive eigenvalue can be represented by two collinear vectors having the same directions. They also approximated eigenvalues by finding the ratio of the lengths of *x* and *Ax*. Thus, given $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, they dragged *x*, found a position where *x* and *Ax* became collinear and then approximated the eigenvalue to be 7 by attending to the ratio of the two lengths. The matrix *A* also has a negative eigenvalue of λ= -4. The geometric representation of a set of eigenvectors associated with λ= -4 can be represented by two collinear vectors having opposite directions. In the dynamic geometric representation, the collinearity of vectors precedes the identification of eigenvalue(s). It also enables a learner to distinguish a positive eigenvalue from a negative one by drawing attention to the direction of collinear vectors. In contrast, the algebraic strategy of finding eigenvalues (i.e. finding roots of the characteristic equation of a square matrix) provides a learner with numeric values (positive or negative) that can be used to find the associated eigenvectors without being explicitly aware that she is identifying a scalar factor and special vectors that are transformed into a scalar multiple of themselves.

## Data analysis

Mike (a graduate student who had taken a linear algebra course during his bachelor's degree but did not recall the concept of eigenvector) used guided dragging to drag *x* from the first quadrant into the fourth quadrant. In doing so, he found a position where *x* and *Ax* were collinear but had opposite directions (*x* was in the fourth quadrant whereas *Ax* was in the second quadrant) and said "this is an interesting point". He then used line dragging to drag *x* along its path collinear with *Ax* in the fourth quadrant. Noticing that *x* and *Ax* preserved collinearity, Mike shifted his locus of attention to the definition, re-read it and said "now we want to *Ax* equal to, um, this would work. Because, in this case lambda would be negative 4". This suggests that the manner of Mike's attention was perceiving properties of eigenvectors as a special vector that lines up with its scalar multiples in the opposite or the same direction.

Jack (a third-year undergraduate student who completed a linear algebra course during his second year of study) used guided dragging to drag *x* in the second quadrant as he carefully attended to the position of *Ax* on the sketch. He found a position where *x* and *Ax* were collinear and said "it's a linear transformation um makes it I guess, oh yeah I guess it is still like the opposite eigenvector". He used his hands and arms to gesture *x* and *Ax*. He then shifted his locus of attention to the definition and then to the sketch. He used line dragging to drag *x* along its collinear path with *Ax* passing through the origin. He then said "oh yeah, yeah. I guess it would be -8" (the actual eigenvalue is -4). It seems that he used line dragging to verify the invariance property of the negative eigenvalue. This is evidence that the dragging tool was transformed into an instrument detecting geometric representation of eigenvectors associated with a negative eigenvalue. The manner of his attention, like Mike, was perceiving properties of eigenvectors.

Kate (a graduate student who had completed a linear algebra course during her bachelor's degree but did not recall the concept of eigenvector) used guided dragging to drag *x* from the first quadrant into the fourth quadrant and stopped dragging when *x* and *Ax* became collinear, and said "they're opposite to each other". This suggests that she shifted her attention to collinearity of opposite vectors. She then tried to approximate the lambda by attending to the ratio of the two vectors, thus she thought the eigenvalue has a positive value. It seems that her attention was blocked by considering the ratio of the lengths as lambda. I prompted her by asking "how is this [representation] different from the previous one?". She then said "oh I see the lambda should be negative four [...] yeah they have opposite directions so the lambda should be negative". This suggests that she reasoned on the basis of agreed properties of eigenvectors.

Tom (a second-year undergraduate student who was taking a linear algebra course at the time of the interview, and was introduced into the concept of eigenvectors) used guided dragging to drag *x* in an anti-clockwise circular direction from the first quadrant into the other quadrants and said "I guess it looks like [*x* and *Ax* are] tracing each other". He stopped dragging when *x* and *Ax* lined up (*x* was in the second quadrant and *Ax* in the fourth quadrant), stared at the screen for a few seconds and said "this one goes to the opposite direction". He then used line dragging to

drag *x* along the line where they were collinear, and said "eigenvalue is probably negative six" (the actual eigenvalue is -4). It seems that Tom attended to the direction of vectors, their ratio, and position on the sketch, therefore he immediately approximated the eigenvalue to be -6. His interaction suggests that the manner of his attention was involved in reasoning on the basis of the properties of eigenvectors (invariant collinearity) and eigenvalues (dilation factor).

Rose (a first-year student who was taking a linear algebra course at the time of interview and was not introduced into the concept of eigenvector) used guided dragging to drag *x* into the all four quadrants as she attended to the changes that occurred to *Ax*. She noticed that the geometric behaviour of *x* and *Ax* was different from the previous ones and said "usually they [*x* and *Ax*] go in the same direction but this [*Ax*] goes opposite direction". She then found a position where *x* and *Ax* were collinear and said "it's completely straight here". In approximating the eigenvalue, she attended to the ratio of vectors and thought that it would be 4. I prompted her saying that four is the ratio of the two lengths. She gazed at the screen, attended to the arithmetic representation of *x* on the sketch, used a numerical example of vector *x* and *Ax*, thus realized that the eigenvalue is -4. She then reflected on her findings and said that "the other one [eigenvalue] is positive, this one is negative. Those are located in these quadrants [points to the second and fourth quadrants]. You have to multiply by a negative so the vector would be opposite direction of what it [*x*] would be". Her reflection on the geometric representation of eigenvectors associated with positive or negative eigenvalues suggest that her attention was involved in perceiving geometric properties of positive and negative eigenvalues.

## Results

The participants all used guided dragging since they were looking to find a position where *x* and *Ax* would become collinear and have the same direction. This modality of dragging resulted from their experience finding eigenvector(s) and associated eigenvalue(s) of the first two matrices, which only had positive eigenvalues.

As they dragged *x*, they all found a position where *x* and *Ax* were collinear but had opposite directions. After finding the position, Mike, Jack and Tom used line dragging to drag *x* along the (invisible) line where *x* and *Ax* were collinear. This suggests that they were looking for the same *behaviour* as for the positive eignvectors: as they dragged vector *x* along the (invisible) line, *x* and *Ax* maintained collinearity but their length changed in a coordinated fashion and they had opposite directions. Mike, Jack and Tom attended to the direction of the two collinear vectors and their ratio, and approximated the eigenvalue to be negative. This suggests that they perceived properties of eigenvectors as a special vector that lines up with its scalar multiples in the opposite or the same direction. It seems that the dragging tool transformed into an instrument of semiotic mediation which caused shifts in the structure of their attention. Their attention shifted to (1) the direction of vectors as well as to the position of vectors on the sketch; (2) the collinearity of two opposite directed vectors; (3) the properties of eigenvectors as a special vector that is collinear with its scalar multiples in the opposite or the same directions; (4) the *behaviour* of vector *x* and

its transformation under matrix *A* that is not necessarily relevant in algebraic or static representations of the concept of eigenvectors.

The other participants, Kate and Rose, also attended to the direction of vectors and noticed that *x* and *Ax* had opposite directions, but in their first approximations of the eigenvalue they both thought that the eigenvalue was positive. Although this suggests that their attention was blocked for a short period of time by the ratio of two lengths, they re-drew their attention to the direction of the vectors and realized that the eigenvalue was in fact negative. After approximating the eigenvalue, they both shifted their attention to the geometric representation of eigenvectors. Rose compared the geometric representation of eigenvectors associated with positive eigenvalues to negative eigenvalues, and reflected on vectors' behaviour in different quadrants. Kate, like Rose, reflected on her findings and emphasized the negativity of the eigenvalue with reference to eigenvectors' position and direction.

In summary, my findings show that the five participants developed an understanding of the concepts of eigenvalue and eigenvector as it is evident from shifts in their structure of attention. Of the five participants, four used their hands and arms to communicate their mental imagery of the concepts during the interview. This makes me to suggest integrating speech and gesture analysis with my findings above to provide an insight of the participant's modes of thinking.

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# Students' Ways of Thinking about Enumerative Combinatorics Solution Sets: The Odometer Category

Contributed Research Report

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*This presentation aims to address students' ways of thinking about the set of elements being counted in enumerative combinatorics problems. Fourteen undergraduates with no formal experience with combinatorics participated in individual task-based interviews in spring 2011. Open coding was used to identify students' ways of thinking about the set of elements being counted, called the solution set. One category of ways of thinking which emerged from the data analysis involves holding an item constant and cycling through possible items for the remaining spots in order to generate all elements of the solution set. This category is known as Odometer thinking and two ways of thinking from this category, Standard Odometer and Wacky Odometer, are presented here. The conjectured Generalized Odometer way of thinking, which involves holding an array of items constant, is introduced as an extension of Wacky Odometer thinking. Its potential to coordinate set- and process-oriented thinking is discussed.*

Keywords: ways of thinking, enumerative combinatorics, counting, solution set

## Introduction and Research Questions

The purpose of this presentation is to discuss one category of students' ways of thinking about the set of elements being counted in enumerative combinatorics problems. According to Piaget and Inhelder (1975) children's combinatorial reasoning is a fundamental mathematical idea based in additive and multiplicative reasoning. Indeed, as Kavousian (2008) said "without much prior knowledge of mathematics, one can solve many creative, interesting, and challenging combinatorial problems" (p. 2). This indicates that students should be able to solve combinatorial problems by employing their additive and multiplicative reasoning. However, the research indicates that students often struggle to solve combinatorial problems (Batanero, Godino, & Navarro-Pelayo, 1997a, 1997b; Eizenberg & Zaslavsky, 2004; English, 1991, 1993; Hadar & Hadass, 1981; Lockwood, 2010; Smith, 2007). In particular, in a study conducted by Batanero et al. (1997b), the majority of students both with and without instruction struggled to give the correct answer. Furthermore, there is evidence that post-secondary students must navigate a variety of pitfalls on the road solving combinatorics problems (Hadar & Hadass, 1981).

In order to address these difficulties, some studies have investigated the student errors (Batanero, et al., 1997a, 1997b; Kavousian, 2008) and which formulae students use to respond to particular combinatorial problems (CadwalladerOlsker, Annin, & Engelke, 2011). Still, however, much of the prior research on combinatorics education has focused on students' actions, not their reasoning and understanding. It is widely accepted by mathematics educators that just because a student can *do* something, this does not mean that the students *understands*, or that the student is applying coherent reasoning. Thus, it is not enough to examine students' actions as they solve particular combinatorics problems – it is essential to understand their reasoning as well. Further, it will be foundational to understand the stable patterns in reasoning that students apply in a variety of combinatorial situations. These coherent patterns in reasoning are known as *ways of*

*thinking* (Harel, 2008). The research study described here aims to answer the following research questions:

1. What actions and reasoning do students reveal when solving each counting problem?
2. What are students' ways of thinking about the set of elements being counted in combinatorial problems?

Lockwood (2011) began to explore student's thinking as they counted. She identified two main perspectives of thinking about combinatorial problems: the *process-oriented perspective*, and the *set-oriented perspective*. In the process-oriented perspective, the act of counting amounts to completing a procedure which consists of individual stages. The student may or may not associate this procedure with a set of outcomes. In the set-oriented perspective, the act of counting amounts to determining the cardinality of the set of objects being counted, known as the *solution set*. Lockwood (2011) claims that being able to coordinate processes and sets is important because though thinking in steps or stages is a necessary part of counting, it is sometimes vital to link the process with a set of outcomes. Framed in this language, the second research question investigates students' ways of thinking about the solution set of combinatorial problems.

## Theoretical Framework

The philosophical perspective underlying this study is that "knowledge is not passively received either through the senses or by way of communication, but it is actively built by the cognizing subject"(Von Glasersfeld, 1995, p. 51). This idea that mathematical knowledge is constructed as the learner engages actively in the tasks is central to this research. Harel (2008) contends that there are two different categories of mathematical knowledge: ways of understanding and ways of thinking. Humans' reasoning "involves numerous *mental acts* such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving" (Harel, 2008, p. 3). *Ways of understanding* refers to the reasoning applied to a particular mathematical situation – the cognitive products of mental acts carried out by a person. *Ways of thinking*, then, refer to what governs one's ways of understanding – the cognitive characteristics of mental acts – and are always inferred from ways of understanding. Reasoning involved in ways of thinking does not apply to one particular situation, but to a multitude of situations. Ways of understanding and ways of thinking thus comprise mathematical knowledge (Harel, 2008).

## Research Methodology

Data for this study comes from a series of individual exploratory teaching interviews (Steffe & Thompson, 2000) conducted at a large southwestern university in the USA. Fourteen students from a Calculus with Analytical Geometry course participated in individual exploratory interviews with the researcher. Each student participated in 2 hour-long interviews with the researcher in a two week period in Spring 2011. The purpose of these interviews was to catalogue students' ways of thinking the relationship between elements of solution sets. Each interview involved the researcher as the teaching agent, one of the students, a series of tasks, and a method of audio-recording the interview.

The tasks were designed with the anticipation that the students might struggle with the problems and that this struggle will illuminate how the student is thinking about the problem. In

general, each task began with the presentation of a situation. The students were then asked questions about the situation so that the researcher might create a model of the situation as the student sees it. Following this, the actual question associated with the task was presented.

There were a few phases of retrospective analysis. Following each interview, the researcher took a few minutes to speak her initial thoughts about the students' ways of thinking aloud while using the a pen which records audio and links it to writing to record some notes regarding each interview. She did not meet with a person with an outside perspective following each interview, but did discuss some of the data with two mathematics education researchers during the study. Content logs including summaries of the video for each task were created for each student following each interview. Relevant portions of the video were transcribed as necessary. At the end of the study, the researcher used open coding (Strauss & Corbin, 1998) to identify and catalogue the ways of thinking in which each student engaged. Finally, the researcher returned to the original data – the audio and video-recorded sessions and the copies of the student work – to confirm her models of student thinking.

**Results**

Several different ways of thinking emerged from the data analysis. One category of ways of thinking was present when students were searching for a systematic manner of enumerating the elements of the solution set. Another involves creating a similar problem, determining the size of the solution set of the new problem, and relating this to the size of the solution set of the original problem. A third category, known as *Odometer* thinking, is discussed here.

In line with English (1993), the term *item* is used to refer to one of the objects involved in the counting process. For example, in a problem involving counting the number of permutations of {A,B,C,D}, A is an item. The term *element* is used to refer to elements of solution sets. In our example of permutations of the set {A,B,C,D}, ACBD is an element of the solution set. In tasks for this study, elements of the solution set can be thought of as having slots. Here, the terms *position* and *spot* refer to a slot. The item in the second position or spot in ACBD is C.

The motivation for the Odometer way of thinking came from English (1991), in which young children employed the odometer strategy to solve tasks involving dressing toy bears. An extension of this strategy is the Odometer ways of thinking where the main idea is to hold one item constant and systematically vary the other items to create all possible outcomes.

It is important to note that in the Odometer way of thinking students are able to anticipate that this idea of holding an item and systematically varying the other items will generate the set of all possible elements of the solution set. In addition, they must know how to systematically vary the other items. They may do so by recursively applying the same Odometer way of thinking, or by using some other system. Essentially, students engaging in the Odometer ways of thinking will have conceptually constructed a tree diagram (or table in the two-dimensional case) and can anticipate how the branches of the trees will be determined. It can be difficult to distinguish between whether students are using the Odometer strategy, as described by English (1991) or engaging in the Odometer way of thinking. It is only through probing the students' utterances and actions, that the researcher is able to determine if the students have simply stumbled upon a plan of action that is currently fruitful, or if the students are truly envisioning the tree diagram and anticipating how the branches might be determined. Two different versions

of Odometer thinking are discussed here. Student's solutions to combinatorial tasks driven by the different ways of thinking are presented.

*Standard Odometer*. In the Standard Odometer way of thinking, one would first hold an item constant in the first position and then systematically (and possibly recursively) vary the other items. Following this, the item in the first position is changed and the process repeats until all possible items for the first position are exhausted.

Consider a 3-digit odometer. First the odometer would hold numbers in hundreds and tens places constant and cycle through digits for units place, thus moving from 000 to 001, 002, and so forth until 009. Then, the digit in the tens place would increase to 1 and the odometer would again cycle through possible digits for the units place, to create 010, 011, through 019. Following this, the digit in the tens place would again increase and the process would repeat until exhaustion of items in the tens place. Thus, all numbers which can be created with a 0 in the hundreds place would have been created. Following this, the odometer would increase the digit in the hundreds place to 1 and repeat the entire process again: 100, 101, 102, …, 109, 110, …

In a similar manner, the Standard Odometer way of thinking can be applied to combinatorial situations. Ben was presented with the Security Code problem below:

- **Situation:** A security code for a computer involves two letters. It is case insensitive, but the two letters must be different from each other.
- **Question:** How many possible security codes are there for this computer?

In his solution, Ben anticipated that a security code of the sort "AA" or "BB" would not be allowed. He determined the answer to the question to be $26 \times 25$. His written work is shown in Figure 1. He explained:

> *"You have, they have to be different. So if you had the first letter A, it would have to go, you could have A and then B through Z for the next letter. So. And then the same, well the same kind of concept for the next letter was B, you could go A or C through Z…"*

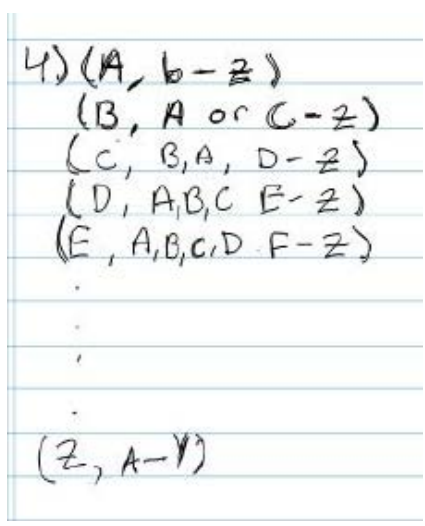


Figure 1: Ben's written work

Ben's explanation shows that he first held the "A" constant as the first letter in the security code. He then cycled through the possibilities for the second letter in the code. Next, he held the "B" constant as the first letter in the security code, and cycled through the possibilities for the second letter in the code. He anticipated that this structure would hold when the letters "C" – "Z" were held constant as the first letter in the code. He recognized that, for each option he held constant as the first letter in the code there were 25 possibilities for the second letter.

*Wacky Odometer*. In the Wacky Odometer way of thinking, an item is still being held constant. In contrast to the Standard way of thinking however, the item being held constant is not necessarily in the first position. Here, the student would hold one item, say *, constant in a given position and systematically vary items for the other positions. The position of * would then change and the process would repeat.

Jack engaged in the Wacky Odometer way of thinking when he was attempting to find the number of permutations of the items {A,B,C}:

> *"So when A is up front, there's two options [moves the cards to create these different permutations].If A is in the middle [moves the A card to the second position], there's two options. That's two – four. If A is in back [places the A card in the third position], there's two options. Six."*

Jack's explanation indicates that he chose the item A to hold constant in the first position. He then cycled through the possibilities for the items in the other positions, physically doing so in this case. He then changed the position of A, and cycled through the possibilities for the other positions, and repeated a third time. He anticipated that there would be 2 ways to position the remaining items when A was in the second and the third positions. Jack had trouble engaging the same way of thinking for permutations of 4 distinct objects, and instead reverted to engaging in the Standard Odometer way of thinking.

**Discussion**

The Odometer ways of thinking were quite prevalent in the research study. Students engaging in Odometer thinking generally tended to the Standard Odometer way of thinking. Jack's reluctance to engage in the Wacky Odometer way of thinking when the permutation problem became slightly more sophisticated supports this idea. However, it may be that the Standard Odometer can be extended to the Wacky Odometer way of thinking, which may, in turn, be extended to a more sophisticated way of thinking, known as the *Generalized Odometer*.

The Generalized Odometer way of thinking is not rooted in empirical data, but rather is one of the researcher's own ways of thinking about the solution set of many combinatorics problems. It is an extension of the Wacky Odometer way of thinking in the sense that though something is being held constant, it is not necessarily in the first position. However, in contrast to the Wacky Odometer way of thinking, an array of items is being held constant instead of just one item. Consider the following problem and solution driven by the Generalized Odometer way of thinking:

*Problem*: How many case-insensitive 8-letter passwords are there with exactly 5 E's?

*Solution*: First, we consider the number of ways to place 5 E's in 8 spots. There are $\binom{8}{5}$ ways to do so. Now consider one of these ways, say E _ E _ E _ E E. Because we can no longer use E's, we only have 25 other item possibilities for each position. Now we can use the Standard Odometer way of thinking (or another way of thinking) to determine the number of ways to fill the remaining positions ($25^3$). See Figure 2. Note that this was for one possible way of placing the E's. In fact, for *each* way of placing the E's, there are $25^3$ ways to fill the remaining positions. Therefore, there are $\binom{8}{5} \cdot 25^3$ total 8-letter passwords with exactly 5 E's.

In the above solution, the process of choosing where to place the E's and then placing the other letters gives structure to the tree diagram in Figure 2. Thus, the Generalized Odometer way of thinking provides a way to coordinate the process-oriented and the set-oriented perspectives about combinatorics problems identified by Lockwood (2011). Since the Wacky Odometer can be thought of as a precursor to Generalized Odometer thinking, it could be fruitful to encourage students to engage in Wacky Odometer thinking before supporting them in deepening that way of thinking to Generalized Odometer thinking.

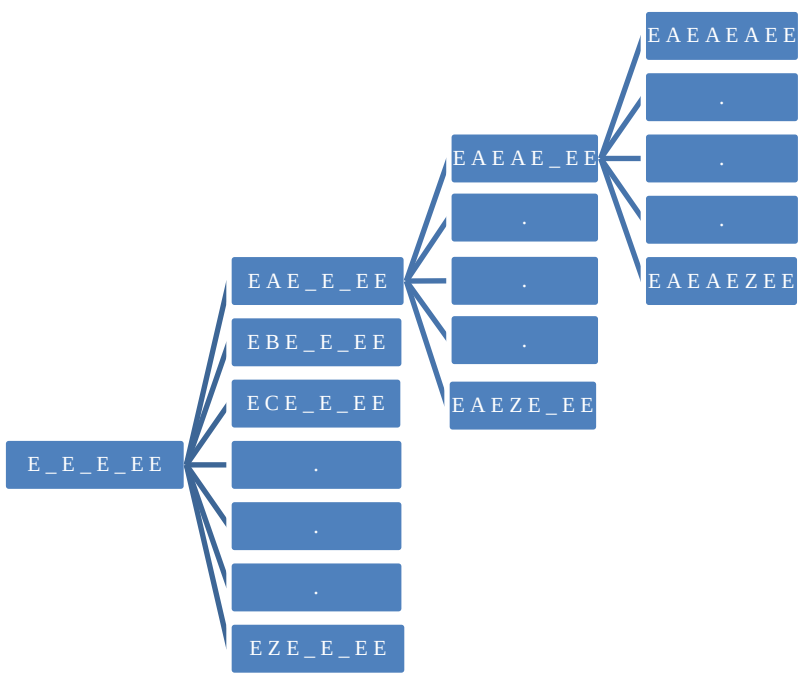


Figure 2: Partial Representation of Generalized Odometer Way of Thinking

## Conclusion

This study which focused on understanding students' ways of thinking about the set of elements being counted and how that thinking expresses itself in their attempts to solve combinatorial problems can be foundational for future studies and for teaching practice. It can serve to assist teachers in implementing instructional interventions designed to help students

develop productive ways of thinking about combinatorics and supporting curriculum developers in organizing tasks to build upon students' ways of thinking. In addition, this study could provide a framework for analyzing how the ways of thinking are distributed across various mathematical populations. This researcher hopes to conduct further studies to investigate how students develop their ways of thinking about the solution sets as they progress through a variety of combinatorial tasks and the instructor implements interventions designed to encourage particular ways of thinking, including Wacky and Generalized Odometer thinking.

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# On mathematicians' different standards when evaluating elementary proofs

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*Abstract.* Many mathematics educators and philosophers of mathematics believe there is an unusually high level of agreement in their evaluations of the validity of a mathematical proof. The data in this paper challenge this assumption. 108 research-active mathematicians were shown an adaptation of a published proof from elementary calculus and were asked to evaluate its validity. 24% of the participants judged the proof to be valid, while 76% indicated the proof was invalid. Applied mathematicians were more likely to judge the proof as valid than the pure mathematicians (43% vs. 17%). Participants who judged the proof to be invalid were more confident in their judgments. These findings suggest that different groups of mathematicians may use different standards in evaluating proofs.

Keywords: Mathematical practice; Proof; Validation

## Introduction

Many mathematics educators remark on the unusual level of agreement amongst mathematicians on whether or not an argument is a valid proof. For instance, Selden and Selden (2003) remarked on "the unusual degree of agreement about the correctness of arguments and the truth of theorems arising from the validation process" (p.7); they contended that validity is a function only of the argument and not of the reader: "Mathematicians say that an argument proves a theorem, not that it proves it for Smith and possibly not for Jones" (p. 11). McKnight, Magid, Murphy and McKnight (2000) asserted that "all agree that something is either a proof or it isn't and what makes it a proof is that every assertion in it is correct." (p.1). Harel and Sowder (2007) claimed that in their research paradigm, the goal of instruction is "to gradually refine students' proof schemes to those shared and practiced by contemporary mathematicians. This is based on the premise that such a shared proof scheme exists", implying that Harel and Sowder assumed a shared standards for proof evaluation. Such viewpoints are not limited to mathematicians. For instance, Azzouni (2004), a philosopher of mathematics, attempts to explain why, "mathematicians are so good at agreeing with one another on whether some proof convincingly establishes a theorem" (p. 84).

## Motivation for study

The work reported here builds on two psychological studies (Inglis & Alcock, submitted, Weber, 2008) that we conducted in response to Selden and Selden's (2003) research on the cognitive processes of undergraduate mathematics majors engaged in proof validation. In their study, Selden and Selden (both published research mathematicians) asked students to evaluate an argument they labeled "the real thing", which they judged to be a fully valid proof, and another they labeled "the gap", which they evaluated as invalid. In separate studies, Inglis and Alcock (submitted) and Weber (2008) found mathematicians did not uniformly agree with Selden and Selden's evaluations. Weber (2008) found that one mathematician (among eight) found the "real deal" proof to be invalid and Inglis and Alcock (submitted) found five of 12 mathematicians thought this proof was invalid. For "the gap", only five of the eight mathematicians in Weber's

study found the proof to be invalid; similarly, only seven of the 12 mathematicians in Inglis and Alcock's study judged this proof invalid. We were surprised that such short, simple proofs could lead to disagreement among mathematicians, but thought this was perhaps due to the awkward ways in which the proofs were written (they were variants of student-based proofs) and potential ambiguity about the student audience for the proof.

A third proof used in Inglis and Alcock's study purportedly established $\int x^{-1}\, dx = \ln(x) + C$ , presented in Figure 1. This proof used was more substantial than Selden and Selden's proofs, but nonetheless was situated in elementary calculus and written in a traditional format. Of the 12 mathematicians in their study, six judged the proof to be valid and six judged it to be invalid. However, as Inglis and Alcock's (submitted) and Weber's (2008) studies (as well as the Selden and Selden, 2003, study) were designed to investigate the processes of proof validation and not level of agreement, each of these studies employed sample sizes too small to make generalizations. To investigate whether these apparent differences in validity judgments were genuine, we asked a large number of research-active mathematicians to judge the validity of this proof.

**Theoretical perspective**

In order to frame both the design and results of our empirical study, we first discuss the ways in which individuals make judgments about validity. We argue that while mathematicians are apparently willing to make positive judgments that particular proofs are valid, these can more accurately be characterized as negative judgments that such proofs are not invalid. That is, a proof is valid if the validator was unable to detect a significantly serious flaw in the proof to render it invalid. Such a flaw might include a deduction that is not a logical consequence of previous assertions or it might be a serious gap, where it is not sufficiently clear that how a new assertion can be deduced from previous assertions. (We note that most proofs have gaps and these gaps are not necessarily problematic. Validators need to judge whether the gap is sufficiently significant to reject the proof as invalid).

For individual validators, the potential existence of errors and gaps has consequences in terms of the balance of confidence with which validity judgments can be made. If a gap or a problematic statement is located, the validator can be confident that the proof (as written) is not correct. If, however, no such gap or statement is found, the validator cannot with absolute confidence conclude that none exists: a problem might simply have eluded detection. This negative characterization of validity judgments leads, therefore, to the following two predictions about validator behavior: First, when validating a purported proof, those who regard it as invalid will be more confident in their judgment than those who regard it as valid (because they have found a problem, rather than merely having failed to find one). Second, it will be easier for validators to justify their response if they have rejected the proof as invalid rather than accepting it as valid (because those who rate it valid have nothing to say beyond that they have failed to find a problem).

**Methods**

*An internet-based study*

Given the general difficulty of obtaining large samples of research-active mathematicians, we maximized our sample size by collecting our data through the internet. We followed the strategies recommended by Reips (2000) and employed by Inglis and Mejia-Ramos (2009) to ensure the valid collection of data while conducting internet studies in mathematics education research.

*Participants*

111 participants agreed to participate in this study. They were recruited through an email sent via their departmental secretary. Those mathematicians who chose to take part in the study clicked a link contained in the email, which directed them to the study website. Of the 111 mathematicians, 108 of these mathematicians specified their broad field of research as either "pure mathematics" or "applied mathematics". The resulting analysis pertains to these 108 participants.

*Procedure*

Participants were first asked to provide demographic information about themselves, including their broad research area (applied mathematics, pure mathematics or statistics)[1]. Next participants were given the following instruction: "Below is a proof of the type that might be submitted to a recreational mathematics journal such as *The Mathematical Gazette*. Please read the proof and decide whether or not you think it is valid." Participants were then presented with the proof of the theorem $\int x^{-1}\, dx = \ln(x) + C$ , as given earlier. After reading the proof, participants were asked two questions: "Do you think the proof is valid or invalid?" and "How certain are you that your answer is correct?" They responded to the first by selecting either "valid" or "invalid", and to the second via a five point Likert scale (from "1 – It was a complete guess" to "5 – I am completely certain"). Finally, they were given the opportunity of explaining their answer via a free response text box.

Participants were then asked to estimate what percentage of mathematicians would agree with their judgment about the validity of the purported proof (they responded by selecting 0-20%, 20-50%, 50-70%, 70-90%, 90-99%, 100%). They were also asked to suggest reasons another mathematician might have for disagreeing with their judgment. The remaining pages were not used in the analysis.

**5. Results**

There was not uniform agreement in how the participants evaluated the proof, with 26 (24%) judging the proof as valid and 82 (76%) judging the proof as invalid. The applied mathematicians were more likely than the pure mathematicians to judge the proof as valid: 13 of the 30 (43%) applied mathematicians judged the proof as valid as opposed to 13 of the 78 (17%) of the pure mathematicians, a statistically reliable difference ($p<0.01$). (Note all tests are two-tailed Fisher exact tests).

Participants who judged the proof to be invalid had a higher level of confidence in their judgment (4.24 vs. 3.53). They also were more likely to give the confidence level in their evaluation a 5 out of 5, indicating they were "completely certain" in their judgment (48% vs. 11%, $p<0.01$) and leave a comment justifying their response (77% vs. 35%, $p<0.001$). These results confirm the predictions made in the theoretical perspective that judging a proof to be valid can be viewed as a failure to find cause to judge the proof invalid.

Only 41 of the 108 participants (38%) thought that 90-99% or 100% of the participants would agree with their judgment, indicating the majority believed there would be some disagreement (at least 10%) amongst mathematicians as to the validity of the proof. This suggests that while some mathematicians are convinced there is a high degree of agreement on mathematicians' judgments as to whether a proof is valid, many mathematicians do not appear to hold such a position.

Many participants who judged the proof invalid commented that a significant problem with the proof was that it commuted the limits and the integral ($\lim_{k\to -1} \int x^k dx = \int \lim_{k\to -1} x^k dx$). For instance, one participant wrote, "The reader needs to justify why he can evaluate the integral

[1] One participant indicated that he or she was a statistician and two did not indicate their broad research area. They were not included in the subsequent analysis. The analysis focuses on the remaining 108 participants.

of the limit by the limit of the integral" and another wrote "The limit of the integral is not necessarily the integral of the limit.". However, another participant who judged the proof to be valid was also aware of this limitation, writing, "There's a couple implicit steps that should be made more explicit (for example, passing the limit through the integral)". Other participants who judged the proof to be invalid thought some mathematicians might disagree with them because they would not find commuting the integral and limit to be a serious flaw. These quotes, and others, suggest that there is not agreement among the participants as to whether limits and integrals commuting are a serious enough flaw to render the proof invalid. Hence, it is likely that at least some of mathematicians' disagreement is not due to oversight or error, but to different standards as to what types of gaps or flaws are significant enough to render a proof invalid.

**6. Discussion and significance**

For mathematics educators, the results illustrate the complexity of the construct of validity. At a theoretical level, several theories of proof rely on the notion that what constitutes a proof to a mathematician is known and obvious. For instance, Stylianides (2007) specifies that a criterion for a school proof is that the methods of inference in the proof would be valid to a mathematician. Harel and Sowder's (2007) proof schemes sets as a goal for students to develop the shared standards of conviction and proof held by mathematicians. At a broad level, Stylianides, as well as Harel and Sowder's perspectives, are reasonable, as (nearly) all mathematicians would desire students recognize the limits of authoritative and empirical reasoning, and appreciate the logical necessity that a deductive argument can reveal. However, within the realm of deductive argumentation, it is not so obvious what methods of inference mathematicians find to be valid. Indeed, the data in this study suggest that this question might not have a unique answer as mathematicians vary on this, even in the domain as simple as elementary calculus.

As a practical concern for mathematics researchers, many researchers seek to determine if students can recognize a correct proof or a flawed argument (e.g., Weber, 2010; Selden & Selden, 2003). To do so, they present proofs that they claim are unambiguously valid or invalid. The results of this study suggest more care needs to be taken in this evaluation. In the future, it may be worthwhile for researchers to check the validity or non-validity of their research items with their mathematical colleagues, to be sure the validity of the proof is as clear as they believe it is.

For teachers of proof-oriented mathematics, this paper suggests that there is some measure of subjectivity in whether or not a proof is valid. Not only did the participants in this study disagree on validity, several remarked that they were not clear as to whether commuting the limit and the integral was a flaw significant to render a proof invalid. It is often believed that in mathematics, an answer is either correct or it is not. While this is arguably true for some calculations, the issue may be more complex with proofs and validity. The level of consistency in mathematicians' grading and instruction on what constitutes a proof is a useful avenue for future research.

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**Figure 1. Proof used in this study**

**Theorem.** $\int x^{-1}dx = \ln(x) + c.$

**Proof.** We know that $\int x^k dx = \frac{x^{k+1}}{k+1} + c$ for $k \neq -1$.

Rearranging the constant of integration gives $\int x^k dx = \frac{x^{k+1}-1}{k+1} + c'$ for $k \neq -1$.

Set $y = \frac{x^{k+1}-1}{k+1}$, and take the limit as $k \to -1$ as follows.

Let $m=k+1$, and rearrange $y = \frac{x^{k+1}-1}{k+1}$ to give $x^m = 1 + ym$ or $x = (1+ym)^{\frac{1}{m}}$.

Set $n = \frac{1}{m}$. Then $x = (1+ym)^{\frac{1}{m}} = \left(1 + \frac{y}{n}\right)^n \to e^y$ as $n \to \infty$, by properties of $e$.

As $n \to \infty$ we have $m \to 0$, so $k \to -1$.

In other words, $x \to e^y$ as $k \to -1$, so $y \to \ln(x)$ as $k \to -1$.

So $\int x^k dx = \frac{x^{k+1}-1}{k+1} + c' = y + c' \to \ln(x) + c'$ as $k \to -1$. So $\int x^{-1}dx = \ln(x) + c'$.

# Student Understanding of Symbols in Introductory Statistics Courses

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**Abstract**

This study explores student understanding of the symbolic representation system in statistics. Furthermore it attempted to describe the relation between student understanding of the symbolic system and statistical concepts that students develop as the result of an introductory undergraduate statistics course. The theory, drawn from the notion of *semantic function* that links representations and concepts seeks to expand the range of representations considered in exploring students' statistical proficiencies. Results suggest that students experience considerable difficulty in making correct associations between symbols and concepts; that they describe the relationship as seemingly arbitrary and that they are unlikely to understand statistics as quantities that can vary. Finally, this study describes students' need for robust knowledge of preliminary concepts in order to understand the construct of a sampling distribution.

**Keywords:** statistical symbols, symbolic representation, symbolic fluency, introductory statistical concepts

## 1. Research Questions

In the field of mathematics, significant importance was placed upon symbolic representations of communication, teaching and learning (Arcavi, 1994). In particular, students at introductory level statistics courses have been found to mix up the symbols for statistics and parameters (Mayen, Diaz & Batanero, 2009), which could hinder them from developing the concepts that such symbols represent. However, our literature search suggests that there have not been any studies published that explore students' understanding of the symbolic system of statistics. Therefore, we investigate the following questions:

- How do students perceive the symbols for mean and standard deviation as the result of a lecture-based introductory undergraduate statistics course? What levels of symbolic fluency do students develop after one such course? How does the level of symbolic fluency influence their understanding of sampling distribution?

The results suggest that students find the choices of symbols arbitrary and difficult to associate with related concepts, and that students need particularly strong conceptual and symbolic understandings in order to make sense of the standard deviation of a sampling distribution. We also found that student understanding of the relation of *statistics* to *parameters* was not satisfactory, and they did not consistently view statistics as variables.

## 2. Literature Review

Onto-semiotic research proposes that "representations cannot be understood on their own. An equation or specific formula, a particular graph in a Cartesian system, only acquires meaning as part of a larger system with established meanings and conventions" (Font, Godino, &

D'Amore, 2007, p. 6). The implication is that the system of practices is complex in that each one of the different object/representation pairs provides, without segregating the pairs, a subset of the set of practices that are considered to be the meaning of the object (Font, Godino, & D'Amore). Within the realm of statistics, even when the object under consideration seems relatively simple, such as the mean, there are often multiple symbolic representations used interchangeably. For example, $\bar{x} = \sum x_i / n$ may be used without consideration of any other type of representation: graphical, verbal, etc. The relationships between object and representations become even more complex when moving toward a more complex idea, such as the standard deviation of a sample mean. Due to a layering of representations, it is conceivable that the different possible pairs of object/representation convey different meanings of the same object.

The onto-semiotic approach requires a discussion of the role of communities of practice in order to describe the representation system, since representations (symbols) are only ascribed meaning by those who work with them. Eco (1976) gave the term ***semiotic function*** to describe the dependence between a text and its components and between the components themselves. The semiotic function relates ***the antecedent*** (that which is being signified) and ***the consequent sign*** (that which symbolizes the antecedent) (Noth, 1995). The members of the statistical community and their representation system define a complex web of semiotic functions. It is important to note that these functions "… the role of representation is not totally undertaken by language (oral, written, gestures…)" (Font, Godino, & D'Amore, 2007, p. 4).

For example, when learning the standard error of a sample mean, students are confronted with the simple looking formula: $\sigma_{\bar{x}} = \sigma / \sqrt{n}$. This formula has a seemingly simple explanation: "*the population standard deviation* of *the sample means* is given by the population standard deviation divided by the positive square root of the sample size." In this case, the representation $\sigma_{\bar{x}}$ draws on the agreed-upon symbols for *the population standard deviation* and *the sample mean* to communicate the meaning "the population standard deviation of the sample means." However, it does not give information about how to determine the value. Moreover, the symbol $\sigma_{\bar{x}}$ requires students to be able to make sense of a mixture of previously separate representational systems: those that represent statistics derived from a sample (for example, $\bar{x}$) and those that represent parameters derived from a population (for example, $\sigma$). When given the representation on the right-hand side of the equation, $\sigma_{\bar{x}} = \sigma / \sqrt{n}$, students read a formula that implies they should perform a calculation by mixing the pieces of symbols up from separate representational systems. Most importantly, the right and left-hand symbols could be interpreted as different meanings of the standard error of the sample mean. So, students are potentially confronted with various possible representations of the same object as described above.

A representation is "something that can be put in place of something different to itself and on the other hand, it has an instrumental value: it permits specific practices to be carried out that, with another type of representation, would not be possible" (Font, Godino, & D'Amore, 2007, p.7). In this case, the standard error of a sampling distribution, the object $\sigma_{\bar{x}}$, can be understood as a necessary concept (Hewitt, 1999) that emerged from a system of practices. It should be considered unique, with a holistic meaning that is agreed upon by the community of practice; however, the concept is expressed by a number of different semiotic functions. Each of these object/representation pairs should be understood as encapsulating a different possible set of meanings and enabling different practices.

This study is in line with the tradition of onto-semiotic research in mathematics education (Font, Godino, & D'Amore, 2007). It is situated in the context of statistics education, and

designed to explore undergraduate students' understanding of the symbolic system of statistics at the conclusion of an introductory statistics course. It will attempt to describe the collection of semiotic functions in this community, focusing on those involving symbolic representations for the concept of the sampling distribution that students have constructed. It will also give a preliminary explanation of why they have constructed this particular set of semiotic functions by describing their understandings of the general representational system.

## 3. Rationale

A literature search suggests that although there have been investigations of students' understanding of measures of center (Mayen, Diaz, & Batanero, 2009; Watier, Lamontagne, & Chartier, 2011), variation (Peters, 2011; Watson, 2009; Zieffler & Garfield, 2009), and even students' preconceptions of the terms related to statistics (Kaplan, Fisher, & Rogness, 2009), no one has yet explored student understanding of the symbolic system of statistics. One paper did draw upon the onto-semiotic tradition to describe student errors related to representations of the mean and median (Mayen, Diaz & Batanero, 2009).

Hewitt (1999, 2001a, 2001b) distinguished those aspects of a concept that can only be learned by being told and then memorizing, which he labeled *arbitrary*, from those that can be learned or understood through exploration and practice, which he labeled *necessary*. This distinction between symbols of the mathematical system suggests the importance of symbolic representations in building conceptual understanding and procedural fluency. Hewitt noted that names, symbols and other aspects of a representation system were culturally agreed-on convention. Although symbols may seem sensible once an individual has an understanding of the culture, "names and labels can feel arbitrary for students…there does not appear to be any reason why something has to be called that particular name"—after all, as he argues, "there is no reason why something has to be given a particular name" (1999, p. 3). Hewitt pointed out that for students to communicate with experts, they must memorize the arbitrary elements and correctly associate them with appropriate understandings of the necessary elements.

Recently, Shaughnessy called for research into "students' conceptions of the interrelationships of the aspects of a distribution" (2007, p. 999). But he focused only on the special place of graphs as a tool in statistical thinking, and did not acknowledge the importance of the representational system in which graphs are situated. The research on students' conceptual understanding of statistical concepts has avoided discussion of the importance of representation; yet, onto-semiotic research claims that descriptions of conceptual understanding are incomplete when pursued only via one or two possible representations of a concept. This study contributes to the growing body of research on student understanding of statistical concepts by describing students' symbolic fluency and the ways they link concepts and symbols.

## 4. The History of Changes in Statistics Courses

In the past 25 years, there has been significant change in the structure of the introductory statistics curriculum; where there once was a focus on learning probability, theory, and formulae, there now is a data-driven approach to content via descriptive statistics, basic probability, and inferential statistics (Garfield & Ben-Zvi, 2008). The focus of reform, especially because of recent technological advances, has been to emphasize statistical thinking. This includes using data, understanding the importance of data production, and appreciating the presence of variability (Garfield & Ben-Zvi). The statistics education community has also adopted the view that the course should "rely much less on lecturing, much more on alternatives such as projects, lab exercises, and group problem solving and discussion activities" (Garfield & Ben-Zvi, p. 12).

One way to define the difference between traditional, lecture-based statistics classes and inquiry or reform-oriented classrooms is to compare how much of the responsibility for mastering cognitive processes is given to the students.

While the curricular organization of the courses in this study conformed to those typically found in a reform-oriented classroom, the instruction itself was essentially traditional. The instructors had almost total responsibility for daily classroom activities and the content was delivered primarily via lecture.

## 5. Methods

Data for this study was drawn from nine participants in a mid-sized public university in New England. Two of the participants were in a lower level introductory statistics class and seven were from an upper level class. The lower level class was designed to allow first-year students to meet the general education requirement of the university, and thus is non-calculus based. The upper level class was designed to serve mathematics majors, and thus is calculus-based. The two courses occurred in the same semester.

The data collection process was conducted in two steps: a survey assessment and a follow-up interview. For the survey, we developed a fourteen-item assessment. Some of these items were based on *Assessment Resource Tools for Improving Statistical Thinking*. The research team created the rest of the items. The assessment items sought to evaluate student understanding of what the symbols represented and their conceptual understanding primarily via their symbolic representations.

The goal of the interview process was to identify how students' understanding of symbol representations and their level of symbolic fluency potentially impacted their understanding of certain symbol-oriented concepts. The interview of the two participants from the lower level class was conducted a few days after the survey; the interview of the participants from the upper level class was conducted immediately after the survey. Both the survey and the interview were analyzed qualitatively. The nine students ranged from low achieving to high achieving based on their work on the survey.

All interviews were audio-recorded and transcribed. For coding, each utterance was assessed to examine the information it gave about symbolic understandings. Then, within each transcript, we categorized and summarized the utterances that deemed informative understandings by the type of concepts and connections it described with their symbolic understanding. We read within and across categories to develop conclusions. We continually rechecked our conclusions against the data that described the students' proficiencies.

## 6. Results

The following results were obtained by analyzing the interviews with the first eight students.

**1. *Students find the choice of symbols seemingly arbitrary and difficult to associate with related concepts***. According to onto-semiotic research, holding various connections that a concept has with its various expressions is essential in internalizing the concept. One of the connections is associated with the symbol that represents the concept. In introductory statistics courses, the concepts of descriptive statistics and their associated symbols are introduced early in the semester/term and many new ideas in inferential statistics are built on them. Thus, if students do not know the concepts and the associated symbols of descriptive statistics, they will be hindered in acquiring new concepts about inferential statistics. Items on the survey were designed to assess if students were able to discern the symbols for statistics from the symbols for

parameters. While students' responses on the assessment instrument were 72% correct, they consistently reported in the interview that they struggled to understand the difference between statistics and parameters and to distinguish between the symbols. Consider, for example, Michael's claims:

> I know μ, I just always associate μ with the mean. I wasn't really sure, I don't remember if it was in the population, if it was the mean of the population or the sample, so I just kind of guessed on that one. And, for $\bar{x}$, I think I've learned that is also the mean…

He continued, "So, μ would be, like, all the data, and then, sorted, from smallest to largest, and then divided by how many were in the sample… And then, $\bar{x}$ is, I think $\bar{x}$ is the same, it's just not sorted by smallest to largest. I'm not really sure." Based on his performance on other items, it appears that Michael knows how to calculate the mean and understands what it implies mathematically. But these are only part of a complete understanding the concept of *mean*. Another aspect of understanding the mean is the ability to pair it with the distinction between sample and population, which Michael was not able to do. Instead, he attributed an incorrect difference of meanings to the two symbols for mean. While he may be able to correctly answer questions that require calculating the mean, the lack of connection may prevent him from acquiring symbolic fluency.

2. ***Students need particularly strong conceptual and symbolic understandings in order to make sense of the standard deviation of a sampling distribution***. The concept of the standard deviation of a sampling distribution was determined to be one of the most difficult concepts for students in our survey. For example, Ian said, "I feel like we just didn't get any of the foundational stuff." When asked to describe what a particular symbol represents, such as $\sigma/\sqrt{n}$, Ian said, "This is the population standard deviation." He continued, "(*s* is) the standard deviation of our sample. I think we used *s* in class. I'm not sure. But we used another thing to separate, just like this, our mean in our sample. And so I thought that was what it was." That is, he was willing to call $\sigma/\sqrt{n}$ the standard deviation of a sample even though the class had used *s* as the symbol for the sample standard deviation . This implies that he was so unsure in his knowledge that he was willing to believe that a different symbol could be substituted for *s* and still mean the same thing. Moreover, Ian's responses to the questions were initially definitive; only after further questioning did he admit having any insecurity of his knowledge. Even then, he did not express concern about mixed understandings or possible misattribution of meaning to symbols. One reason this is happening is that students are trained to distinguish statistics from parameters through in-class learning. Once students establish the distinction, they habitually try to discern statistics from parameters; yet their work shows that they admit to struggle in doing this. It should be noted that the expression $\sigma/\sqrt{n}$ has a great potential to confuse new learners because the symbol $\sigma$ represents a *population* standard deviation, but the process of dividing by radical *n* is associated with a *sample*. Students can be easily confused as to what $\sigma/\sqrt{n}$ is associated with because they are trained to distinguish samples from population in order to be able to distinguish statistics from parameters.

**3. *Students had difficulty viewing statistics as a variable.*** One of the items was designed to find out if students were able to view statistics as variables and parameters as fixed constants. This skill is an essential aspect of understanding the relationship between statistics and parameters and lays the groundwork for understanding the sampling distribution. We found that many students had difficulty holding this view. For example, Michael said, "I think a statistic is a calculated value, and a parameter is a, like a, it would be like a boundary that satisfies a value. S, so, I think S would be a, I think S would be a parameter, because sigma is the statistic. It's

[measured estimating?].” Also, Brian said, “because it (*s*) is representative of standard deviation. I guess that varies, but—”, and Ian said, “I didn’t understand that at all. I didn't know what we were looking at as, what was changing and what wasn’t changing.” These statements show that the students’ understanding of statistics as a variable was minimal or nonexistent.

## 7. Discussion

Students must know the symbols used by statisticians and associate them with their accepted statistical meanings. Similarly, they must be able to distinguish statistics and parameters and to view statistics as variables when they are embedded in a certain context. The results show that these skills are difficult for students to learn. Although we need further research on this subject, our results suggests that the lack of symbolic fluency and the inability to view statistics as variables hinder students from understanding the concept of sampling distribution and thus the broader conceptual domain of inferential statistics. Without improved practices and more instructional focus, students are likely to continue to create incorrect semiotic links and experience great difficulty in developing conceptual understanding.

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# Role of Faculty Professional Development in Improving Undergraduate Mathematics Education: The Case of IBL Workshops

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Abstract

Professional development opportunities might provide the tools that faculty need to transition from lecture to the research-based student-centered instructional methods. With data from two offerings of faculty professional development workshop aimed at educating faculty in the use of Inquiry Based Learning (IBL) instructional techniques, we examine what changes in the instructional practices can—or cannot—be accomplished through the means of professional development workshops. Faculty participants report strong immediate benefits in the form of increased knowledge, skills, and motivation to use student-centered instructional methods. The long term outcomes are also very encouraging. The vast majority of respondents to the follow-up survey report that they had implemented IBL techniques to some degree and about half of these achieved full implementation. This self-reported implementation rate is even more encouraging, given considerable changes in practice indicated by respondents on the follow-up survey and corroborated in the interviews. The shifts in practice indicate that workshops have an impact not only on participants' learning, but also on their application of methods learned.

*Key words*: research-based teaching, inquiry-based learning, faculty change, faculty development

## Conceptual Framework

While education research shows that students learn best through active participation and engagement with the material, most undergraduate mathematics instructors still use lecture as their primary means of instruction. Disseminating teaching techniques that foster students' active engagement and educating mathematics faculty in their use could provide a research-based alternative and thus become a vital part of improving undergraduate mathematics education. Faculty professional development could play an important role in such reform-oriented dissemination.

We have examined two offerings of a five-day faculty professional development workshop aimed at educating faculty in the use of Inquiry Based Learning (IBL). IBL is a student-centered, active learning approach to undergraduate mathematics. Rather than emphasizing rote memorization and computation skills, IBL approaches seek to help students develop critical thought processes by analyzing ill-defined problems, constructing and evaluating arguments (Dewey, 1938; Bruner, 1961; Prince & Felder, 2007; Savin-Baden & Major, 2004). Such activities also support deep learning of mathematical ideas (Moon, 2004; McCann, et al., 2005). By building students' confidence in their abilities to generate and critique ideas independently, IBL methods can foster creativity and persistence (Zimmerman & Schunk, 2001).

In order to learn to teach this way, faculty need considerable outside support. This support must come in several forms: being exposed to methods of course selection and script construction, learning the mechanics of running an IBL classroom, and creating a network of connections with colleagues and places where it has already been established. All these needs could be fulfilled by professional development opportunities for faculty interested in the method.

While professional development of K-12 teachers has been the subject of many studies, much less literature exists on the professional development of undergraduate faculty (Sell, 1998). Their higher degree of autonomy and greater content mastery, as well the division of work time between teaching and research (Fairweather, 2008), make college faculty quite different from their K-12 colleagues. Although research has begun to identify some of the internal and external barriers to faculty change (e.g. Walczyk, Ramsay & Zha, 2007; Henderson, 2005; Henderson & Dancy, 2007; Dancy & Henderson, 2010; Austin, 2011), there is yet much to be learned about these issues, and especially, how to help faculty overcome them.

The literature on faculty professional development identifies some key features of successful workshops, especially informal, interactive approaches that model the teaching approaches being shared, and time for participants to work on their own applications (Foertsch et al., 1997; Penberthy & Connolly, 2000; Marder et al., 2001; Burke, Greenbowe & Gelder, 2004). Time to work on their own applications is especially important, since most faculty prefer to adapt or even re-invent the research-based instructional strategies they learned in their implementation (Henderson & Dancy, 2008).

Despite these insights, it is still unclear how much impact professional development workshops make on faculty instructional practices. Thus, our study examines the research questions:

> What is the role of professional development for mathematics undergraduate faculty in propagating student-centered teaching practices?
>
> What changes in faculty instructional practices can—or cannot—be accomplished through the means of professional development workshops?

We use five levels of impact of instructional development programs to organize our findings: participation, satisfaction, learning, application, and overall impact (Colbeck, 2003).

Study Sites and Research Methods

The study sites were two universities with IBL Mathematics Centers where an extensive menu of IBL courses had been developed and taught over several years. A cadre of faculty with expertise on IBL methods and experience in teaching specific courses was thus available to lead workshop. Funded by a grant from the National Science Foundation, the universities had developed and implemented IBL workshops for two cohorts of math faculty new to IBL, in summers 2010 and 2011. Both workshops spanned five days, with invited talks, open discussions, and hands-on exercises the most common activities. The 2010 was conducted in highly interactive fashion, while the 2011 workshop was somewhat more conference-like.

As evaluators for the workshop project, our team conducted pre- and post- workshop surveys at each workshop. Both surveys included both quantitative items and open-ended questions. Likert-scale items were developed to reflect participants' knowledge, skills, and beliefs about inquiry teaching, as well as their motivation to use inquiry methods and their perceptions of the overall quality of the workshop. By assessing these items both before participants attended the workshop and afterwards, we could identify significant changes in their knowledge and perceptions. Open-ended questions addressed the costs and benefits of using inquiry strategies, participants' impressions and learning from the workshop, and how they may use that learning in their own instructional activities. Participants also reported personal and professional demographic information such as career stage, institution type, gender, and race/ethnicity. We also conducted a follow-up survey with the first (2010) cohort of workshop participants, collecting data on the

extent to which they had implemented IBL teaching methods in the first academic year following their workshop attendance, and on the obstacles they faced in their implementation. Interviews (now underway) with a subset of implementers will provide detailed data to enrich our understanding of implementation successes and challenges.

Results

1. Participation

Workshop participants (N=103) came from diverse institutional backgrounds and represented a variety of career stages. The largest portion taught at four-year colleges (43%), followed by PhD-granting research universities (35%), masters-granting comprehensive universities (21%), and two-year colleges (3%). Most workshop participants (71%) were tenured or untenured faculty in tenure-track positions. They had varied degrees of teaching experience. Most of them (60%) had less than five years of teaching experience; 12% had 6-10 years of experience, 13% had 11-20 years, and 16% had taught for more than 20 years. Overall, the workshops included a high proportion of early-career participants with a healthy mix of more senior faculty.

The group was mostly male (52%), although the percentage of women (48%) was higher than among math faculty as a whole (NSF, 2006). Most participants were of European descent (74%). As a group they were slightly more diverse than mathematics faculty as a whole.

The workshop participants constituted a particular subset of mathematics faculty who were interested in active engagement instruction and motivated to improve their teaching strongly enough to attend five-day intensive workshop. Yet they primarily used traditional, lecture-based teaching practices. The pre-workshop surveys showed that half of participants lectured every class, and 47% solved problems on the board every class. While some workshop participants did report using a variety of student-centered teaching approaches, many did not: almost half of instructors never used student-led whole class discussions, 26% never used small group discussions, and 25% never used student presentations. Thus, workshop participants' initial teaching practices were primarily aligned with traditional mathematics instruction.

2. Satisfaction

Workshop attendees rated the overall quality of the workshop quite highly. The majority of participants (60%) rated the workshops as "excellent" and another 36% rated them as "good" compared to other professional development workshops they had attended. Only 4% of attendees rated the workshops as 'fair or average', and none rated them 'below average' or 'poor'.

3. Learning

Mathematics faculty who participated in the two workshops made some impressive immediate learning gains as a result of the workshop. Both pre- and post- surveys included items asking participants to rate their current knowledge about inquiry, their skills in inquiry-based teaching, their belief in effectiveness of IBL as an instructional tool, and their motivation to use IBL in their own teaching practice. The pre-to-post changes on all these indicators were positive and statistically significant.

Table 1: Immediate Workshop Outcomes

| | | none | a little | some | a lot | mean rating | pre/post change sig |
|---|---|---|---|---|---|---|---|
| knowledge about inquiry | pre | 12% | 44% | 35% | 9% | 2.40 | |
| | post | 0% | 11% | 59% | 30% | 3.19 | p<0.005 |
| skill in inquiry-based teaching | pre | 30% | 41% | 27% | 2% | 2.03 | |
| | post | 10% | 39% | 48% | 4% | 2.45 | p<0.005 |
| | | don't know | not very effective | somewhat effective | highly effective | mean rating | pre/post change sig |
| belief in effectiveness of IBL | pre | 26% | 0% | 35% | 40% | 2.88 | |
| | post | 1% | 0% | 18% | 81% | 3.78 | p<0.005 |
| | | not at all | a little bit | somewhat motivated | highly motivated | mean rating | pre/post change sig |
| motivation to use IBL | pre | 1% | 5% | 28% | 65% | 3.58 | |
| | post | 0% | 0% | 12% | 88% | 3.88 | p<0.005 |

*Knowledge about Inquiry*

On the pre-survey, the largest group of participants indicated knowing 'a little' about IBL, followed by 'some', and only 9% indicated knowing 'a lot.' However, in the post-survey, most participants indicated knowing 'some', and about a third stated they knew 'a lot.' The pre-survey mean of 2.40 (on a 4-point scale) rose to a post-survey mean of 3.19.

*Skill in inquiry-based teaching*

On the pre-survey, the largest group of participants indicated having 'a little' skill, followed by 'none', and then 'some.' Only 2% of participants reported having 'a lot' of skill in inquiry. On the other hand, on the post-survey almost half of participants indicated having 'some' IBL skill and 4% stated that they knew 'a lot.' The pre-survey mean of 2.03 (on a 4 point scale) rose to a post-survey mean of 2.45.

Participants' gains in the inquiry-based teaching skills are lower than their gain in knowledge about IBL. This is not surprising, since at the time of the post-survey, attendees had not had a chance to practice the newly-learned techniques.

*Belief in effectiveness of inquiry strategies*

Participants entered the workshop with already strong beliefs in the effectiveness of IBL: three quarters reported believing IBL is 'somewhat' or 'highly' effective. Such beliefs are not surprising among faculty who chose to attend a five-day IBL workshop, indicating a belief in the usefulness of IBL. Nevertheless, attendees left the workshop even more persuaded: 99% reported believing IBL is 'somewhat' or highly' effective.

*Motivation to use IBL*

Participants were already highly motivated going into the workshop, with almost two thirds of

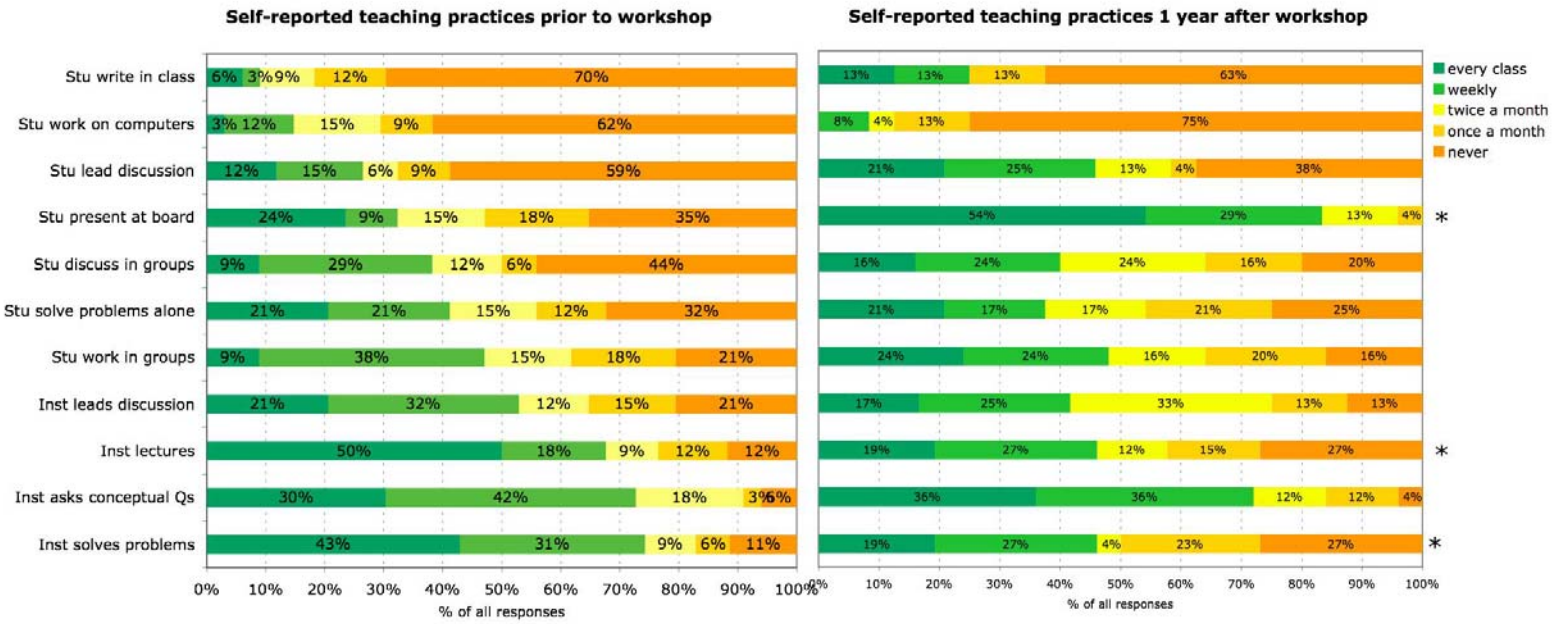
respondents identifying themselves as ‘highly motivated’ to incorporate inquiry into their courses. However, even more participants were ‘highly motivated’ after the workshop. In fact, all workshop participants who answered the question were “highly” or “somewhat” motivated to use inquiry in their educational practices. The pre-survey mean of 3.58 (on a 4 point scale) rose to a post-survey mean of 3.88.

4. Application

While participants reported some impressive immediate gains, the long-term outcomes of the workshops are also encouraging. On the follow-up survey, the 2010 cohort of workshop participants reported on their implementation of IBL within the first academic year following their workshop attendance. Only 13% of the respondents did not implement at all. The largest group of respondents (45%) indicated that they had not implemented a full IBL course, but had applied some IBL approaches to their teaching. Moreover, 42% of respondents reported implementing one (23%) or more (19%) IBL courses. The results may be affected by response bias, as only 83% of the 2010 workshop participants responded to the follow-up survey, and implementers are more likely to respond. If all non-respondents are assumed to have implemented nothing, this yields a conservative estimate that 37% of participants had applied some IBL approaches to their teaching and at least 35% fully implemented IBL in their classrooms. Overall, these numbers point to partial but pervasive implementation of IBL methods by the 2010 workshop cohort.

This self-reported implementation rate is even more encouraging, given the changes in practice indicated by respondents on the follow-up survey and corroborated in the interviews. Figure 1 summarizes the teaching practices reported by the 2010 workshop participants on the pre-workshop survey and on the follow-up survey a year after the workshop. It illustrates some considerable shifts in teaching practice.

Figure 1: Change in Teaching Practices Reported by the 2010 Workshop Cohort



As discussed above and as shown on the left side of Figure 1, the initial teaching practices reported by this cohort of workshop participants were fairly in line with traditional mathematics teaching. On the follow-up survey, some of their reported practices have shifted toward more student-centered instructional activities. Almost half (46%) of those who answered this question reported incorporating student-led whole class discussion every class or weekly, compared to 27% on the pre-survey. Moreover, 83% of follow-up respondents reported using student presentations every class or weekly, and no respondents indicated never having this activity in class. This is a drastic change from the pre-survey, where only 33% reported having students present every class or weekly, and 35% reported never using this activity. The shift in mean rating for the frequency of student presentation is statistically significant.

Moreover, the follow-up survey indicates that many faculty moved away from instructor-centered instructional activities. The proportion of respondents who reported lecturing every class or weekly went from 68% on the pre-survey down to 46% on the follow-up. Similarly, the proportion reporting that the instructor solved problems on the board every class or weekly dropped from 74% on the pre-survey to 46% on the follow-up. The drops in mean frequency for instructor lecture and instructor solving problems are statistically significant.

5. Overall impact

While we did not directly collect any student outcome data, in the follow-up interviews we asked instructors about the differences they observed in their students' learning outcomes. Faculty reported positive changes in students' understanding of the material, their involvement in the class, and their independent and critical thinking,

Implications

First, it is encouraging that our survey instruments are fairly sensitive to changes in respondents' attitudes, skills, and practices, even where changes are slight and participants entered the workshop already rating themselves high on certain indicators (such as motivation to implement IBL). The changes detected here make sense in the context of an intensive, one-time professional development workshop. For example, gains in IBL knowledge are higher than in IBL skills, consistent with the workshops' emphasis on presentation and discussion of IBL approaches but absence of opportunities to practice these skills. Similarly, changes in some instructional activities but not the others indicate that our instrument is fine-tuned enough to detect a change in faculty practice. This seems to indicate a sort of content validity of the instruments.

The results provide an interesting answer to our research questions. IBL professional development workshops seem to provide a strong immediate benefit to faculty in the form of increased knowledge, skills, and motivation to use student-centered instructional methods. In the long term, the results are also very encouraging. The majority of respondents to the follow-up survey reported that they had implemented IBL techniques to some degree and about half of these achieved full implementation. We were able to 'verify' respondents' level of implementation, by comparing their self-categorization (full or partial implementer) to the instructional practices they reported in the follow-up survey. Their overall shifts in practice indicate that workshops have an impact not only on participants' learning, but also on their application of methods learned. Although student outcomes are harder to verify given the scope of this study, the instructor interviews also suggest that implementation of the IBL techniques presented at the workshop has a positive impact on the students' learning outcomes.

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# Outcomes of Inquiry-Based Learning for Pre-Service Teachers: A Multi-site Study

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## The Research Problem

Children's mathematical learning depends on their teachers. To be effective in the classroom, teachers need a variety of types of knowledge: not only deep mathematical knowledge of the content they will be teaching, but knowledge of how to teach that specific content—as well as general pedagogical knowledge for engaging students and managing a classroom. Moreover, teachers' confidence in their mathematical abilities—or lack of it— can have a powerful effect on students, as early as the first grade (Beilock et al., 2010). While some of these abilities are developed over the course of a career, the formal preparation of future teachers is also essential.

We have examined university mathematics content courses that are based in math departments, targeted to future teachers, and taught with inquiry-based approaches. Inquiry-based learning (IBL) is a student-centered, active learning approach to undergraduate mathematics. Rather than emphasizing rote memorization and computation skills, IBL approaches seek to help students develop critical thought processes by analyzing ill-defined problems, constructing and evaluating arguments (Dewey, 1938; Bruner, 1961; Prince & Felder, 2007; Savin-Baden & Major, 2004). Such activities also support deep learning of mathematical ideas (Moon, 2004; McCann, et al., 2005). By building students' confidence in their abilities to generate and critique ideas independently, IBL methods can foster creativity and persistence (Zimmerman & Schunk, 2001).

Inquiry learning is especially important for pre-service teachers, because understanding why and how certain mathematical rules work is critical in teaching mathematics (CBMS, 2001). Pre-service education should rekindle prospective teachers' own powers of mathematical thinking "with classroom experiences in which *their* ideas for solving problems are elicited and taken seriously, their sound reasoning affirmed, and their missteps challenged in ways that help them make sense of their errors" (Conference Board, p. 17). Thus our study examines the question:

> How do IBL learning experiences affect (or fail to affect) the knowledge, attitudes, beliefs, and confidence of undergraduate pre-service teachers, especially in relation to their future teaching careers?

## Prior Work on Inquiry-Based Learning in Teacher Preparation

There is a broad research base on prospective teachers' difficulties with mathematics, but surprisingly few studies have evaluated courses or programs to remedy this problem. Of nineteen outcomes-oriented studies identified by Hough (2010) in a commissioned literature review (see Authors, 2011), only three offer data comparing outcomes from inquiry-based or active learning courses to those from more traditional formats. While the evidence is scanty, the existing studies do suggest that IBL experiences are beneficial to future teachers.

Studies of prospective teachers' *subject matter knowledge* generally find that IBL approaches help prospective teachers to make gains in specific subject matter knowledge (e.g., proportional reasoning, arithmetic operations). The single comparative study found that students who took a problem-based course outperformed the traditionally taught group on a researcher-developed test of Cantorian set theory (Narli & Baser, 2008).

Among studies of students' understandings of the *nature and processes of mathematics*, authors report generally positive outcomes. For example, students learn to participate in mathematical discourse and develop more sophisticated conceptions of creativity in mathematics. In the only comparative study, Yoo and Christian Smith (2007) surveyed students who took inquiry or lecture-based courses on proof. The inquiry students developed a significantly more humanistic and process-oriented view of proof than did their peers in the lecture-based course.

Several studies offer data related to changes in prospective teachers' *beliefs, attitudes and efficacy*. Authors report some gains in confidence and modest shifts in students' beliefs about how mathematics should be taught and learned. In a single comparative study, students who took an active-learning algebra course focused on learning and analyzing alternative algorithms held more positive attitudes about mathematics (Mathews & Seaman, 2007).

Conceptual Framework

In considering how IBL experiences as undergraduates shape future teachers' ability to teach mathematics effectively, we use the notion of "mathematical knowledge for teaching" (MKT) (Ball, Hill & Bass, 2005; Hill, Ball & Schilling, 2008; Shulman, 1986). MKT is that special amalgamation of content and pedagogical knowledge a good teacher draws upon to make instructional decisions. This includes deep understanding of how particular ideas and concepts connect to one another and how children develop understanding of these ideas; children's prior knowledge or misconceptions; and the representations and strategies that children create and teachers can build upon to foster understanding. Hill, Ball and colleagues have devised a measure of MKT, the *Learning Mathematics for Teaching* (LMT) instrument, that shows a positive relationship between practicing teachers' MKT and their own students' mathematics achievement (Hill & Ball, 2004; Hill, Rowan & Ball, 2005). Thus the concept of MKT appears useful in explaining teachers' abilities to teach mathematics effectively.

Study Sites, Study Samples, and Research Methods

The study sites were two universities with IBL Mathematics Centers supported by the Educational Advancement Foundation. Each had developed and implemented IBL courses for future K-12 teachers at both elementary and secondary levels. Though developed independently, the courses emphasized many common pedagogical elements. Students solved difficult problems alone or in groups, shared solutions, and critiqued each others' work. Their ideas and explanations drove progress through a sequence of problems that led students in small steps to big ideas. The instructors' role was to select problems, manage classroom dynamics, and shape discussion at key moments, as "guide on the side" instead of "sage on the stage" (King, 1993). Small class sizes (20-30) facilitated ample interaction among students and with the instructor.

The study samples included students who completed one of three two-term course sequences taught at the two Centers and targeted to pre-service teachers: one for elementary grades (K-6), one for elementary/middle school (K-8) and one for secondary grades (6-12). About 85% of the students were women; nearly all the men were in the secondary course. About 80% of the students were White. Nearly all were juniors and seniors: about 40% were math majors (most in the secondary and K-8 courses), while most of the rest majored in humanities or social science fields. Because none of the courses was also offered in a non-IBL format, we could not compare outcomes of IBL vs. non-IBL instruction for pre-service teachers, but we did have a non-IBL sample of courses for math-track (math major) courses.

Multiple measures were used to explore students' growth in knowledge, attitudes, beliefs, and confidence. Pre/post measures were given at the start and end of the two-term sequence.

1) the *Learning Mathematics for Teaching* (LMT) instrument (Hill & Ball, 2004). Test items are based on an extensive development process (Hill, Schilling & Ball, 2004). The LMT is a suite of instruments on different subjects; we used the elementary *Number Concepts and Operations* test. The pre-test and post-test items are paired but not identical. Standardized item response theory (IRT) scores are based on a scale provided by the developers. We also collected demographic data on students' gender, class year, prior math background, and career plans. A total of 109 students took both pre- and post-tests.

2) pre- and post-surveys of students' learning gains and mathematics-related attitudes. Learning gains items were based on the Student Assessment of their Learning Gains (Seymour et al., 2000; Weston et al., 2006) and the attitudinal scales (addressing goals for studying mathematics, confidence, approaches to problem-solving, etc.) were based on a review of theoretical and empirical literature (Authors, 2011). The surveys also included demographic items. Multiple-choice and open-ended numerical and text items were subjected to a variety of statistical and qualitative analyses using SPSS and Excel software. The sample comprised 220 post-surveys and 184 matched pre/post survey pairs.

3) Interviews with 24 students who were taking (or had recently completed) each of the three courses, and with 7 instructors and teaching assistants of these courses. The interview protocols examined students' learning gains (as reported by both students and their instructors) and the teaching and learning processes that took place in the courses. Detailed coding was done on verbatim transcriptions of interviews using *N'Vivo 8* software.

Results

First, the LMT instrument was successfully used to measure growth in MKT among pre-service teachers. Though developed to assess professional learning of in-service teachers, we found the items suitable for pre-service elementary and secondary teachers. We know of no such use of this test previously.

Second, the LMT results indicate that students' MKT increased during the course. The three groups started at different initial levels of knowledge for teaching elementary number concepts and operations. On average, Group 3 outperformed Group 1 on the pre-test ($p<0.05$), likely due to the higher number of prior college math courses they had taken ($p<0.001$). But all three groups made statistically significant score increases from pre- to post-test (Table 1). Students with a stronger math background improved their score more than students with less previous college math experience. The increases showed effect sizes in the range (above 0.70) that is generally considered to be large (Cohen, 1988).

**Table 1: Average Changes in Scores for a 24-item LMT Test, by Course Group**

| Student Group | Pre/post Change in Average Score (# of correct answers) | Significance Level | Effect Size for Pre/Post Change |
|---|---|---|---|
| Group 1 (*N=27*) | +2 | $p<0.01$ | 0.67 |
| Group 2 (*N=64*) | +2 | $p<0.001$ | 0.79 |
| Group 3 (*N=18*) | +3 | $p<0.01$ | 0.90 |

Moreover, the extent of learning did not depend on the level of teaching certificate students were pursuing. Since teachers' increased ability to understand and communicate key mathematical ideas (as measured by the LMT) has a positive effect on their actual classroom instruction (Hill, Schilling & Ball, 2004), these test score gains imply that these pre-service teachers are being prepared well for their future classroom duties.

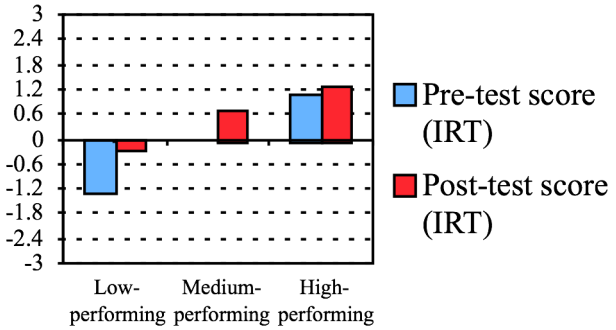
The correlation between initial test score and test score gain was negative ($r= -0.362$, $p<0.01$), meaning that students with initially lower scores had higher test score gains than students who performed better at the start of the course. Figure 1 shows the average change in standardized test score for three groups of students, divided by initial test score. Clearly, initially low- ($p< 0.01$) and medium-scoring ($p< 0.05$) students improved more than did high-scoring students.

**Figure 1: Changes in the Standardized IRT Test Score, by Performance Group**

low-performing: pre-test score ≤ 50% correct (N=22)

medium: pre-test score 51-74% (N=60)

high: pre-test score ≥ 75% (N=27)




On the learning gains survey, pre-service teachers reported strong gains in important domains such as applying mathematics to other domains, confidence, collaboration, and comfort in teaching. Similar to the pattern seen for LMT scores, low-achieving students (based on prior GPA) reported the highest gains. This pattern contrasted with the pattern for math majors taking lecture-based courses (not targeted to pre-service teaching), where already-strong students reported greater gains than did initially weaker students. That is, in traditional math courses, the "rich got richer"—but these and other data (Authors, 2011) suggest that IBL approaches provided a particular boost to students whose mathematics achievement is initially lower.

Results from the attitudinal component were mixed. Compared to students in non-teaching-related courses for math majors, IBL pre-service teachers started with weaker motivation to study mathematics, although they had higher interest in teaching. The effects of IBL courses on their beliefs, motivation, and strategies were mixed. Changes generally supportive of mathematics learning included (with effect size, pre to post):

- Less emphasis on extrinsic learning goals ($d=0.15$) and
- Higher preference for group work ($d=0.18$) and less for instructor-driven instruction($d=0.18$).

Weakened emphasis on extrinsic goals and instructor-driven activities suggest that students were developing more mature approaches to learning mathematics. As IBL pre-service teachers' goals for communicating about mathematics and use of collaboration were already strong, these changed little. But after an IBL course, students also reported some less positive changes. They:

- less often used self-regulatory strategies in learning ($d=0.36$),

- preserved but did not gain confidence in their own math ability, and
- were less willing to take additional math courses or study hard for mathematics (d=0.41).

The decline in students' use of important self-regulatory strategies for learning mathematics is disappointing. However, the lack of change in confidence may be a more positive outcome than it appears: in (non-teaching) courses using lecture approaches, we found that confidence generally declined, especially for women. The decline in students' willingness to take additional math courses likely reflects the fact that they did not need to study more mathematics: completion of the targeted, two-course IBL sequence satisfied their degree requirements.

The gains reported on numerical items are corroborated by open-ended comments: over 40% of pre-service teachers wrote in comments identifying at least one learning gain, and 20% reported three or more gains. In interviews and write-in comments on surveys, students described:

- cognitive gains: learning and understanding math concepts, thinking and problem-solving;
- changes in their learning, especially how they learned math and solved problems, and in learning from others;
- affective gains, such as confidence and enjoyment; and
- communication skills.

Pre-service teachers described how these gains would transfer to their future teaching, helping them "break things down for kids" and developing a habit of curiosity and skepticism that they saw as beneficial to their future students. Their ideas about how to teach math shifted, as one student commented:

> [This class] definitely twisted my mind about how to teach math…. It's less structured in the sense of, like, processes and procedures. It's definitely thinking outside of the box, working with others—and, like, why things are the way they are, rather than, 'They are the way they are. Just accept it.' So this class allows us to question a lot of things: Why does this algorithm work? And we don't know; we were just taught in schools about it. I think, after taking this class, if we were placed in junior high schools and high schools, I would want to teach them this way…. I think students learn way more doing it this way than how we were taught it.

Students reported certain classroom practices as particularly helpful to their learning. Compared to traditional courses, tests were less important as drivers of learning. Rather, students benefited from a steady pace of everyday work, high peer interaction and their own active participation.

Interviews further reveal that the "twin pillars" of learning in IBL courses were deep engagement with mathematics and peer-to-peer collaboration. Deep engagement fostered deep understanding and relied on students' motivation and effort, while peer collaboration made IBL classes enjoyable, fostered confidence, and required communication that developed skills and deepened understanding. Because students felt accountable to their peers, they invested significant effort outside class in solving problems and preparing for class. In turn, discussing the problems in class clarified ideas: "Once you spend some time alone with it, then talk to other people—that *really* helps solidify it." Their prior individual work on group or class problems fostered interest in others' insights and appreciation for approaches that differed from their own. As one student put it, "IBL opened my eyes to [the notion] that math isn't necessarily something that's one way

only, which is something that I've always been taught. …Math doesn't have to be this looming, scary subject for students—us or elementary students."

Implications for Teaching and Research

Triangulating across multiple data sources yields a rich picture of MKT development among prospective teachers as well as highlighting areas for future study. Inquiry-based learning can deepen future teachers' mathematical knowledge for teaching, as well as broaden their notions about how math can be taught. This study suggests that inquiry-based approaches to courses for future teachers are productive—but studies that compare IBL to more traditional approaches across multiple courses and institutions are still needed.

Acknowledgments

We thank Anne-Barrie Hunter for her contributions to the analysis of the student interview data, and Marina Kogan for helpful discussions. The work was supported by the Educational Advancement Foundation.

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# A Model of Students' Combinatorial Thinking

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*Combinatorial topics are prevalent in undergraduate curricula, and research indicates that students face difficulties when solving counting problems. The literature has not sufficiently addressed students' ways of thinking about combinatorial concepts at a level that enables researchers to understand how students conceptualize counting problems. In this talk, a model of students' combinatorial thinking is presented that emphasizes relationships between formulas/expressions, counting processes, and sets of outcomes. The model serves as a conceptual analysis of students' thinking and activity related to counting; it sheds light on relevant aspects of students' combinatorial thinking, and it provides language to describe and explain aspects of students' counting activity. In this way, the model has practical implications, both for researchers (providing a lens through which to examine data on combinatorics education) and for teachers (providing an aid to instructional design based on student thinking).*

*Key words*: combinatorics, counting, grounded theory, model, discrete mathematics

## Introduction and Motivation

The importance of combinatorics in K-12 and undergraduate curricula is well-established in the mathematics education literature (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; English, 1991; NCTM, 2000), both for its rich potential as a problem solving context, as well as for its applications in probability and computer science. As such, knowledge and pedagogy related to combinatorics is of great import. One aspect of combinatorics, counting, is among our earliest intellectual processes. However, as students advance mathematically, they tend to experience considerable difficulties with complex counting problems (e.g., Batanero, et al., 1997; Kavousian, 2006; Martin, 2001). In spite of efforts to improve the implementation of combinatorial topics in the classroom (e.g., Kenney & Hirsch, 1991; NCTM, 2000), students continue to struggle with understanding such concepts.

In order to help students succeed combinatorially and to help assuage the difficulties students face, I maintain that researchers need a better understanding of how students *think about* counting. That is, a fundamental aspect of helping students overcome the difficulty of solving combinatorial problems is to understand students' conceptualizations of such activity. To this point, literature on combinatorics has not addressed such ways of thinking at a level that enables researchers to identify, describe, or explain how students conceptualize counting problems.

To this end, I conducted research on post-secondary students and attempted to learn more about their ways of thinking about counting problems. Thompson (2008) points out that conceptual analyses can be used in part "to generate models of knowing that help us think about how others might know particular ideas" (p. 57). In this paper, I present such a model of students' combinatorial thinking[1]. The model represents a conceptual analysis of students' thinking and activity related to combinatorial enumeration (counting); it has been refined and elaborated through analysis of student data.

[1] By student thinking, I mean my interpretation of students' thinking via their observable mathematical activity and statements.

## Data Collection and Analysis

In order to present the model and to contextualize subsequent discussion of the model, I briefly describe the study from which it emerged. I interviewed twenty-two post-secondary students in 60-90 minute individual, videotaped interviews as they solved five combinatorial tasks. In these interviews, students were encouraged to explain their reasoning as they first solved all five problems on their own, giving their best initial attempt at an answer. Then, I revisited the problems with the students and gave them alternative expressions that I asked them to evaluate. The purpose was to have students make sense of the alternative expression and to make some judgment as to the correctness of their original answer as compared to the alternative expression. During the interviews, my intent was not to instruct the students or to measure learning, but rather to examine their activity and thinking.

Because not much is known about students' ways of thinking about counting, the methodological framework of grounded theory (Auerbach & Silverstein, 2003; Strauss & Corbin, 1998) was adopted for the study. At the core of grounded theory is the premise that researchers may study phenomena about which no previously existing theory exists. According to this perspective, raw data is carefully analyzed, relevant phenomena and themes from the data are identified and organized, and theory emerges as the end product of such work. My analysis consisted of transcribing the interviews, searching the transcripts and videos for episodes that highlighted particular phenomena, labeling and structuring the phenomena, and ultimately developing theory out of the analysis process.

## Results and Findings

In this proposal I focus on one outcome of the study described above – a model of students' combinatorial thinking. I take a model as a framework for identifying, describing, and explaining phenomena related to a particular topic – in this case, combinatorial thinking. The purpose of this model is to shed light on relevant elements of students' counting and to provide language by which to describe and explain aspects of such counting activity, with the end goal of ultimately highlighting ways in which students might think about combinatorial ideas.

(Insert Figure 1)

I begin by explaining each of the components of the model (see Figure 1): Formulas/Expressions, Counting Processes, and Sets of Outcomes. By *Formulas/Expressions*, I mean mathematical expressions that yield some numerical value. The formula could have some inherent combinatorial meaning (such as a binomial coefficient *C(8,3)*), or it could be some combination of numerical operations (such as a sum of products $9\times13+3\times12$). Two expressions may be mathematically equivalent, but I consider them to be different if they differ in form. By *Counting Processes*, I mean the enumeration process (or series of processes) in which a counter engages (either mentally or physically) as they solve a counting problem. These processes consist of the steps or procedures the counter does, or imagines doing, in order to complete a combinatorial task. As examples, the implementation of a case breakdown or successive applications of the multiplication principle represent counting processes that a counter might enact. By *Sets of Outcomes*, I mean those sets of elements that one can imagine being generated or enumerated by a counting process. In the context of a counting problem, this may be the set whose cardinality represents the answer to that counting problem, but sets of outcomes could also refer to any set that can be associated with a counting process (even if that set is not the answer to the counting problem at hand). For example, in a counting problem that asks for the number of 10-letter sequences (repetition allowed) that contain exactly two consecutive A's, the

desirable set of outcomes is all such 10-letter sequences that satisfy the constraint; those sequences could be conceived of as a set of outcomes. Additionally, that set could be considered in light of another set – the set of all possible 10-letter sequences.

For a given counting problem, a student may work with one or more of these components and may explicitly or implicitly coordinate them. The model also includes key relationships between these components. While the relationships are bidirectional, for the sake of brevity I do not explicitly discuss each direction separately. Instead, I briefly describe the relationships and provide an example from the data for one of the relationships.

## Key Relationships Between Components of the Model

### *Counting Processes and Formulas/Expressions*

In the context of a counting problem, a given mathematical expression can often naturally be associated with a counting process. As an example, we may consider the expression *C(5,2)*C(5,3)*. This product of binomial coefficients can represent a number of different things. From one perspective, it is a just number; we could calculate the product to arrive at 100. However, in the context of counting, this same product typically signifies a particular process. Specifically, it is an instance of the multiplication principle in which a typical element that is being counted is constructed in two stages. In the first stage, two objects are chosen from five distinct objects, and in the second three objects are chosen from five distinct objects; the multiplication indicates that the two stages are performed independently. We can further specify a context, such as a problem about answering five of ten questions on a test; in the context of such a problem, the expression may represent choosing two of the first five questions and then choosing three of the second five questions. Regardless of the specific context, counters can attribute combinatorial meaning to a mathematical expression in the form of a counting process.

We could also conceptualize a counting process that generates an appropriate formula. If we wanted to count the number of ways of arranging give objects from a set of ten distinct objects, there is a counting process that would allow us to do that, which could be expressed through a formula. We could consider the number of options for which objects could go in the respective positions, and using the multiplication principle we could arrive at an answer of $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$. There are thus formulas and mathematical expressions that can be generated by a particular counting process. There may be more than one counting process associated with a single formula or expression, and there may be more than one formula associated with a given counting process (see Lockwood, 2011 for more discussion). The appropriate counting process would depend on a given person's way of thinking about the problem.

### *Sets of Outcomes and Formulas/Expressions*

In the diagram of the model above (Figure 1), the arrow representing this relationship is dotted because I conjecture that this relationship is less clearly linked than the other two. I suspect that for some particularly experienced counters, there may be certain sets of outcomes that could be directly connected to certain formulas or expressions without having to consider a counting process. A possible example of this is an expression for a binomial coefficient, *C(n,k)*. While there is an underlying counting process that it represents (choosing a subset of *k* objects from a set of *n* distinct objects), for some counters it may be an expression with encapsulated set-theoretic meaning. Specifically, it can be seen as the set of all possible *k*-element subsets whose elements come from some larger *n*-element set.

*Counting Processes and Sets of Outcomes*

A counting process can be seen as generating some set of outcomes, and in fact different counting processes can result in different structures of the set of outcomes. Additionally, a counter could also consider the set of outcomes first, make a decision about how to organize that set of outcomes, and that decision could influence the counting process he or she employs. The point to make about this relationship is that counting can be seen as more than simply manipulating particular procedures, strategies, or formulas. Rather, solving a counting problem can essentially consist of determining the cardinality of the set of outcomes, and though it is not often utilized, this relationship is a fundamental aspect of counting.

To demonstrate the relationship between counting processes and sets of outcomes, I provide an example of a student's work on the Passwords[2] problem. Peter initially arrived at the correct answer

$$26^8 - \left[\binom{8}{2}\cdot 25^6 + \binom{8}{1}\cdot 25^7 + \binom{8}{0}\cdot 25^8\right],$$

and his reasoning is found in the excerpt below. Peter went on to draw the diagram in Figure 2 as well, which reflects the strategy he described in his language below.

P: I want to know how many contain at least 3 E's … just because the counting is easier, I'm going to probably turn that around, and say I want to know how many contain 2 or 1 or 0 E's, and then subtract that from the total.

Peter's language and his diagram below (Figure 2) reflect what might be called a *total minus bad* approach, because he subtracted the "bad" outcomes from the total number of outcomes. In the excerpt above, I interpret that Peter implicitly utilized the notion of sets of outcomes by organizing the set of all 8-letter passwords into two parts – those that contain three or more E's and those that contain fewer than three E's. Peter made a decision about how he would organize the set of outcomes, and that organization led him to implement a particular counting process, ultimately yielding a correct expression.

(Insert Figure 2)

## Conclusion

As discussed above, domain-specific models of the ways in which students think about or approach counting problems do not currently exist. The model presented in this paper offers a first attempt at addressing what kinds of concepts might be underlying students' combinatorial thinking, and in doing so it addresses a gap in the mathematics education research on the area of combinatorics. In addition to the overall potential of the model, I suggest that the model is innovative in emphasizing the significance and role of sets of outcomes. Data from the study discussed in this paper (described in detail in Lockwood, 2011) suggest that utilizing sets of outcomes can be particularly fruitful, and thus the model's emphasis on this aspect of counting is something that could be used effectively by mathematics education researchers. Paying attention to sets of outcomes as a vital aspect of combinatorics/counting has the potential to be significant and useful for researchers and teachers.

[2] The Passwords problem states, "A password consists of 8 upper case-letters. How many such 8-letter passwords contain at least 3 E's"

Researchers could use the components of the model (and the relationships between the components) as a lens through which to describe and analyze students' counting activity. By highlighting relevant phenomena related to students' combinatorial thinking (and by facilitating the common articulation of such phenomena), the model may assist researchers in developing their understanding of students' conceptualizations of combinatorial ideas. Additionally, by getting a better sense of what aspects of counting students think about, understand, and struggle with, researchers may be more poised to conduct experiments to facilitate the improvement of teaching and learning related to combinatorics. While my study examined undergraduate students, I suspect that the components of the model may extend to K-12 student populations as well, and the model could serve as a tool for researchers at any level of investigation related to combinatorics education. In sum, the model elaborated in this paper is meant to put forth an initial attempt at characterizing students' combinatorial thinking, providing ideas and common language that researchers can utilize in evaluating their own students' thinking and activity. While the model can certainly be further developed and investigated, by presenting the model I hope to offer the mathematics education community a starting point for the deeper investigation of students' combinatorial thinking.

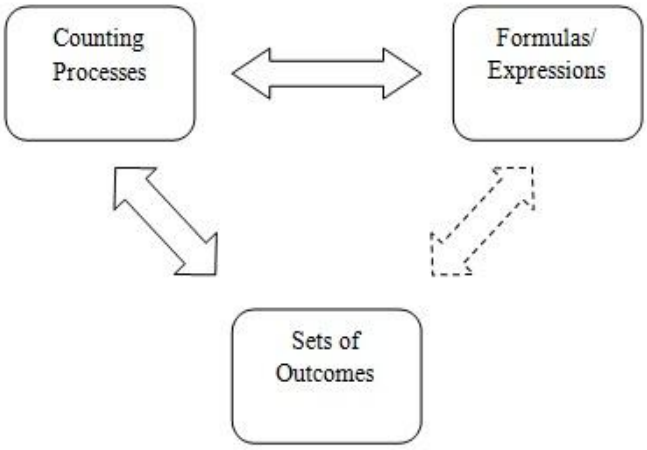


Figure 1

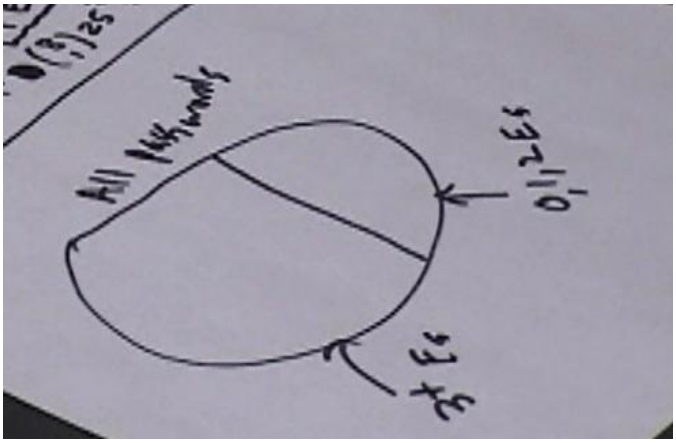

Figure 2

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# CHALLENGING CONVENTION: MATHEMATICS STUDENTS' RESISTANCE TO THE UNCONVENTIONAL

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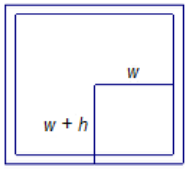
*This article explores undergraduate mathematics students' responses to tasks that deal with areas, perimeters, volumes and derivatives, and their abilities to transfer appropriate knowledge to a novel and unconventional situation. Specifically we focus on an unconventional use of parameters in familiar formulas and investigate how students react to such a change and whether they proceed to implement a similar one. Our analysis attends to the specific mathematical ideas and connections transferred by our participants. We suggest that considering and accepting the unconventional is part and parcel to an appreciation of the overarching structure of mathematics.*

Key Words: convention; transfer; derivative-relationship

This paper presents part of a broader study which examined the influences of prior experience and aesthetic sensibilities on university mathematics students' appreciation of mathematical structure. We focus here on participants' responses to tasks that deal with areas, perimeters, volumes and derivatives, and their abilities to transfer appropriate knowledge to a novel and unconventional situation.

Student's ideas of derivative have been noted in prior research to be conceptually disconnected (Przenioslo, 2004), to lack relational understanding (Lither, 2003), and to be influenced by an instructor's priorities regarding different aspects of derivative, such as rate of change or tangent line aspects (Bingolbali & Monaghan, 2008). In our study we focus on an unconventional use of parameters in familiar formulas and investigate how students react to such a change and whether they proceed to implement a similar one. The importance of considering unconventional representations was emphasized by Zazkis (2008), who illustrated that challenging conventions can aide in developing richer mental schemas.

We were interested in exploring students' understanding of the relationship between the derivative of the area of a shape and its perimeter (or circumference) – namely that $\frac{dA}{dx} = P$ provided an appropriate parameter $x$ is chosen. Zazkis, Leikin and Sinitsky (in press) referred to this as the "derivative-relationship", and while the derivative-relationship of a circle and sphere is generally accepted as "common knowledge", it is tempting to conclude that a similar derivative-relationship does not hold for a square. Symbolically, for a square with side $s$, area $A$, and perimeter $P$: $A = s^2$, $P = 4s$, and therefore $\frac{dA}{ds} = 2s \neq P$. However, this lack of analogy is inconsistent with a mathematical sense of structure. As such, an analogy is sought and achieved by considering the area and perimeter of a square with respect to half of its side. That is, if a side of a square is $2w$, then $P = 4(2w) = 8w$, $A = (2w)^2 = 4w^2$, and $\frac{dA}{dw} = 8w$. So the desired derivative-relationship holds. The symbolical manipulation has a geometrical reason behind it, as illustrated in Figure 1:




$$\lim_{h\to 0}\frac{4(w+h)^2 - 4w^2}{h} = \lim_{h\to 0}(8w+4h) = 8w.$$

Figure 1: Derivative-relationship in a square

*Research Questions*

Based on our prior research, and the mathematical analysis provided above, we designed tasks that are presented in Appendix A. We were interested to explore:

(a) How do participants explain the derivative-relationship for a circle? (Task 1A)
(b) Do they recognize the derivative-relationship for a square on their own? (Task 1B)
(c) How do they react when introduced to the derivative-relationship for a square? (Task 2A)
(d) Are they able to extend the derivative-relationship to a cube? (Task 2B)

For the purposes of this proposal, we focus on two tasks (Task 2A and 2B), and analyse participants' responses via the theoretical framework of actor-oriented transfer.

**The Study**

Thirty-two upper-year undergraduate university students volunteered as participants, all of whom were studying towards a major or minor in mathematics, with at least two courses completed in calculus. Participants were given Tasks 1A and 1B, and then two weeks later, were asked to respond to Tasks 2A and 2B. During each session, participants were given approximately 30 minutes to address the tasks individually in writing.

We briefly mention trends in participants' responses to Tasks 1A and 1B before turning our attention to the latter tasks. In response to Task 1A, 28 participants were able to give support for why the derivative of the area of the circle equals its circumference either with limit or derivative computations (22 of 28) and/or with verbal explanations that made use of the provided diagram (12 of 28). Some common themes to participants' responses included (i) derivative as rate of change, (ii) a loss or gain in (infinitesimal) area, and (iii) a uniform change in area.

Responses to Task 1B regarding a possible derivative-relationship for the square were, in a word, conventional. 31 of the 32 participants claimed the derivative-relationship for the square did not exist or was not possible; the large majority of whom (26 out of 31) based their conclusions on the conventional formulation for the derivative of the area of a square with the length of a side as the parameter. Of the "not possible" responses, 14 were based solely on computations. Of the participants who included verbal explanations for why the derivative-relationship does not hold for a square, 15 included diagrams, 9 of which were analogous to the circle construction. Despite the analogous diagram, participants argued that the lack of a 'uniform ring' around the square negated the derivative-relationship. Responses alluded to problems with the diagonal and with the corners. Notable, the sole individual who claimed that the derivative-relationship for the square was possible based his response on a flawed calculation, mistaking the perimeter of the square to be $2s$, where $s$ was the length of a side.

Focusing now on Task 2A, of the 32 participants, 17 indicated that the alternative approach was valid, 10 indicated it was flawed, and the rest were unsure. Conclusions were based predominantly on computation. While we found the specific features of participants'

computations interesting, our attention was drawn much more powerfully by the conceptual justifications provided. There were 9 participants (of 17) who reasoned about why the derivative-relationship was valid for a square. These 9 participants attended to the rate of change of the area between the two squares, with some noting that the change would be equal for all sides since *a* was being treated as a "radius":

Julie-2A: "VALID. This alternative approach is valid because measuring *a* as a 'radius' defines the smaller square from the inside to the outside. ... This is in contrast to the last [conventional] approach, where (if I remember correctly) extending the length of a square's side simply 'extended' one of its corners… The result of this is that the growth of the square is evenly distributed to all sides of the square, rather than just one direction of length and one direction of width. This is why it works."

Participants who argued that the alternative approach was flawed tended to acknowledge that the computation would work for the "inner square", or sometimes for both squares, but took issue with one of two themes: (i) the corners, or (ii) generalizability. For example, Christina-2A wrote:

Christina-2A: "FLAWED… This approach definitely works on the following 2 squares above, where we have the area and perimeter as the derivative of area. However, it would not work in all such cases. For example if we let a full side length of the square equal to *a* instead of ½ a side length, we wouldn't get the derivative of area to give you the formula for perimeter. $A = a^2,\ A' = 2a.\ P \neq 2a,\ P = 4a.$ Therefore this approach works for this case but is flawed if we label the square differently."

Other participants believed the argument held for the specific case provided, but that it lacked sufficient explanation, or would not generalize to other shapes:

Margo-2A: "FLAWED. I identify this as flawed as it may work for a specific case, the area of the circle in to the perimeter of a circle. But it will not work for the area of a triangle."

Task 2B appeared on the back of the page of Task 2A, and the large majority, 22 out of 32, were able to answer part (a) correctly. Those who could not extend the derivative-relationship to a sphere either did not know the corresponding formula for the volume or for the surface area. Responses were by and large computational. Since the case of a sphere was part of the repertoire for most participants, we focus on their engagement with part (b). In total, 9 of 32 participants could extend the argument, or at least the computation, to the case of a cube; all of whom had responded "valid" to Task 2A. Most of these participants attended to the change in volume between the two cubes:

Michael-2B: "The derivative of the V of a cube also equals the SA. Using the same analogy as above, we have two cubes with half side-lengths of $L$ and $L+h$ respectively. Again, if we let $h \sim 0$, then $L \sim L+h$. Then what is the volume in between the two squares [*sic*]? It's an infinitesimal volume, or in other words area. Since its 3D area [*sic*], it is therefore SA."

Responses that denied the derivative-relationship for the cube were again primarily based on computations – 15 of the 23 participants responded solely with computations, all of which used the standard formulas for volume and surface area of a cube exemplified by Christina-2B:

Christina-2B: The volume of a cube is $s^3$, where $s$ represents the [length of the] side. $V(cube) = s^3, V'(cube) = 3s^2$. The surface area of a cube is $SA(cube) = 6s^2$. The derivative of the cube corresponds to only 3 faces of the cube. Therefore the derivative of the volume of a cube gives you the formula for the surface area of only <u>3 faces</u> of the cube (therefore half the surface area of a full cube). Therefore the derivative of the volume of a cube is half the surface area of a full cube."

The most common objection to the derivative-relationship for a cube was with a disproportionate 'gain' or 'loss' in surface area due to the corners:

Julie-2B: "The derivative of the volume of a cube, on the other hand, is not equal to the SA of the cube. This is for the same reasons as last time's approach [Task 1B]. As we've just explored with the square, adding to the length of a cube does not make it 'grow evenly' like it would with a circle or a sphere. The derivative of the volume of a cube is $3l^2$, where $l$ = length. We know that $l^2$ is the area of a face and that a cube has 6 faces, thus the SA of a cube $= 6l^2$, which $3l^2$ is half of. I think that we end up with the derivative being half (just like last time) because when $V = l^3$ and we add to $l$ to make the cube bigger, only one of two directions in each degree of freedom 'takes the expansion'. This is why it doesn't work, while a 'radius-based' model would."

It is interesting that despite acknowledging that "a 'radius-based' model would" achieve the desired derivative-relationship, Julie-2B did not proceed to consider such a model. In what follows we analyse major themes in participant responses via the lens of actor-oriented transfer.

## Theoretical Framework: Actor-oriented transfer

The focus in *actor-oriented transfer* (AOT) – to distinguish from observer-oriented transfer in a traditional approach – is on what the learner sees as similar between two tasks, rather than what the researcher/expert identifies as similar structural features. From this perspective, transfer is defined as the generalization of learning or more broadly as the influence of prior experiences on learners' activity in novel situations. In other words, by adopting an actor's perspective, we seek to understand the ways in which people generalize their learning experiences rather than predetermining what counts as transfer using models of expert performance (Lobato, 2006). As such, in the following discussion of what is transferred we focus on the prior experiences that influenced participants' interpretations of the tasks.

A well known (and frequently cited) example from Schoenfeld's research (1985, 2011) describes a situation in which students did not use their relevant and recently reviewed knowledge (a proof for a theorem) in a new geometry task. He analyzed this behaviour as "context bound" (2011, p.30), that is, the context shapes the way the task is interpreted, the associated goals for solving, and the knowledge evoked. Schoenfeld suggested that "the students developed *certain understandings of the rules of the game* [italics in the original]" (p.30) and that these rules are invoked according to students' interpretation of the task. This explanation is applicable to participants' responses to our tasks and, although the language is different, we see this as strongly connected to AOT. In particular, individuals' prior experiences establish expectations of what are the 'rules of the game' and accordingly influence their interpretations of novel situations as they attempt to generalize their learning experiences. This perspective ties into the major themes emergent in our data, which we discuss in the following section.

## Results and Analysis

*Participants' resistance to or lack of confidence with the alternative approach*

With respect to participants' responses to Task 2A, we interpret via AOT an implicit desire to transfer familiar knowledge regarding the derivative of the area of a square. This resistance and lack of confidence is well exemplified by participants who conceded that the alternative approach was valid, but then drew conclusions of "flawed" or "unsure". For instance, Margo-2A wrote "I identify this as flawed as it may work for a specific case.... But it will not work for the area of a triangle." Margo-2A's rejection of the derivative-relationship was not

based on the argument itself as applied to the square, but rather on her scepticism regarding its generalizability. Incidentally, the argument does hold for an equilateral triangle, as well as every regular polygon (Zazkis, Sinitsky & Leikin, in press). Like Margo-2A, other participants criticized the argument as not being generalizable, claiming it "worked, but" and noting that the relationship would not hold if "we label the square differently" (see for example Christina-2A). They seemed to respond to whether or not the relationship $A' = P$ was true, as opposed to whether or not the argument was valid. We suggest that those participants' prior experiences with and expectations for derivative questions were transferred and subsequently influenced their interpretation of our task, since if the parameters were changed (e.g. if the square were labelled differently), then the argument itself would also be different (though no one acknowledged this).

*Participants' inability or unwillingness to reason by analogy or extend the argument*

As previously mentioned, the majority of participants (23 of 32) reasoned that the derivative of the volume of the cube was not equal to its surface area. All of these participants fell back to the familiar $V = s^3$, and $SA = 6s^2$ formulas for their justifications (where $s$ is the length of the edge of a cube), transferring the traditional interpretations of volume and surface area. Notably, of these 23 participants, 17 drew conclusions from computation only. Further, recalling Task 2A, of the 17 participants who accepted as valid the argument for the square, 9 were not able to extend the argument or computation to the case of the cube. This is in accord with Schoenfeld's (2011) observation that the "tacit but strong lesson[s] they had learned from their classroom experience with such problems" (p.30) can overshadow new knowledge.

Turning our attention to the 8 participants who included explanations with their computations for why the derivative-relationship did not hold for the cube, we observed a transfer of ideas raised in Task 1B. For example, Julie-2B transferred her objection to the derivative-relationship to the case of the cube despite recognizing the analogy with the "radius-based model". In response to Task 1B Julie wrote that the derivative-relationship was not possible because "extending $l$ [the length of a side] infinitesimally would be 'like' attaching two sides to the square. This is why $\frac{d}{dl}l^2 = 2l$, and <u>not</u> $4l$." She reasoned later that "measuring $a$ as a 'radius'" allows "all sides of the square [to] increase equally" and that the "result of this is that the growth of the square is evenly distributed to all sides of the square" and "this is why it works" (Julie-2A). Interestingly, in Julie's explanation for why the derivative-relationship for a circle "makes sense" and for why it "actually works" also hinged on the idea of a uniform increase in area, "extending the radius (all around the circle) by an infinitesimal amount".

Julie-2A further observed that her response was "in contrast to the last approach", that is, her judgment of the argument in Task 2A as "valid" was in contrast to her response to Task 1B. In her response to Task 2B, Julie-2B acknowledged both approaches though she seemed to prioritize her original reasoning, that is, the familiar formulation experienced in prior work with areas and volumes of squares and cubes. Recalling the excerpt quoted above, Julie-2B wrote:

> "The derivative of the volume of a cube, on the other hand, is not equal to the SA of the cube. This is for the same reasons as last time's approach. … I think that we end up with the derivative being half (just like last time) because … only one of two directions in each degree of freedom 'takes the expansion'. This is why it doesn't work, while a 'radius-based' model would."

It is interesting that while Julie-2B recognized that "a 'radius-based model would" work, she seemed to assume that the question posed ("How does [the volume of a cube] relate to the SA?") was asking for a particular representation of the formula for volume or surface area of a cube. In

addition to transferring her prior knowledge regarding conventional approaches (in using conventional formulas) and her new knowledge regarding an alternative approach (in claiming that a radius based approach would work), Julie seemed to transfer her expectations regarding what knowledge is prioritized (e.g. conventional over alternative).

## Conclusion

We presented tasks based on the derivative-relationship to a group of year 3 and 4 university students studying towards a major or a minor in mathematics. The results suggested that none of the participants was able to generalize for a square the derivative-relationship evident in a circle. Further, when such a relationship was presented, only about half of the participants considered it as valid, and very few were able to extend the argument to a cube. Our study extends research on understanding derivatives by focusing on unconventional use of parameters in familiar formulas, and we highlight student difficulties in moving away from conventional representations. The importance of considering unconventional representations, and as such challenging basic assumptions, was posited by Zazkis (2008) as a vehicle towards constructing a "richer or more abstract schema" (p.154) and "understand[ing] better what has been already understood" (ibid). In considering conventions Zazkis focused on other-than-ten bases for representing numbers and other-than-Cartesian coordinates for graphing functions. We add here other-than-standard use of parameters in familiar formulas for perimeter, area, surface area and volume. Moreover, we recognize an important component in considering the unconventional: we consider the flexibility in accepting the unconventional and acknowledging the analogy with the conventional as part of an individual's appreciation of the overarching structure of mathematical concepts and relationships.

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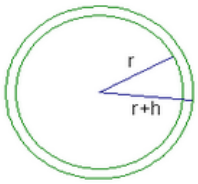
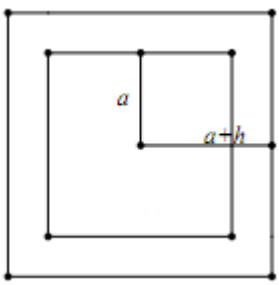
## Appendix A – List of Tasks

| | |
|---|---|
|  | During a calculus class, one student noticed that when working with the circle, the derivative of the area formula yields the formula for circumference. That is, $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$. The student asked why this relationship held for the circle, and not in other cases such as with the square.<br>1A. Use the diagram to show why the derivative of the area of a circle yields the formula for the circumference.<br>1B. Is it possible to represent the derivative of the area of a square as the formula for its perimeter? If so, explain how. If not, explain why not. |

Tasks 1A and 1B – derivative-relationship for a circle and square

| | |
|---|---|
|  | Consider the following argument and diagram:<br>Imagine $a$ as an analogy to the radius of a circle. In this way, we can describe the perimeter and area of the inner square as a function of $a$:<br>$P(a) = 4(2a) = 8a$, and $A(a) = (2a)^2 = 4a^2$.<br>Similarly, the area of the outer square can be described by<br>$A(a+h) = (2a + 2h)^2$.<br>Then we can express the derivative of the area of the square as its perimeter in the following way:<br>$A'(a) = 2 \times (4a) = 8a = P(a)$.<br>Reflect on this "alternative approach". This approach is (circle one):<br>VALID FLAWED NOT SURE<br>If you circled VALID, please explain why this approach makes sense. If you circled FLAWED, please identify the flaw.<br>If you circled NOT SURE, please explain why you are unsure. |

Task 2A – the derivative-relationship for a square

| |
|---|
| Consider the derivative of the volume of<br>a) a sphere, and<br>b) a cube.<br>How does it relate to the surface area? |

Task 2B – extending the derivative-relationship

# Reinvention Six Months Later: The Case of Megan

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Contributed Research Report

*This research is a part of a larger project to gain insights into how calculus students might come to understand formal limit definitions. For this study, a pair of students participated in a guided reinvention teaching experiment where they had to wrestle with and resolve many problems in creating a formal sequence convergence definition. Six months after the initial teaching experiment, the students returned for individual interviews in which they were both asked to reproduce their sequence convergence definition. In this paper, we highlight one student's activities to recreate this definition. The definition was not immediately recalled, but instead, particular phrases and relationships were remembered. Furthermore, problems were re-engaged, and we claim that because of prior experiences with these problems, solutions were more readily available and reconstruction was thereby quickened.*

Keywords: Limit, Definition, Guided Reinvention, Sequences, Retention

## Introduction and Research Questions

Some recent studies have begun to detail *how* students come to understand formal limit definitions (Cory & Garofalo, 2011; Cottrill et al., 1996; Martin et al., 2011; Oehrtman et al., 2011; Roh, 2010; Swinyard, 2011). Cottrill et al. (1996) *conjectured* as to how a student might move from an informal understanding to a more formal understanding of limit of a function at a point and then called for more research to enhance limit instruction. Recently, via the process of guided reinvention, Swinyard (2011) and Oehrtman et al. (2011) detailed some of the challenges that students face while making the move from an informal to a formal understanding of limit of a function at a point and limit of a sequence, respectively. Roh (2010) and Cory and Garofalo (2011) used specifically designed activities to explore how students' images of limit of a sequence can influence their emerging formal understandings of convergence. Furthermore, Cory and Garofalo (2011) were able to demonstrate how activities utilizing dynamic computer visualizations of sequences can help to support not only more coherent understandings of limit but understandings that can be retained 15 months after activity completion. This paper adds to this body of research by providing more empirical evidence of what students retain after having gone through a rigorous process of having to reinvent formal limit definitions. Specifically, this study addresses:

1. What might an individual student recall six months after the guided reinvention of a formal definition for sequence convergence?
2. Of those things that a student recalls, how can they be evoked, and how are they effective in helping a student recreate their definition?

## Theoretical Perspective

This study is a part of a larger project to gain insights into how calculus students might come to understand formal limit definitions. For the project we adopted a *developmental research* design (Gravemeijer, 1998) and incorporated a *guided reinvention* heuristic described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237). In the spirit of the *proofs and refutations* design heuristic (Larsen & Zandieh, 2007), instead of having students reason about a *provided* definition, task design centered around having students rigorously articulate and refine their personal concept definition (Tall & Vinner, 1981) of sequence convergence. This positioned the project team to identify *authentic* challenges students encountered as they formalized their intuitive understandings.

Oehrtman et al. (2011) further detailed student activity during reinvention as an iterative refinement process (*Figure 1*). During the multiple iterations of this process, Oehrtman et al. (2011) observed students identifying, engaging, and resolving *problems* that typically arose due to conflicts between the students' concept image and their currently stated definition. Specifically, the identification of problems can be greatly aided by the presence of examples that can serve as guides for features to incorporate or as sources of cognitive conflict when a student generated definition failed to accurately capture the example. During the refinement process, experts might identify other *problematic issues* with the emerging definition that have yet to be recognized as problems by the students. Oehrtman et al. (2011) claim that it is the explicit resolution of identified problems that are most meaningful for students as they support the formation of integral ideas that remain stable throughout the remaining iterative refinement process. This process can lead students to feel strong ownership of each component of "their" definition as they become able to cite problems that each component resolved (Oehrtman et al., 2011).

## Method

This study specifically extends the work of Oehrtman et al. (2011), Hart-Weber et al. (2011), and Martin et al. (2011), as it continues to follow the same two students, Megan and Belinda (pseudonyms), six months after their initial reinventions of sequence, series, and pointwise convergence. Details behind the methods used in the initial teaching experiment that paired Megan and Belinda together to create a sequence convergence definition can be specifically found in Oehrtman et al. (2011).

Six months following their initial reinventions, both Megan and Belinda participated in two 70 to 90-minute individual interviews in which each was prompted by the facilitator to complete the statement, "A sequence converges to $U$ provided…" Megan and Belinda were then paired together for five more teaching experiment sessions in which they discussed convergence in the context of $\varepsilon$-$N$ proofs, defining series convergence, and defining power series convergence. For this paper, the focus will be on Megan's individual work during the first two days of her return. By focusing on one student, we detail what an individual can recall and what might evoke productive recollections in re-creating a limit definition. It should be noted that even though the focus is on Megan, both Megan and Belinda followed very similar lines of reasoning during the recreation of their definitions.

For each individual interview, detailed content logs were created containing time-stamped descriptions of the current activity of the student and facilitator and theoretical notes about how that activity was progressing toward recreating a formal definition. The video timeline was coded

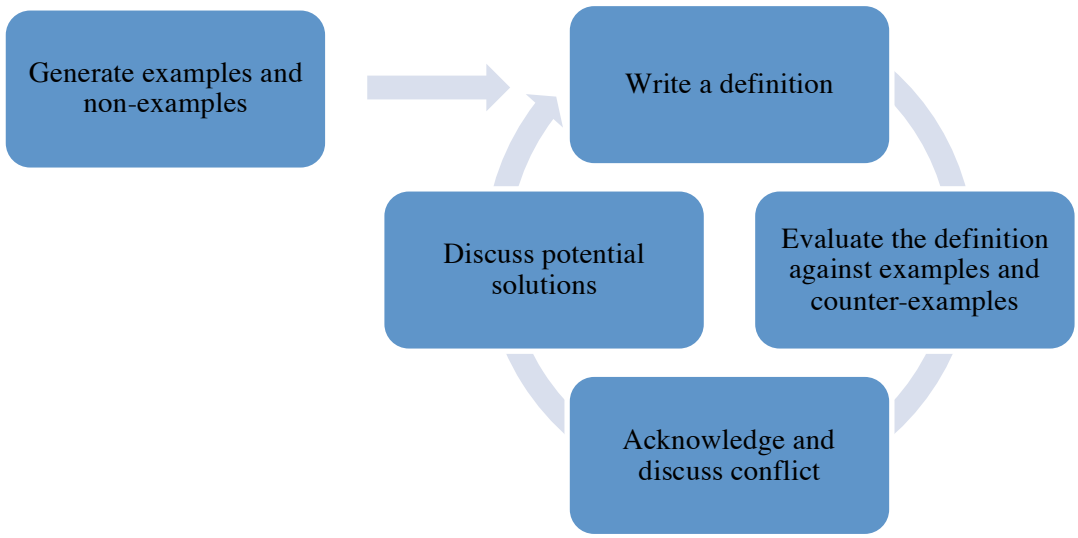


*Figure 1*. Iterative refinement in the process of guided reinvention of a formal definition.
*From From intuition to rigor: Calculus students' reinvention of the definition of sequence convergence by Oehrtman, M., Swinyard, C., Martin, J., Hart-Weber, C., & Roh, K. (2010). In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 2, p. 327). Portland, OR: Portland State University.

for instances in which students appeared to recall their prior definition or their experiences related to creating their definition. Attempts were made to determine the origin of those recollections, and detail how such recollections may have aided each student in making progress in recreating their definition.

## Results

*Remarks on the Initial Teaching Experiment*

Six months prior, it took Megan and Belinda working together for almost three 90-minute sessions before eventually creating a definition that they felt appropriately captured sequence convergence. Their final definition was: "A sequence converges to $U$ when $\forall \varepsilon > 0$ there exists some $N$, $\forall n \geq N$, $|U - a_n| < \varepsilon$."

This definition evolved from over 23 cycles of evaluating and refining their definitions against examples of sequences. The first explicit problem they engaged was determining how a definition should capture sequences with "bad [random] early behavior?" The graph in *Figure 2* played the most prominent role in their identification of this problem and was subsequently used to justify the incorporation of their notion of "at some point $n$" into subsequent definitions (a notion that eventually evolved into their conception of $N$ consistent with a standard $\varepsilon$-$N$ definition). Another problem they engaged was expressed as, "How close is close?" This can be tied to them attempting to refine their notions of "approaches" and "becomes closer to" found in earlier definitions. Their resolution was to introduce their idea of an "acceptable error range" that eventually

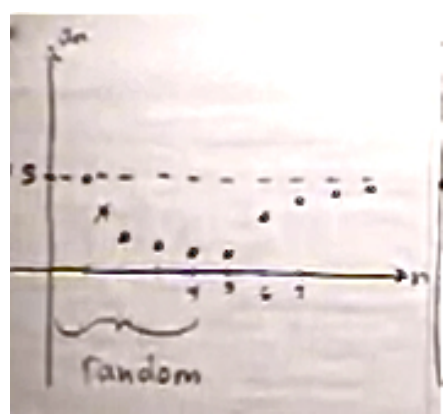


*Figure 2*. A sequence with "bad early behavior."

evolved into a conception of a universally quantified $\varepsilon$.

A major problematic issue for Megan and Belinda was their persistent desire to require monotonically decreasing errors that they represented by "$|5-a_n|<|5-a_{n+1}|$" in their sequence convergence definitions. Furthermore, as Hart-Weber et al. (2011) detail, Megan and Belinda consistently either viewed the "dying sine graph" (*Figure 3*), as divergent or conceived of the graph as eventually behaving like another convergent sequence, such as becoming alternating or constant. Once they accepted the "dying sine graph" as convergent even though it continued to oscillate, they immediately removed the extra condition of monotonically decreasing errors, never appearing again in subsequent definitions.

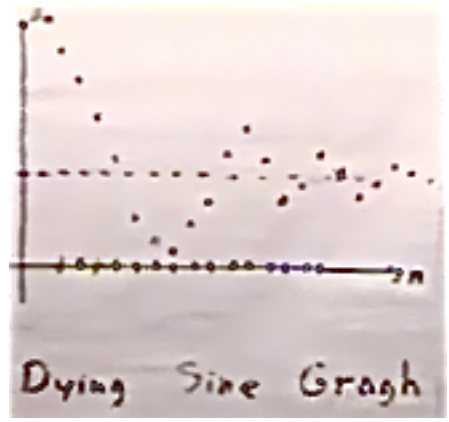

*Figure 3.* The "Dying Sine Graph."

More details behind student interaction with these problems, and other problems, can be found in Oehrtman et al. (2011) and Hart-Weber et al. (2011).

*Megan Six Months Later*

During the first individual interview, Megan was given the prompt to complete the statement, "A sequence converges to *U* provided…" Megan quickly responded "Oh geez, I can see the pieces of it in my head [with eyes closed], I just cannot put it all together." Indeed for 9 minutes, Megan articulated phrases of *their* final definition such as "there exists," "such that," "*n*," "error," and "epsilon." She even illustrated ideas about the sequence not leaving an error bound after some *n* and that *n*'s dependence upon epsilon. After 3 refinements, *her* definition went from "Something about as *n* or as the *x* increased… the *y* got closer to this value that it was converging to," to "$\exists n$ s.t. [such that] for all $n<N$, $\varepsilon$ decreased." Clearly there are many problematic issues with this definition. Even though Megan had expressed several phrases and ideas suggestive of concepts from the formal definition, this definition did not contain all of them. Furthermore, the phrases her definition did contain were not integrated in a way to effectively convey her ideas. In addition, other ideas, such as the universal quantification of the error bound, were noticeably absent from all of Megan's early recollections.

After being moved to the activity of creating several graphs of sequences converging to 5, Megan eventually produced graphs for monotonically increasing, monotonically decreasing, alternating and constant sequences. Furthermore, Megan recalled the "dying sine graph" but then expressed no need to create this graph because she felt the alternating sequence was "basically" the same. She also produced examples of sequences not converging to 5 that included: increasing to 6, decreasing to a number below 5, alternating but not dampened, and increasing and decreasing to infinity.

Once she produced all of these graphs, Megan was again prompted to define sequence convergence. Initially Megan moved back and forth from looking at her graphs to looking at her emerging definition. After 3 more iterations, Megan wrote: "$|U-a_n|$ decreased as $n \to \infty$." Following the creation of this definition, except for a 2 second glance, Megan did not attend to her graphs. After a few mumblings, such as "No that was… [trails off]" and a veiled reference to "for all," she expressed dissatisfaction with her current definition. After two more iterations, she expressed an idea of monotonically decreasing errors: "$\forall n > N, |U-a_n| < |U-a_N|$." After going to the graphs, she quickly identified problems generated by the constant sequences at 5 and

at 3. Furthermore, the facilitator graphed a monotonically increasing sequence to 3. Megan concluded that her current definition did not correctly exclude this new graph, and after a long moment of silence, stated, "I remember having this problem before when we were doing the teaching experiment, but I'm trying to remember how we solved it before."

After some inactivity, the facilitator directed her attention to a prior definition where she had written, "approaches," and asked, "What does approaches mean?" After framing her response in "closer and closer" language, she mumbled, "It's just not small enough." She then articulated that "we were using terms before, like infinitely small, and things like that, and we came up with a better way of saying that." Her utterances suggest that better articulating "approaches" was tied to a previous resolution, such as of "How close is close?" and that she knew it had been resolved before but was unable to satisfactorily recall the solution. After another attempt at a definition did not resolve this issue for Megan, she left Friday's day 1 interview with this problem in mind.

On the following Monday, Megan returned claiming that she had had an "ah-ha" moment particular to remembering another phrase of their definition. After being encouraged to write, Megan produced "$\forall \varepsilon \geq 0, \exists n > N$ s.t. $|U - a_n| < |U - a_N|$." Megan went on to add that for the definition, "the [$\varepsilon$] we're interested in is like .0000001." The issue of "How close is close?" or the meaning of "approaches" was not again raised by Megan, and the "$\forall \varepsilon$" appeared in all subsequent definitions. Moreover, as Megan applied her definition to graphs, she was clear about $N$'s dependence upon $\varepsilon$ even though, capturing this relationship was a problematic issue with her current definition. After applying her definition to the sequence increasing to 3, and noticing that for $\varepsilon = 0.5$, the sequence was "not within that error range," Megan addressed the problem of how to capture her notion of getting within an "acceptable error range."

At this point, the facilitator graphed a convergent sequence with "bad early behavior." When asked "Why this [sequence] might have been important last time?" she immediately responded that "we wanted to say that *after this point*, it didn't leave the epsilon again" (emphasis added). She further elaborated and related it to the notion of the "cap $N$" that corresponded to where the sequence initially entered the "acceptable range" determined by $\varepsilon$. Using the "bad early behavior" graph, the facilitator then focused her attention on two points successively decreasing toward 5, after which, she stated that "based on my definition, this area [circling the two points] would be convergent […] but if I go a little further out, it goes away again." Her utterance of "goes away again" suggests that she was attending to the monotonically decreasing part of her definition instead of the "acceptable range" idea. Furthermore, when the facilitator moved her back to talking about the "acceptable range" she expanded that idea to include where the graph "behaves the way you want it to for convergence" and that this behavior was where the errors at a point "is larger than the next [point's error], which is larger than the next, which is larger than the next." Even though she stated that her current definition "was not quite there yet," after these discussions she made no progress on starting another definition.

The last graph drawn by the facilitator was a reproduction of the "dying sine graph" with the added requirement that it always had "three [points] above [5] and three below." She quickly remembered this graph as "de-validating one of the definitions we had come up with." After moving up to the graph, she quickly stated that her current definition did not work because the errors in this graph were not monotonically decreasing. She then added, "but if you keep going, eventually they are going to be… [trails off]" and never completed her thought. It seems conceivable that she may have contemplated the validity of this sequence's convergence, like she had previously contemplated with Belinda. Yet, in this case, she didn't articulate any such conception and there are no other indications suggesting that she questioned the sequence's

convergence. Instead there is a period of silence where she looked intently at her definition and this newly produced graph. When she broke the silence she said, "The error of one being smaller than the other might be a little bit too restrictive." She now identified that having monotonically decreasing errors in her definition created a problem in correctly capturing this sequence. Within 3 minutes she produced her 12th and final definition: "$\forall \varepsilon \geq 0, \exists N$ s.t. $\forall n > N, |U - a_n| < \varepsilon$." The remainder of the time during day 2 was spent applying her definition to her examples and non-examples of convergent sequences to 5. It should be noted that once paired with Belinda on day 3, Megan readily accepted Belinda's explanation for why $\varepsilon$ was strictly greater than 0 because "we would end up excluding the sequences that never actually get to the number it's converging to." The pair's 1st and final sequence definition from day 3 was: "A sequence converges to $U$ provided $\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N, |U - a_n| < \varepsilon$."

## Conclusion and Discussion

Megan almost did in 12 definitions what it took the pair of students over 23 definitions to complete. Megan almost did in 2 75-minute sessions what it took the pair of students almost 3 90-minute sessions to complete. So what allowed Megan to make such rapid progress? This paper detailed how Megan utilized an iterative refinement process (Oehrtman et al., 2011), and during this process, Megan actively sought to recall key phrases from their prior definition and resolutions to previously engaged problems.

By placing Megan in the situation of having to reproduce a sequence convergence definition, she attempted to remember the definition created 6 months before. It should be noted that even though Megan's early definitions were far from a standard $\varepsilon$-$N$ definition, she quickly evoked several key phrases and relationships found within their definition from 6 months before. These phrases and relationships provided Megan with building blocks upon which to recreate her definition. By her second definition, Megan had already captured ideas that did not appear until much later for the pair.

Many of the problems for the pair were again problems for Megan, but Megan also remembered many of these problems as they arose with her current definition. Solutions were sometimes quickly recalled, but even when Megan faltered in remembering solutions, something as simple as being re-exposed to a certain graph (like the "dying sine graph") could bring Megan into a state of recognition that supported her in reconstructing solutions. This further supports the importance that Oehrtman et al. (2011) placed on student engagement of problems as opposed to problem avoidance. If Megan had not previously wrestled with "bad early behavior," articulating what close means, or classifying the "dying sine graph" as convergent, she would have to go down uncharted paths to address these issues six months later.

We also feel a worthy question is, "Why did it take Megan so long?" The extent to which these phrases and relationships, and problems and resolutions had remained meaningfully organized and connected had eroded over time. To some degree this erosion should be expected, as Megan had had no significant exposure to formal limit definitions in the six months following the initial teaching experiment. Even so, we see room for improvement of retention. These results suggest that ground might be gained by supporting students in better remembering certain iconic images that played a vital role in the creation of their definition. We are currently investigating the reinforcement of these images through dynamic visualization techniques adapted from Cory and Garofalo (2011) (see Cory et al., 2012).

We acknowledge that our results from this single student don't necessarily generalize to others. Nevertheless, this provides rich data about what a student *might* remember months after a

guided reinvention of a limit definition. Furthermore, over the next few months we will be following up with other students that have completed similar teaching experiments. The data gained from these studies can begin to better support more generalizable conclusions.

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# To Reject or Not to Reject: One Student's Non-Normative Decision Procedure for Testing a Null Hypothesis

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## Abstract

The purpose of this study was to gain insight into how engagement with hands-on and computer resampling methods affected a statistically naïve student's emergent understandings of statistical inference. In this study, simulation design activities provided a vehicle for engaging a student with the core ideas of hypothesis testing. The results highlight challenges the student experienced in coordinating the components of the logic into a coherent scheme of ideas and sheds light on aspects of engagement which need to be emphasized in order to resolve the inherent conceptual difficulties associated with reasoning that invokes a modus tollens-like argument. Moreover, I report on a heuristic the student used to make his inferential decisions—one that does not produce correct inferences. I've termed this the "similarity heuristic" because of a specific similarity relationship the student would look for and then use as a method for rejecting or not rejecting the hypothesis being tested.

**Keywords:** Statistical inference, Statistical reasoning, Hypothesis testing, Resampling, Simulation

## Background and Prior Research

Statistical inference is arguably one of the most important schemes of ideas we expect students to understand. It is now applied in a wide range of scientific disciplines and given its extraordinary range of applications the question of how to support the development of a coherent understanding of statistical inference has taken on an increased importance. The traditional approach to teaching statistical inference is through the probability based normal distribution couched in abstract theory and formal language. Statisticians of generations past invented these parametric methods because direct simulation through an empirically obtained sampling distribution was simply too slow to be practical (Cobb, 2007). But modern computing has now made simulation both fast and practical and this has created a growing movement concerned with how we teach statistical concepts. Many educators (e.g., Chance (2006), Erickson (2006), Cobb (2007), Rossman (2008) and Garfield & Ben-Zvi (2008)) consider parametric methods as too formal an introduction for most students and they advocate a new equilibrium that opens the door for computer simulation activities as a way to help students understand the difficult concepts which underlie how statistical decisions are made.

Mills(2002) provided an overview of the literature on the use of computer simulation methods from 1983 to 2000. In her summary of 48 articles, many of the authors recommended the use of computer simulation methods to teach abstract concepts in statistics. One advantage which was suggested was simulation's ability to utilize the power of concrete illustration to ease logical difficulties and enhance understanding. The consensus among the authors was that computer simulations are arguably instructionally productive. delmas, Garfield and Chance (1999), for instance, demonstrated a powerful effect of using computer simulation on students reasoning about sampling distributions and the Central Limit Theorem. Several more recent publications have also suggested that improved instructional results can be achieved by using

good simulation tools and activities (Lipson, 2002; Chance, delmas & Garfield, 2004). The quantitative results in her review of the literature indicated that "something" important happened between pre and posttest measures but without additional evidence it's not possible to reveal exactly what that was.

Saldanha (2004) reported on a series of classroom teaching experiments that engaged high school students with instructional tasks in which they designed the components of the simulation in the context of modeling contextual scenarios involving hypothesis tests. The intent was to use computer simulations and the interactions that flowed out of that engagement to help students understand the vital connections between sample, population, and the sampling distribution on which an inference is based. The study is notable because it is one of the first to actually characterize the reasoning that emerged as students engaged in computer simulations. As part of this larger study (Saldanha 2004), Saldanha & Thompson (2007) reported on key developments and critical shifts that unfolded over a series of 3 consecutive lessons as student's engaged in both concrete and computer simulated sampling activities. The report characterizes how simulation helped shape the students conceptions of sampling distributions and the inferences that can be made based on these collections. More recently, Saldanha (2011) reports on a single simulation activity in which a group of high school students encountered severe conceptual difficulties as they grappled hard with the crucial process of turning a phenomenon of interest into first a statistical question and then into a stochastic experiment in order to judge whether a particular event was unusual.

**Research Questions**

My review of the literature suggested the need for studies which contain highly dense and detailed analysis of students reasoning as they engage in simulation activities centered on advancing the logic of statistical inference. The goal of this pilot study was to move in this direction by exploring how engagement with simulation activities affected a single student's emergent understandings of hypothesis testing. Questions of interest included: 1) what ways of thinking--interpretations, understandings and imagery--express themselves as the student engages in the instructional activities? 2) What conceptual difficulties did the student experience? 3) What aspects of engagement in these activities hindered or moved his thinking forward in productive ways?

**Theoretical Perspectives**

Four basic theoretical perspectives underlay this study and were drawn on extensively. First, I drew upon radical constructivism as elaborated by von Glasersfeld (1995). By adopting this perspective I constrained myself to the idea that whatever sense the student made of his experience in this study, he constructed it for himself, in spite of my efforts to influence his thinking in particular ways. Secondly, I drew upon Thompson's (1994) theory of quantitative reasoning. Thompson's theory provided a frame for thinking about the unusualness of any particular sampling outcome as a statistical quantity; that is, as a measurable attribute of the sampling outcomes frequency of occurrence within a distribution of outcomes. Thirdly, I drew upon Thompson's instructional design theory of creating tasks as *didactic objects* (Thompson, 2002). When instructional activities produce environments that foster reflective goal-directed, interactive discourse they become didactic objects; that is, they become tools for generating observable information about student understanding. Lastly, I followed Saldanha (2004) by conceiving the basic structure of the inferential process in terms of a population, a random

procedure for selecting objects from the population, a resulting sampling distribution, and an inference from the sample back to the population.

**Methods and Subject**

To explore the participating student's thinking as he engaged with instruction I conducted a one-on-one teaching experiment. The participant in this study was a statistically naïve freshman who had had little or no formal experience with making statistical-based arguments. He was recruited from an undergraduate pre-calculus class, at a large southwestern university. The student participated in 9 sessions in an out-of class setting. Each lesson unfolded over a 75-90 minute period. A written pre-assessment queried the student's initial intuitions and understandings and a post-activity interview queried his thinking about the key ideas and interconnections among them that were addressed in the designed instructional activities. The teaching experiment itself unfolded in a sequence of 7 lessons over a two and one-half week period. During the teaching experiment the student was prompted to explain his thinking both verbally and in written responses in order to gain insight in his reasoning processes. The data corpus includes video-taped discussions around the activity sequence. An analysis of the video produced annotated transcriptions identifying critical events in the students reasoning. The student's utterances were triangulated with his written responses in an attempt to determine the mental actions and ways of thinking that contributed to his observable behavior. In this way, the descriptions and analysis of the student's understandings were grounded in his participation in instruction.

**Summary of Results**

The broad finding was that the student experienced overwhelming difficulty generating and composing the requisite images necessary to coordinate the logic of hypothesis testing into a coherent scheme of ideas. Against this background two specific findings emerged: 1) the student's use of sampling distributions as *comparison devices,* and 2) the student's adoption of a logically unsound *similarity heuristic* by which he made his inferential decisions.

*Sampling distributions as comparison devices.* Early on I had the student begin looking at distributions of outcomes of many samples drawn from a single population in order to judge whether a given sample of data is likely to have come from the known population that produced the distribution. By examining the place of the observed sample in the distribution of samples I intended for the student to quantify the degree to which the sample is or isn't a surprising outcome. If it turns out to be highly improbable then he can reasonably conclude that the observed sample didn't come from such a population. I wanted him to internalize that in order to make inferences from a single sample he must first observe the behavior of samples from the known population. His engagement in these activities, however, did not have the intended effect. Instead of using the probability of seeing the observed sample in the distribution as evidence that the population it came from was likely or unlikely to have been the one that produced the distribution, the student immediately developed a comparison procedure in which the observed sample's place in the distribution—the most vital of information-- was of no importance at all. In the full report I describe the emergence of this comparison procedure in detail elaborating on how it would later morph into what I call his *similarity heuristic*.

*The emergence of a similarity heuristic.* In later activities, the student was presented with contextual scenarios that involved testing a hypothesis about where a sample of data came from. The student's task was to first model the problem in the scenario, then to investigate his models behavior in terms of the samples it produces, and finally to interpret the results in terms of how

far out in the tail the observed sample of data falls. The intended logic is that he can assess the strength of evidence against a hypothesis by quantifying how unlikely the observed result would be if in fact the hypothesis was true. Since the observed sample is a fact, a result that is rare must mean that the null hypothesis is inconsistent with the observed outcome. The student, however, would immediately lose sight of what population his samples were being drawn from and why he was even sampling from that population in the first place. The key idea--that he had set up a null hypothesis for the very purpose of making it susceptible to a probability estimate--was nowhere in his reasoning processes. In fact, he created his own decision making procedure based on a similarity heuristic in which he was literally deciding if the empirically produced sampling distribution was sufficiently similar to the distribution that he imagined should be produced under the null hypothesis. Never mind that the empirical sampling distribution is exactly what he should expect to see if the initial hypothesis is true—for the very reason that he set it up to be that way. In the full report I describe how deeply embedded in his thinking this similarity heuristic became and how it defied remediation and essentially disabled him from assimilating the logic of statistical inference. I will also elaborate on the implications of this for teaching practice and further research.

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# The Unit Circle and Unit Conversions

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*The unit circle is a central concept of trigonometry. Yet, students and teachers are often tied to using right triangles to reason about trigonometric functions with only superficial connections to the unit circle. In an attempt to identify ways to better support student learning in trigonometry, we conducted a teaching experiment investigating two pre-service secondary teachers' notions of the unit circle. Initially, both students had difficulty relating circle contexts to the unit circle. The students' actions suggested that their calculations stemmed from memorized procedures, as opposed to reasoning about quantitative relationships. In an attempt to foster connections between novel circle contexts and the unit circle, we implemented tasks that developed the unit circle as the result of conceptualizing a circle's radius as a unit of measure. We report on the students' progress during these tasks and the subsequent improvements in their ability to apply trigonometric functions to circle contexts.*

Key Words: Unit Circle, Trigonometry, Pre-service Secondary Teachers, Teaching Experiment, Quantitative Reasoning

## Introduction

Pre-service and in-service mathematics teachers frequently hold limited and fragmented understandings of concepts central to trigonometry (Akkoc, 2008; Fi, 2006; Thompson, Carlson, & Silverman, 2007; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006). Research has also characterized students as constructing disconnected understandings of trigonometric functions (Brown, 2005, 2006; Weber, 2005). In light of these findings, recent efforts (Moore, in preperation, submitted; Thompson et al., 2007; Weber, 2005) have sought to identify the critical reasoning abilities and understandings necessary for developing robust trigonometric understandings (e.g., understandings containing flexible connections between trigonometry contexts). Collectively, the body of research on student (and teacher) learning in trigonometry highlights that individuals' notions of topics foundational to trigonometry (e.g., angle measure and the unit circle) significantly influence their conceptions of trigonometric functions.

In an attempt to better understand ways to support students in constructing connected understandings of trigonometric functions, we explored two secondary pre-service teachers' (who we refer to as students) thinking during an instructional sequence grounded in mathematics education research. Specifically, we used a teaching experiment methodology (Steffe & Thompson, 2000) to explore the students' notions of the unit circle. Our research questions were:

- What are the students' notions of the unit circle and how do these notions influence their use of trigonometric functions?
- What are the critical ways of reasoning involved in using the unit circle to apply trigonometric functions to novel situations?

Our findings illustrate the important role notions of measurement play in students' ability to leverage the unit circle when reasoning about trigonometric functions. For instance, stemming from shifts in their measurement conversion schemes, the students were better able to use the unit circle to apply trigonometric functions in novel circle contexts. Despite shifts in their notions of the unit circle, the students' previous notions of unit conversion and the unit circle

continued to inhibit their progress at various points during the study, highlighting the deep-rooted nature of their previous understandings.

## Background

Research (Akkoc, 2008; Fi, 2006; Thompson et al., 2007; Topçu et al., 2006) on pre-service and in-service teachers' knowledge of trigonometric functions suggests that teachers are lacking the content knowledge necessary to support their students in constructing robust understandings of trigonometry. Teachers are often tied to discussing trigonometric functions in a right triangle context with only superficial connections to circle contexts (Akkoc, 2008; Thompson et al., 2007; Topçu et al., 2006). Complicating the issue, several studies have argued that teachers hold disconnected and shallow understandings of angle measure that restrict their ability to create connections between trigonometry contexts. For instance, stemming from the findings in a sequence of related studies (Akkoc, 2008; Topçu et al., 2006), the authors argued that teachers' reliance on degree angle measure restricts their understandings of trigonometric functions to right triangle contexts (a context that predominantly uses degree angle measure). When compared to the aforementioned findings, one should not be surprised to find that students' (some of who will become teachers) predominant notions of trigonometric functions frequently lie within right triangle contexts (Weber, 2005) and that students encounter difficulty reasoning about trigonometric functions in a circle context (Brown, 2005).

Recent studies (Moore, in preperation, submitted; Weber, 2005) have made progress in supporting students in constructing connected understandings of trigonometric functions. Weber (2005) identified that students' understandings of trigonometric functions are influenced by their ability to leverage the geometric objects (e.g., the unit circle and right triangles) of trigonometry. Weber argued that trigonometry instruction could benefit from future investigations that explore how to support students in conceptualizing the geometric objects of trigonometry in ways that posit students to use these objects in novel situations. Following Weber's suggestion, as well as several calls (Bressoud, 2010; Thompson, 2008) for revising trigonometry instruction, Moore investigated students' angle measure conceptions (submitted) and the role of angle measure in a student's construction of the sine function (in preperation). The two studies illustrated that arc length images of angle measure, in combination with reasoning about the radius as a unit of measure, create foundations for coherence between the trigonometry contexts.

In addition to drawing on the body of research on student learning in trigonometry, we draw on theories of quantitative reasoning (Smith III & Thompson, 2008; Thompson, 2011) to inform the present study. A central premise of quantitative reasoning is that a quantity exists in the mind of the beholder and students' conceptions of quantities should not be taken as given or trivial (Thompson, 2011). Notions of measurement contribute significantly to theories of quantitative reasoning, with Thompson (2011) suggesting that reasoning about magnitudes is an often overlooked, but critical, aspect of quantitative reasoning and measurement. Thompson roots his explication of magnitude reasoning in Wildi's (1991) description of magnitudes, which is based on the notion that the magnitude of a quantity is not dependent on the unit used to measure the quantity. That is, given that the measure of a quantity is $a$ units, the magnitude of a quantity is $a$ times as large as the magnitude of the unit used to obtain the measure.

In order to offer students repeated experiences in magnitude reasoning, Thompson (2011) identifies an approach to unit conversion that centers on reasoning about how the measure of a quantity varies as the magnitude of the unit used to make the measure varies. To borrow an example from Thompson, "if the measure of a quantity is $M_u$ in units of $u$, then its measure is $12M_u$ in units of magnitude $(1/12)\|u\|$ and its measure is $(1/12)M_u$ in units of magnitude of $12\|u\|$

” (2011, p. 21). We conjectured that such reasoning – reasoning about relationships between units’ magnitudes and measures in those units – provides one possible conceptualization of the unit circle; the unit circle results from changing the magnitude of the unit being used to measure various attributes of circular motion (e.g., those attributes that trigonometric functions relate). To say more, if the radius of a circle is 4.2 feet and all given measures are in feet, one can reason that the radius is a magnitude that is 4.2 times as large as the magnitude of a foot. It follows that measures in radii will be 1/4.2 times as large as the corresponding measures in feet. Hence, to convert a measure in feet to a measure in radii, one merely multiplies by 1/4.2 (or divides by 4.2). By engaging in such reasoning, any circle whose radius length is given in any standard unit can be viewed as the unit circle (Fig. 1).

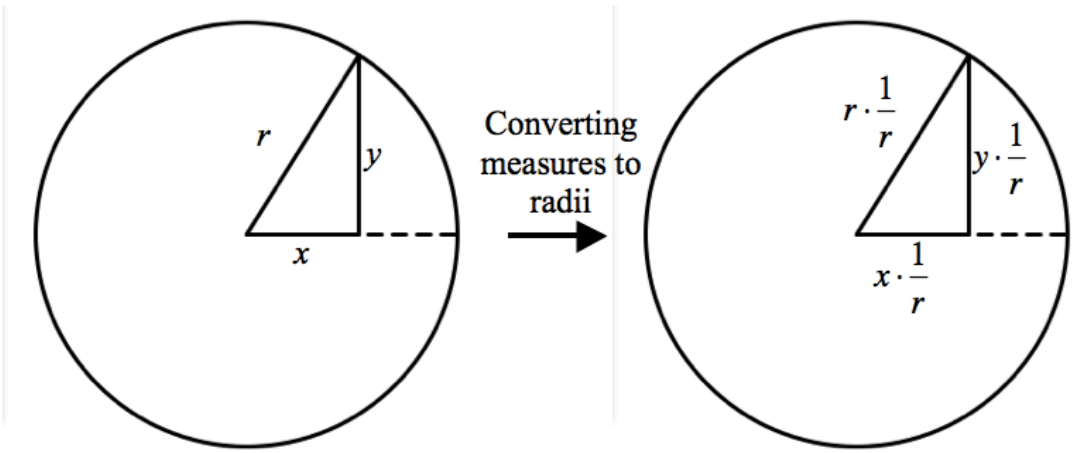


Figure 1 – Unit Circle, Ratios, and Units of Measure

To compare Thompson’s articulation of unit conversion to a common approach to unit conversion, consider that of dimensional analysis (or unit-cancellation), which is found in mathematics, engineering, and science courses. Dimensional analysis typically consists of starting with a measure (e.g., 4.5 feet), identifying two equivalent measures (one in the given unit and the other in the desired unit), and then using unit-cancellation to determine what ratio to multiply the given measure by. For instance, converting a measure of 4.5 feet to a number of centimeters would be as follows:

- I have a measure of 4.5 feet and wish to find the equivalent measure in centimeters.
- There are 30.48 centimeters in 1 foot, or there are 0.0328 feet in 1 centimeter.
- The desired measure is $4.5\ \cancel{\text{feet}} \cdot \frac{30.48 \text{ centimeters}}{1\ \cancel{\text{foot}}} = 137.16 \text{ centimeters}$, or

  $4.5\ \cancel{\text{feet}} \cdot \frac{1 \text{ centimeter}}{0.0328\ \cancel{\text{feet}}} = 137.16 \text{ centimeters}$.

The calculations performed in dimensional analysis might be identical to those used when reasoning about magnitudes (the magnitude of a centimeter is 1/30.48 times as large as the magnitude of a foot, and thus a quantity’s measure in centimeters is 30.48 times as large as the quantity’s measure in feet), but dimensional analysis circumvents the meanings for the operations of division or multiplication, and the method instead treats units as if they are things that can be discarded through procedural rules. On the surface, dimensional analysis appears to provide a straightforward approach to unit conversions. However, there is evidence (Reed, 2006) that dimensional analysis can lead to decreases in student performance.

## Methods and Subjects

The study's participants (Bob and Mindy) were second year undergraduate students enrolled in a pre-service secondary mathematics education program at a large state university in the southeast United States. We chose the students on a voluntary basis while they were enrolled in their first course in the education program. Bob and Mindy were the only students (out of 10) to volunteer for the study.

Stemming from radical constructivist theories of knowing and learning (Glasersfeld, 1995), we consider each individual's knowledge fundamentally unknowable to any other individual. We sought to build and test models of the students' thinking in an attempt to obtain viable models of the students' mathematics. To accomplish this goal we used a teaching experiment methodology (Steffe & Thompson, 2000). Each student participated individually in three 60-minute teaching sessions (six sessions total) taking place within a span of fourteen days. During their participation in the study, the two students did not attend the regular class sessions of the course in which they were enrolled. The lead author acted as the instructor for each teaching session, with the second and third authors acting as observers.

We used an open and axial coding approach (Strauss & Corbin, 1998) in combination with a conceptual analysis (Thompson, 2000) to analyze the data. We first characterized each student's thinking over the course of the study. We then compared and contrasted each student's thinking in order to determine how his or her thinking progressed during the teaching sessions. For instance, we compared and contrasted Bob's notions of the unit circle over the course of the study in an attempt to document shifts in his understanding of the unit circle. After conducting the same analysis of Mindy's notions of the unit circle, we juxtaposed the two students' progress in order to gain deeper insights into their ways of thinking.

## Results

During the first interview session, both students described the unit circle as a circle with a radius of "one," but struggled to use the unit circle when solving tasks involving angle measure and trigonometric functions. For example, when asked to determine an angle measure given a radius length and arc length measured in a number of inches, the students drew a second circle with a radius of "one" to represent the unit circle. After drawing a separate circle, Bob suggested dividing all given measures by the radius length, claiming that dividing the radius by the radius gave a numerical result of "one." He then divided the arc length (1.2 inches) by the radius (3.1 inches) to obtain 0.387. However, when attempting to give a meaning to this value, he became confused and explained, "I think it's in inches…could be in degrees" (Fig. 2), suggesting that his act of division did not stem from reasoning about the radius as a unit of measure.

When Mindy attempted to determine the measure of an angle in radians that cuts off an arc length of 6.6 inches given a circle with radius 2.4 inches, she responded, "I could simplify this by creating a unit circle." She followed this statement by drawing two distinct circles and claiming, "So this is our original circle and this is going to be a unit circle. We know that by nature, a unit circle is going to have a radius one. Because we are already given the unit, we can go ahead and say one inch." Similar to Bob's actions, Mindy drew a distinct circle to represent the unit circle. Differing from Bob, she claimed that this circle had a radius of one inch (the stated unit in the problem). Such an action suggests that she, like Bob, had not conceptualized the radius of the *given* circle as a unit of measure.

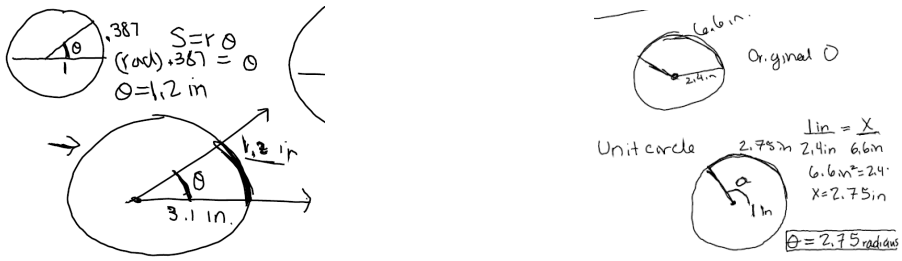


Figure 2 – Bob's (left) and Mindy's (right) Drawings of Unit Circles

As the interviewer pressed the students to justify their calculations during the initial teaching sessions, we observed the students comparing units for the values constituting the calculations. For instance, when the interviewer asked Mindy to explain her calculations, she immediately compared and cancelled the values' units, while claiming, "It's really about comfort because I don't like to do something unless I can see the units perfectly dividing out."

When attempting to model the circular motion of a Ferris wheel with a given radius in feet, the students' conception of the unit circle and unit conversion schemes hindered their ability to use the sine function to relate the arc length traveled by an object on the Ferris wheel and its vertical distance from the ground. Specifically, Bob and Mindy both attempted to relate the given circle (a circle with a radius of 36 feet) to the unit circle (which they again drew separately of their drawn Ferris wheel) by multiplying and dividing by the radius. However, the students expressed an uncertainty relative to the correctness of their calculations and failed to relate the input and output of the sine function to the Ferris wheel. As an example, when Bob calculated the output of the sine function for a specific input, he was unable to determine how the output value on the "unit circle" related to the given circle.

After the first interview sessions, we sought to support the students in connecting the unit circle to a circle of any given radius length by focusing on establishing new unit conversion schemes. To achieve this goal, we designed tasks that asked the students to reason about relationships between the measure of a quantity and the magnitude of the unit used to make the measure. For instance, we asked the students to compare the magnitude of one foot to that of one inch, and then use this relationship to convert a measure in feet to a measure in inches.

After exploring relationships between measures and magnitudes for standard units of measure, we had the students reconsider circular contexts, where both students showed progress in conceptualizing the radius as a unit of measure. For example, when given a radius of 3.5 feet and an arc length of 4.9 feet, Mindy explained, "We know that the radius is 3.5 feet so if the arc length is 4.9 feet we are trying to find how many radii that is. So we need to divide the 4.9 by 3.5 to figure out ok if 3.5 is our measure it's like our unit then how many of those 3.5's are going into 4.9." As opposed to drawing a separate "unit circle," Mindy provided this description by referring to the given circle and its radius. Similarly, Mindy solved a context-rich trigonometric problem (Fig. 3) without drawing a separate unit circle. Instead, she labeled the coordinates (1, 0) on the given circle and completed the problem by converting between measuring lengths in radii and the given unit. Mindy's actions suggest that her solutions, which consisted of reasoning about a circle of radius "one," stemmed from conceptualizing the radius as a unit of measure.

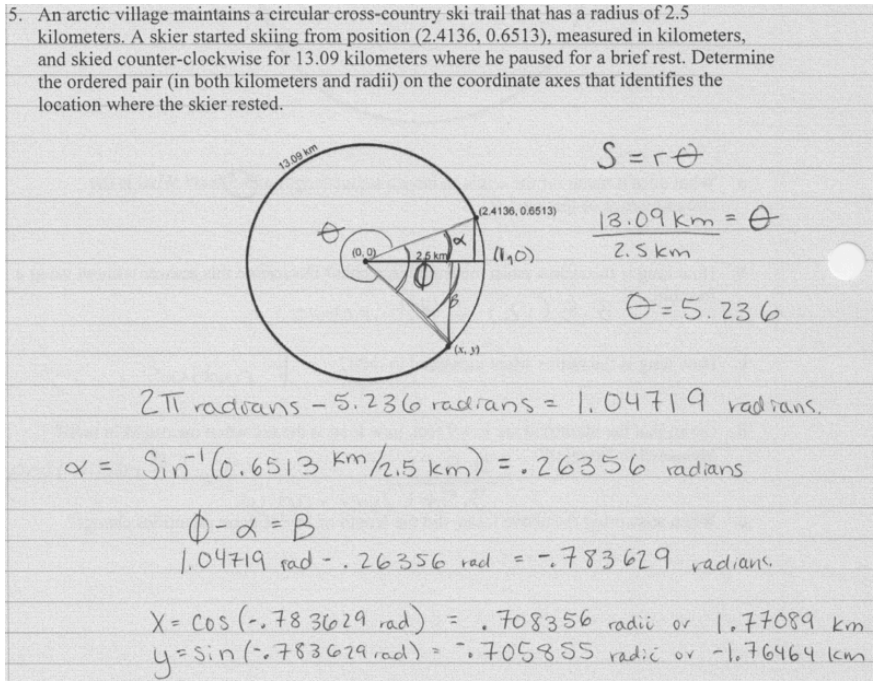


Figure 3 – Mindy Relabeling a Circle as the Unit Circle

Despite the shifts in the students' perception of the unit circle and radius as a unit of measure from the first session (where the unit circle was conceived as a distinct circle), we continued to observe relics from their ways of thinking exhibited during the first interview. As an example, Bob abstracted dividing by the radius as the calculation relating given measures to "the unit circle," but at times he did not conceive of the resulting values as measures in radii. For example, when given circles with different radius lengths and asked if he could determine which ones (if any) were unit circles, he stated, "Well…I guess in retrospect, or in theory, they could all be unit circles just be dividing by their corresponding lengths. 3 feet divided by 3 feet is one, 2.1 divided by 2.1 is one." Although this statement showed his ability to view each circle as the unit circle, it wasn't without continued interview prompting that he identified each "one" as representing one radius length.

## Conclusions and Implications

The students' difficulties at the onset of the study suggest that their initial conceptions of the unit circle did not support flexible reasoning about trigonometric functions or radian angle measures. Specifically, the students encountered obstacles in relating the unit circle (and trigonometric functions) to circles with a radius measure other than "one." We determined that their attempts to relate the unit circle to the given circles were not rooted in reasoning about the radius as a unit of measure, but rather the unit circle existed as a circle distinct from the given circle. Furthermore, their methods for unit conversions relied on unit-cancellation (or dimensional analysis) and did not provide a foundation for conceptualizing a circle's radius as a unit of measure. Our finding is compatible with Reed's (2006) observation that unit-cancellation can mask important mathematical ideas.

As we worked with the students to base unit conversions in reasoning about how the measure of a quantity changes as the magnitude used to measure the quantity changes, we noted shifts in their unit circle conceptions. The students' actions suggested that by conceptualizing a circle's radius as a unit of measure, they were able to view any given circle as the unit circle. The students no longer approached the unit circle as an object separate of a given circle. Stemming

from this shift in their notion of the unit circle, the students more fluently used trigonometric functions in novel circle contexts by reasoning about the input and output of these functions as representing measures in radii. Weber (2005) emphasized the importance of students coming to view the unit circle as a tool of reasoning, and the influence of students' unit conversion schemes on their notions of the unit circle provides insights into how to accomplish this goal.

Although we saw significant shifts in the students' ways of thinking over the course of the study, their previous notions of the unit circle and unit conversions remained apparent throughout the study. The deep-rooted nature of the students' ways of thinking (e.g., unit conversion through unit-cancellation) highlights the significant impact of pre-service teachers' schooling on their mathematical content knowledge. Previous to participating in a teacher preparation program, pre-service teachers likely encounter 12 to 15 years of mathematics courses, each of which influence their content knowledge. As our study reveals, these experiences can create obstacles that are necessary to address when attempting to shape the pre-service teachers' content knowledge.

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# Learning to Play Projective Geometry: An Embodied Approach to Undergraduate Geometry Learning

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## Abstract

*Research is beginning to investigate the role of embodiment in undergraduate mathematics education. This study builds on this intellectual trend by developing embodied theories of tool use in the context of an undergraduate geometry course. During this course, students explored the mathematics of projective geometry through (a) a designed object called Alberti's Window, (b) a Geometer's Sketchpad sketch, and (c) an art project inspired by the geometry of linear projection. This preliminary report relays early analyses of classroom and interview video recordings. The research team is analyzing the data from the emerging theoretical construct of a mathematical instrument, a material or semiotic object together with a diverse collection of perceptuomotor activities involved with its use. We explore how students come to coordinate multiple mathematical instruments in order to 'play' projective geometry. Audience discussion will address questions related to perception, embodiment, and mathematical art in relation to undergraduate geometry education.*

**Keywords:** embodied cognition, undergraduate education, projective geometry, tool use, mathematical imagination

## Theoretical Background

Our research team is engaging in ongoing efforts to develop an embodied theory of mathematical thinking and learning at the undergraduate level, a project we feel is particularly significant given current debates within cognitive science about the prospects of "scaling up" embodied theories to account for the kinds of complex cognitive activity found in, for example, an undergraduate mathematics classroom (Kirsh, 1991). Indeed, a small but growing educational literature is currently taking on this challenge of understanding undergraduate-, graduate-, and professional-level mathematical thinking and learning in a way that dissolves age-old dualisms between mind and body, and conceptual and perceptual (Marghetis & Núñez, 2010; Nemirovsky, Rasmussen, Sweeney, & Wawro, In press; Nemirovsky & Smith, 2011). A related area of research examines the role of gesture in undergraduate mathematics learning (Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004). In this study we aim to build on this important intellectual trend by exploring a novel way of theorizing the emergence of fluency with mathematical tools in the context of an undergraduate geometry class.

## Data Sources

This preliminary report is based on early analyses of video data from an undergraduate geometry class. During the fall semester of 2010, students enrolled in a Foundations of Geometry course at a large southwestern university. Data for this project include video recordings of six class sessions as well as video-recorded interviews with individual students. The students enrolled in the course were primarily pre-service secondary mathematics teachers, with the exception of a few mathematics majors and a few mathematics education masters students. The class culminated in student art projects inspired by the geometry of linear projection.

Projective geometry was motivated by the study of perspective art and the desire to create pieces of art true to linear perspective. In general, projective geometry involves how objects on one plane project onto a second plane, where the two planes intersect. The projection is determined by extending lines from a given point (your eye, for example) through one plane to the individual points on the object on the second plane. The points at which the extended lines intersect the first plane form the projection of the object on the second plane onto the first plane.

Undergraduates in the Foundations of Geometry course began their exploration of projective geometry ideas through the use of a physical *Alberti's Window*, consisting of a 12x12-inch rectangular piece of clear acrylic that stands perpendicular to a table and a moveable eyepiece through which the students view drawings or objects (see Figure 1). Students look through the eyepiece and trace on the window with a marker the drawing or object seen in front of them. Class discussions and additional activities also worked to extend the activity to the geometric theory of projection, including imagining the projection of points behind the eye and between the eye and the window.

Students were then introduced to a Geometer's Sketchpad version of an *Alberti's Window* to further explore the geometry of linear projection (see Figure 2). This GSP sketch consists of 3 manipulable lines – a base line, a horizon line, and an eye distance line – as well as a built-in transformation that sends a point P to its projection on the window, P'. Users create objects in the sketch, highlight the objects, then obtain the projection of the object by carrying out the built-in transformation.

Using the GSP program, students created an artistic design in the form of a GSP sketch fitting within a 9 in. x 14 in. frame. Students were required to use aspects of projective geometry discussed in the course to create their designs, however not all aspects of the design needed to be projected objects. Stencils with adhesive backings of each student's design were cut. Finally, students attached the stencil to airbrush paper and airbrushed their design in any way they desired.

## Early Analysis and Theoretical Developments

Our preliminary analysis efforts have focused on an effort to build new, embodied theories of tool use in the context of mathematical thinking and learning. Specifically, our commitment to mind-body nondualism inspires us to move away from dialectical theories of tool use that distinguish and seek correspondences between physically extended artifacts and their mental representations, including Vygotskian (1978, 1987) and neo-Vygotskian mediational theory and the instrumentalist theory proposed by Vérillon and Rabardel (1995). As an alternative, we offer the construct of a *mathematical instrument*, a material or semiotic object together with a diverse collection of perceptuomotor activities involved with its use, such as overt physical manipulation of the object, quasi-covert actions involved in imagining or anticipating using the object, and so on. In contrast to dialectical approaches, gaining fluency with a mathematical instrument does not entail the development of mental representations or schemes of utilization; instead, instrumental fluency emerges as a transformation in the lived bodies of mathematics learners (Noble, DiMattia, Nemirovsky, & Barros, 2006). The use of the term *instrument* intentionally evokes the culture of music; just as musicians learns to play mathematics through the incorporation of her chosen instrument into her lived body, mathematics learners come to 'play' mathematics as their bodies incorporate the tools of the discipline.

Early analyses of these data are being conducted using the techniques of grounded theory and microethnography (Corbin & Strauss, 2008; Erickson, 1996; Strauss & Corbin, 1994;

Streeck & Mehus, 2005). The analysis efforts aim to further elaborate on the notion of a mathematical instrument in the context of the undergraduate Foundations of Geometry course in which students are learning to 'play' several mathematical instruments in concert: the Geometer's Sketchpad sketch, *Alberti's Window*, and the artist's airbrush. To better understand how students learn to coordinate multiple instruments, we again draw on the culture of music to develop the construct of *polyphony*. In music, polyphony refers to a phenomenon in which multiple instruments play different parts to produce a musical whole; although each instrument yields a distinct melody, the multiplicity of melodies hangs together within the unity of the song. Similarly, we suggest that the undergraduate mathematics learners in our data must come to be able to 'play' projective geometry through a polyphone performance that involves the coordinated, fluent use of the Geometer's sketchpad sketch, *Alberti's Window*, and the airbrush.

Preliminary findings suggest two features of polyphony that we feel are significant for research in undergraduate mathematics education and, in particular, embodied approaches to understanding mathematical thinking and learning at the undergraduate level. First, analyses of both interview and classroom data from our undergraduate geometry class prompt us to question the often-naturalized assumption that bodies are single, unified entities (Mol, 2002). Instead, our findings begin to suggest a view of the body as irreducibly manifold, and of mathematical activity as comprised of multiple simultaneous streams of bodily involvement, such as talking, gesturing with left and right hands, posturing, moving the eyes, and so on. It is this manifold of ongoing bodily activity that, together with the material and/or semiotic objects at hand, is able to produce the complex polyphony that we feel (a) is a hallmark of mathematical expertise and (b) may provide an important link in contemporary efforts to 'scale up' embodied theory to complex cognitive domains like undergraduate mathematics. Second, analyses of interview data are beginning to suggest that polyphony is an important interactional site for the negotiation of the mathematical status of a given material or semiotic tool, and concomitantly, an important expressive avenue by which students articulate their own views about the nature of mathematics. For example, we are currently comparing interviews with students who did and did not view the artist's airbrush as explicitly mathematical.

## Audience Discussion Questions

We aim to engage the audience in a discussion about perception, embodiment, and mathematical art, in relation to undergraduate geometry education. Discussions will be grounded in short segments selected from video recordings of classroom activity and student interviews. Our three questions are:

1. How are students' everyday perceptual practices used and transformed by experiences with *Alberti's Window*?
2. What is the role of the body in undergraduate-level geometric thinking and learning? Specifically, how do students use their bodies and the material environment to understand visible projective relations as well as to imagine projections they can't see (e.g. the projection of points behind the eye)?
3. What new mathematical understandings can undergraduate students gain through the construction of original, mathematically inspired artworks? How do art-related activities prompt undergraduates to reveal, revisit, or alter their perspectives on the nature of mathematics?

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## Figures

*Figure 1:* Students using *Alberti's window.*

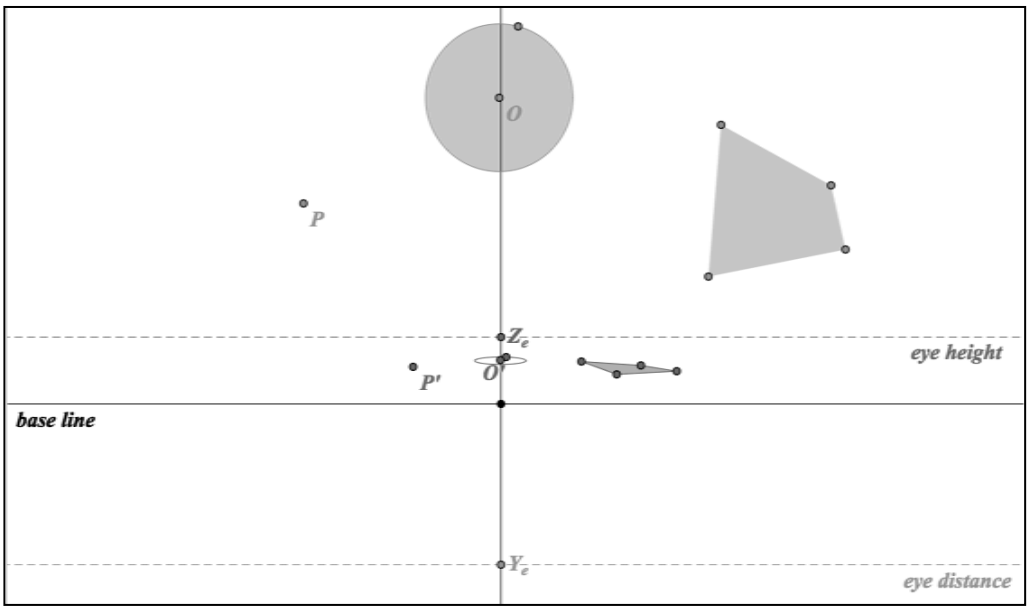


*Figure 2.* Geometer's Sketchpad sketch used to explore projective geometry.

# DISCURSIVE APPROACH TO STUDENTS' THINKING ABOUT THE DERIVATIVE

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*This study investigated characteristics of university calculus students' discourses on the derivative using a communicational approach to cognition. The data were collected from a survey and interviews in three calculus classes at a large public university in Midwest. During the interview, students were asked to explain their solution processes on the survey problems. The analysis of interviews focuses on students' descriptions about the derivative and the relationships between a function, the derivative function, and the derivative at a point. The results show that their descriptions were closely related to how they think about the derivative as a number and as a function. A common description of the derivative as a tangent line, which is a point-specific object but also a function defined on an interval was identified. This description was closely related to their use of word, "derivative" for both "the derivative function" and "the derivative at a point.*

*Key words:* Communicational Approach to Cognition, Calculus, and Derivative

## Introduction

Research in collegiate mathematics education has been growing over the past few years, especially about calculus learning (e.g., Carlson, Oehrtman, & Thompson, 2008; Speer, Smith, & Horvath, 2010). Among calculus concepts, the derivative is known as a difficult concept because its definition contains various other concepts such as ratio, limit, and function and the derivative can be represented in multiple ways (e.g., Thompson, 1994, Zandieh, 2000). Related to previous research, this study explores how students described the derivative and used the descriptions in tasked-based interview settings focusing on their use of the word, *derivative*. Unlike some languages (e.g., Korean or Japanese) *derivative* is colloquially used for both "the derivative function" and "the derivative at a point" in English. This observation suggested an ambiguity about what *derivative* refers to and the possibility for miscommunication between speakers using the words. This word use provided a motivation for this study that addresses following questions:

1. How do students describe the derivative at a point and the derivative of a function?
2. How do students describe or use the relationships between a function, the derivative at a point, and the derivative function?
3. How do students use the derivative function as a function?

Investigating students' thinking through their discourses can add new understanding to the current literature about the role that mathematical language plays in students' learning. There has been research about how word use is related to children's thinking about early mathematical concepts (e.g., Fuson & Kwon, 1992; Sfard, 2008), but few studies have been done in advanced concepts. An explanation about how their use of the word and their thinking about the derivative may extend our understanding of the role that language plays in students' learning of an advanced concept, the derivative, and suggest instructional guide for use of mathematical terms.

## Literature Review

Existing research has explored how students think about the concepts included in the derivative and its various representations. For example, studies found that students' thinking

about the derivative is related to their procedural understanding of the average rate of change (Hauger, 1998; Orton, 1983), and how they relate the rate of change at a point on an interval [a, b] to the change in a function on [a, b] (Thompson, 1994) and think about the variables of a function (Thompson, 1994). They also explored students' thinking about the derivative function as a function (Monk, 1994), and differentiability of functions (Ferrini-Mundy & Graham, 1994). Other studies reported students' levels of thinking about different representations of the derivative, and trouble appreciating their relationships (Santos & Thomas, 2002; Zandieh, 2000).

Although these studies have contributed to our understanding of how students think about the derivative, they have not explored this topic in terms of a) students' thinking about a function b) and their use of the terms related to the derivative. Students' misconceptions about velocity or acceleration (Bezuidenhout, 1998) can be addressed in terms of how they relate a position function to its first and second derivative functions. Since the derivative is derived from a function, how students relate these concepts and use the relations would be crucial to expand our understanding about their thinking about the derivative.

Existing studies also have not explored use of mathematical terms related to the derivative in a systemic way. In English, the word *derivative* can be used both for *the derivative at a point* and *the derivative function*, and such word use may lead to students' incorrect thinking (Park, 2011). For example, a student's error in determining the differentiability of a piecewise function $f(x) = \begin{cases} g(x), & x > a \\ h(x), & x \leq a \end{cases}$ (Ferrini-Mundy &Graham, 1994) may come from her confusion between *the derivative function* that has two equations for each side of $x = a$, and *the derivative at a point* that has the same value from each of the sides, which leads to an incorrect statement that a discontinuous function is differentiable. Exploring students' word use may provide explanations about their incorrect thinking about the derivative. To increase our understanding about these two unexplored questions, this study investigates students' discourse on the derivative while they explain the meaning of the derivative and apply their explanations to solve derivative problems.

## Theoretical Background

This study explores students' discourse on the derivative based on the *communicational approach to cognition,* which views thinking as an "individualized version of interpersonal communication" and mathematics as a form of discourse (Sfard, 2008, p. xvii). Mathematical discourses have four components: *Word use, Endorsed narrative, Visual mediators,* and *Routines* (Sfard, 2008). Among these, this study focuses on the first two components. Mathematical *words* signify objects such as numbers and geometrical shapes. A *word* used by different speakers could refer to different objects. *Narratives* are utterances that speakers can endorse as true or reject as false, and *endorsed narratives* refer to ones believed as true by speakers (Sfard, 2008, p.134). This study addressed students' endorsed narratives about the derivative, and their use of the word, *derivative*, when they described their concepts of the derivative. It also addresses their uses of four topics: a) relation between a function $f(x)$ and its derivative function $f'(x)$, b) relation between $f(x)$ and the derivative at a point $f'(a)$, and c) relation between $f'(x)$ and $f'(a)$, and d) $f'(x)$ as a function (Table 1) while they were justifying their solution processes of the survey items.

Table 1

*Cases for the Relationships among $f(x)$, $f'(x)$,* and $f'(a)$ and $f'(x)$ *as Function*

| Categories | | Cases |
|---|---|---|
| Relationship between $f(x)$ and $f'(x)$ | | Graphing or describing $f(x)$ using the sign of $f'(x)$ on an interval |
| | | Proving the differentiation rules |
| | | Interpreting the chain rule as a product of rates of change |
| | | Finding anti-derivative algebraically or from graphs |
| | | Finding the concavity of $f(x)$ using if $f'(x)$ increases or decreases |
| Relationship between $f(x)$ and $f'(a)$ | | Graphing $f'(x)$ using values of the slope of tangent line to the graph of $y = f(x)$ at points on the domain |
| | | Determining the differentiability of $f(x)$ using $f'(a)$ |
| | | Specifying $f'(a) = 0$ when $f(x)$ has an extreme value at $x = a$ |
| Relation-ship between $f'(x)$ and $f'(a)$ | $f'(a)$ as a value of $f'(x)$ | Mentioning that $f'(a)$ is a value of $f'(x)$ at $x = a$ |
| | | Difference between non-differentiability of function and at a point |
| | | Interpreting a point on the graph of $f'(x)$ as the slope of a tangent line |
| | Transition: $f'(a)$ to $f'(x)$ | Mentioning that several values of the derivative at points form the derivative of a function |
| | Transition: $f'(x)$ to $f'(a)$ | Finding the equation of $f'(x)$ and then substituting a number to evaluate $f'(a)$ |
| $f'(x)$ as a function | | Mentioning that $f'(x)$ is a function as a part of its definition. |
| | | Mentioning that $f'(x)$ is a function that one could graph. |
| | | Mentioning that $f'(x)$ is as a function that has its own derivative, $f''(x)$ |

## Design of Study

This study is part of a larger study consisting of classroom observation, student survey, and interviews with instructors and students. Three calculus classes at a large public university in Midwest were observed for six weeks for the derivative unit. In the end of the unit, a survey was administered to the students in the classrooms, and interviews were conducted with instructors and students after the survey. The students were selected for interviews based on the survey data. Sfard's (2008) framework was used to analyze instructors' and students' discourses. This paper reports students' responses on the survey and their discourses during interviews. This section addresses a) survey and scoring, b) recruiting and interviewing students, and c) analyzing data.

The survey consisted of questions about students' mathematics background and mathematical items involving a function, the derivative function, and the derivative at a point (Appendix 1). Most items came from the Calculus Concept Inventory (Epstein, 2006), which included item reliability. Other items were reviewed by three mathematics professors. In the three classes, 88 of 99 enrolled students took the survey for 20 minutes in an exchange of 20 extra credit points out of 700 total. Two types of scores, raw and frequency, were calculated. Raw scores were based on correctness, and frequency scores were based on all students' responses in each class. For open-ended items, I coded students' responses into categories using the rubric I created. The maximum possible raw score was 23. For the frequency scores, I assigned 2 points for the most popular responses for an item (say $n$ students select that response). If there was a response selected by

more than $n/2$ students, I assigned 1 point for the response. If there were two (or more) most popular responses (say $m$ students select each of those choices), I assigned 2 points for each response, and 1 point for a response that more than $m/2$ students selected. A student whose answers coincided with the most popular answers on all the problems received 32 points.

From each section, four students were invited for interviews based on their survey responses. The raw scores were used to find a heterogeneous performance group based on their survey performance. Frequency scores were used to find students whose answers were similar to the answers most commonly chosen by other students in the classroom. As shown in Table 1, most students interviewed from Instructor Alan's class had high raw scores (above 16 out of 23), most students interviewed from Instructor Ian's class had low raw scores (below 17), and there was a wide range of scores in interviewees from Instructor Tyler's class. Ten of the 12 students had studied the derivative in Advanced Placement Calculus in high school (Table 2).

Table 2

*Students Interviewed*

| Instr-uctor | Name | Gender | Major | First Math Class Including Derivative | Raw Score | Frequency Score |
|---|---|---|---|---|---|---|
| Alan | Cole | M | Pre-med | Pre-calculus in HS | 17 | 25 |
| Alan | Zion | M | Chemical Engineering | Pre-calculus in HS | 17 | 28 |
| Alan | Bill | M | Engineering | Pre-calculus in HS | 18 | 27 |
| Alan | Joe | M | Civil Engineering | AP calculus in HS | 21 | 26 |
| Tyler | Roy | M | Mathematics | Calculus I | 8 | 20 |
| Tyler | Liz | F | Med-Tech | Calculus in HS | 11 | 22 |
| Tyler | Zack | M | Computer Science | Pre-calculus in HS | 15 | 26 |
| Tyler | Neal | M | Computer Science | Calculus in HS | 20 | 31 |
| Ian | Sara | F | Biology | Calculus I | 8 | 17 |
| Ian | Mary | F | Genomics and Genetics | Pre-calculus in HS | 13 | 21 |
| Ian | Mona | F | Natural Science | Pre-calculus in HS | 13 | 24 |
| Ian | Clio | F | Astrophysics | Pre-calculus in HS | 16 | 21 |

*Note.* In the table, AP, and HS refer to Advanced Placement and high school, respectively.

Task-based semi-structured interviews were conducted individually lasting for about an hour. During the interview, students were asked to answer warm-up questions (Appendix 2) about the derivative using their own words, and how they solved survey problems. Follow-up questions to their initial responses were focused on whether and how they used the relationships among a function, the derivative function, and the derivative at a point in their problem solving processes. Interviews were transcribed and coded with Transana (Woods & Fassnacht, 2007). Table 2 was used as a coding table to identify the cases when students described or used these topics to solve problems. The cases were analyzed focusing on their word uses and endorsed narratives

## Findings

*Definitions of $f'(x)$ and $f'(a)$*

During the interview, students were asked, "What is the derivative?" and then asked if their description was closer to the derivative function $f'(x)$ or the derivative at a point $f'(a)$. After choosing one, they were asked to describe the other concept. Table 3 shows students' choices between $f'(x)$ and $f'(a)$ with frequencies and examples.

Table 3

*Students' choice between $f'(x)$ and $f'(a)$, Number of Students, and Examples*

| Student Choice | Number of Students (out of 12) | Examples |
|---|---|---|
| $f'(x)$ | 7 | "You take it [$f(x)$] and derive the new equation…it describes…the slope," "Graphical indication of every single point throughout a graph" |
| $f'(a)$ | 3 | "Slope of a function at a point, "Instantaneous slope at the point" |
| Both $f'(x)$ and $f'(a)$ | 2 | "The slope…it could be both [$f'(x)$ and $f'(a)$]" and "What the coordinates [of $f'(x)$] are at the given point…just plug it in" |

The most common interpretation of the derivative function, and the derivative at a point was the slope. The second and third common interpretations were differentiation rules and the rate of change of a function, respectively. Ten of the students explained $f'(x)$ and $f'(a)$ in the same way. The other two students stated that "*the derivative*" was defined only on an interval, and were not able to explain the derivative at a point. This description suggests that these students considered *the derivative* only as a function on an interval, not a number, a point-specific value.

When the students asked if and how $f'(x)$ and $f'(a)$ are related, nine students explained $f'(a)$ as a value of $f'(x)$ at a point (e.g., "slope at a point" & "velocity at a point" for $f'(a)$, and "slopes at all the points" & "velocity over time" for $f'(x)$). Two students mentioned how to calculate $f'(a)$ by "plugging-in." One of them also incorrectly explained that $f'(a)$ could be also calculated by "plug[ging]" $x = a$ in $f(x)$. To explain if and how $f'(x)$ and $f'(a)$ are related to $f(x)$, they mostly repeated their descriptions of the derivative. Four students interpreted that $f'(x)$ and $f'(a)$ as the indicator of the behavior of $f(x)$ (e.g., "If $f'(x)$ on an interval is negative/positive, $f(x)$ decreases/increases on the interval," or "If $f'(a)$ is negative/positive, $f(x)$ decreases/increases at $x = a$"). A student incorrectly explained $f'(x)$ as "extension or contraction of" (stretching/shrinking) $f(x)$.

*Relationships among $f(x)$, $f'(x)$, and $f'(a)$*

Analysis of students' justification on their solution process focused on whether and how they applied the relationships among $f(x)$, $f'(a)$, and $f'(x)$. They used the relationship between $f(x)$ and $f'(x)$ while applying the differentiation rules to compute $f'(x)$. Graphically, they described how $f(x)$ behaved based on the sign of $f'(x)$. However, one student, who correctly used and explained this relationship, could not apply it to the relation between $f'(x)$ and $f''(x)$. Only three students could correctly use the concavity of a function in relation to the behavior of its derivative.

Some incorrect uses of the relationship between $f(x)$ and $f'(x)$ were also identified. Two students stated that, "they [the graphs of $f(x)$ and $f'(x)$] move to the same direction," but applied it inconsistently; they went back and forth between the sign of the derivative as an indicator of behavior of $f(x)$ and the resemblance of the graphs of $f(x)$ and $f'(x)$. In problems 4 and 5, they stated and tried to use, "if the derivative is positive, then the function should be increasing," but

also said, "It [$f'(x)$] should be increasing here because $f(x)$ is increasing." Another student stated that all graphs of derivative functions are "linear or a piecewise linear" in problem 4, but used a concave-up curve for $f'(x)$ in problem 7. The most common incorrect notion (five students), was that $f'(x)$ increases/decreases if and only if $f(x)$ increases/decreases.

In the items about the relationship between $f(x)$ and $f'(a)$, ten students described $f'(a)$ as the slope of the tangent line to $f(x)$ at $x = a$, and used this description to find the extrema of $f(x)$ or describe the behavior of $f(x)$ at such points. However, when a problem asked to interpret $f'(a)$ in a context (Problem 1), only a few students distinguished the derivative at a point $f'(a)$ ("marginal cost at $q = 2$" in dollars/mile) from a value of a function at a point $f(a)$ (the cost in dollars) and recognized the different units. Five of 10 students, who correctly interpreted the relationship between $f(x)$ and $f'(a)$, could not distinguish $f'(a)$ as the rate of change of $f(x)$ (e.g., "The rate of price change of 2 miles of rope") from the change in $f(x)$ from $x = a$ to $x = a + 1$ (e.g., "How much more cost would be added for that one more unit…If we were to go to 3…$C'(2)$ would be the cost that we would have to add on"). Though $f'(a)$ can be estimated as $f(a + 1) - f(a)$, these answers show that students did not appreciate the difference between the two values. Graphically, five students interpreted $f'(a)$ as a tangent line of $f(x)$ at $x = a$. For example, a student, Joe who correctly explained most problems, wrote the derivative at a point as a tangent line in problem 8. He interpreted $f'(1)$ as the tangent line at $x = 1$ and said that it could not be compared to the graph of $f(x)$ because the tangent line is point-specific, whereas $f(x)$ is a whole function (Figure 1).

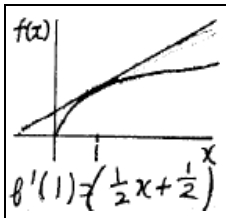
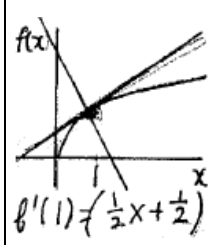
| | |
|---|---|
|  | Joe : I can try the tangent line at $x$=1, $y = ½\,x + ½$.What this means is that $f'(1)$ equals 1/2$x$+1/2, so the function they gave you is particular to the point, $x$ =1…If this only works if $x$=1, you need to have $f(1)$ somewhere…I didn't quite understand what these [choices] were trying to say. As far as manipulating this, I didn't see any $f(1)$…I can't really relate the function for a [tangent] line to the entire function of $f(x)$. It's only relevant at the point $x$ = 1.<br><br>… |
|  | Joe: This is graph of $f$ prime of $x$ (adding a decreasing line to his graph). That tells you the value for the slope at that point. The slope would be 1, if you plug in 1 for $½\,x + ½$.<br>Interviewer: What you wrote here, '$f'(1) = ½\,x + ½$' would be $f'(x)$?<br>Joe: I don't think so. To say that $f'(x)$ equals something…the slopes [would] cover the entire domain…I don't think this [line]…has any other connection to the graph of $f(x)$ besides the slope at that one point. |

*Figure 1.* Joe's Graphs of a Tangent line and $f'(x)$ and Explanation

As shown in Figure 1, he drew a decreasing line for $f'(x)$ and said, "this [line] tells you the value for the slope at that point." Later, he incorrectly found the slope of $f(x)$ at $x = 1$ by substituting $x = 1$ in the equation of the tangent line not in $f'(x)$, and gave another slope ½ from the equation, $y = ½\,x + ½$. When I asked which one is correct, he chose latter but changed the answer by saying "the tangent line is a representative of the slope at this point…I guess that this whole thing [pointing to $y = ½\,x + ½$] is the slope as opposed to just ½…it might be pretty wrong." In the same problem, two other students integrated the equation of the tangent line to find the equation

of $f(x)$, which suggests their inability to conceive of $f'(a)$ as a number.

All 12 students correctly addressed the relationship between $f'(x)$ and $f'(a)$ by interpreting or calculating $f'(a)$ as a value of $f'(x)$ at $x = a$. They also correctly applied it while calculating local extremes of $f(x)$. However, they did not use the substitution when a question asked beyond a simple computation. In problem 9, most students, who previously mentioned the derivative as a slope and calculated $f'(a)$ from $f'(x)$ using substitution, answered incorrectly or changed their choices several times. Three students incorrectly stated that "$a$" in $f'(x) = ax^2 + b$ is the slope and thus $a$ should be positive. Such response suggests that their ability to use substitution in a simple computation problem, does not always show that they consider $f'(a)$ as a value of $f'(x)$ at $x = a$.

*The Derivative Function as a Function*

Many students appeared to experience trouble explaining the derivative function $f'(x)$ as a function and an indicator of the behavior of $f(x)$. Five students stated that $f(x)$ increases/decreases if and only if $f'(x)$ increases/decreases, and thus their graphs resemble each other. Three of the five students described "derivative as a tangent line," because $f(x)$ and its tangent line at a point locally move in the same direction, which suggests that they conceived of "the derivative" as a mixed notion of a point-specific object and dynamic object over an interval without appreciating the relationship between $f'(x)$ and $f'(a)$. In other words, they did not appreciate a) $f'(a)$ as a number, a point-specific value of $f'(x)$; and b) $f'(x)$ as a function that consists of the derivative at points on the domain. Such descriptions were mostly identified when they used the word "derivative" without specifying it as "the derivative function" or "the derivative at a point."

Students also showed the difficulty identifying the independent variable of the derivative function. One student interpreted the independent variable of all derivative functions as time regardless of problem contexts. In problem 1, she interpreted $C'(2)$ as how fast the company made rope and gave the unit miles/sec. Another student stated that the independent variable of the derivative function was the rate of change of the independent variable of the original function.

## Discussion and Conclusion

This study contributes to the field of mathematics education by showing the importance of word use in relation to students' thinking about the derivative. Existing research in this area has shown that students have various *misconceptions* of the derivative and some possible reasons (e.g., lack of understanding of the concept of limits and their procedural understanding of the rate of change). Some other studies related specific types of students' misconceptions (e.g., assuming resemblance in the graphs of $y = f'(x)$ and $f(x)$) to the limited contexts used in calculus books (e.g., increasing distance function whose velocity is also increasing). Research has also reported students' thinking about a function focusing on its co-varying nature (Monk, 1994; Thompson, 1994). This current study expands our understanding about the derivative by looking at the features of their discourses about the derivative. Mathematically, two terms, *the derivative of a function,* and *the derivative at a point* are consistent with *function* and *function at a point.* However, the results of this study showed students' lack of understanding this consistency, which was closely related to their use of the word, *derivative.* Students showed a mixed concept of the derivative being a function defined on an interval, and a point-specific object simultaneously in graphical situations, which provide an explanation of their well-known students' misconception of the derivative, a tangent line. While describing or using this misconception, students used "derivative" without specifying the word as "the derivative at a point" and "the derivative function," which allowed them to change what the word referred to frequently even in one sentence. In their discourses, the word "derivative" was used not only as these two concepts, but also as "the tangent line" at a point. Also, students performed well on the items asking them to

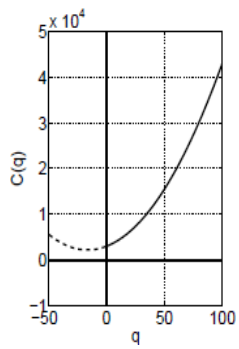
find the derivative at a point when the equation of the derivative of a function was given. However, their explanations on their solution process showed that they did not appreciate mathematical aspects behind the "plug-in" process or a sign of $f'(x)$ in relation to the behavior of $f(x)$ such as a) the derivative as a rate of change (the slope) describing the function behavior, b) the derivative at a point $f'(a)$ as a number, c) the derivative function $f'(x)$ as a function defined on an interval, and d) the relationship between $f'(a)$ and $f'(x)$: the former as a point-specific value of the latter. This lack of understanding was related with their incorrect endorsed narratives (e.g., if a function increases, the derivative increases) based on their concept of the derivative as the tangent line.

These results suggest that calculus instructors should be careful about the use of the mathematical terms such as *function, the derivative, the derivative function,* and *the derivative at a point* especially when they introduce the concept of the derivative of a function and the derivative at a point, make a transition between these two concepts, and address what these two concepts represent in terms of the original function. The analysis of calculus instructors' discourse is included in the paper for the larger project (Park, 2011).

## Appendix 1: Survey Questions

*Please solve the following problems and show your work.*

1. *C(q)* is the total cost (in dollars) required to set up a new rope factory and produce *q* miles of the rope. If the cost satisfies the equation $C(q)=3000+100q+3q^2$, and the graph is given as follows.




(a) Find the value of *C(2)*

(b) What are the units of 2 in (a)?

(c) What are the units of *C*(2)?

(d) What is the meaning of *C(2)* in the problem context?

(e) Find the value of *C(2)*.

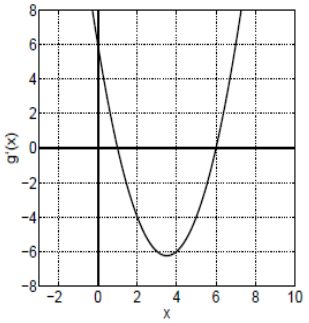
(f) What are the units for 2 in (e)?

(g) What are the units of *C'(2)*?

(h) What is the meaning of *C'(2)* in the problem context?

2. The derivative of a function *f,* is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?

3. The graph of the derivative, *g'(x)* of function *g* is given as follows. What is the value of *g'(2)?*




a) -4

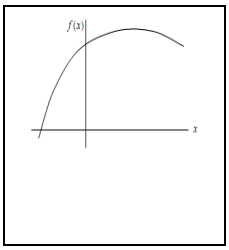
b) -2

c) 0

d) 2

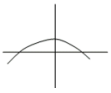
e) 4

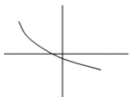
4. Below is the graph of a function *f(x)*, which choice a) to e) could be a graph of the derivative, *f '(x)*?




a) 

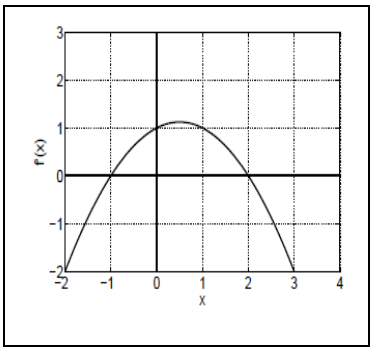
b) 

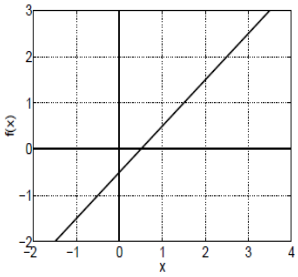
c) 

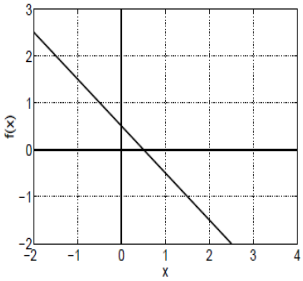
d) 

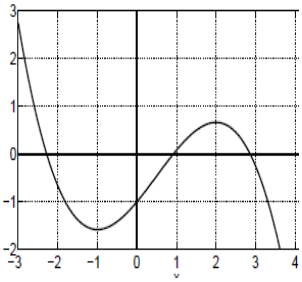
e) 

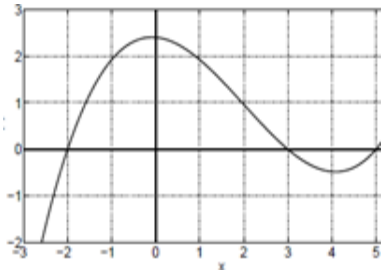
5. Below is the graph of the derivative *f '(x)* of a function *f(x)*. Which choice a) to e) could be a graph of the function *f(x)*?

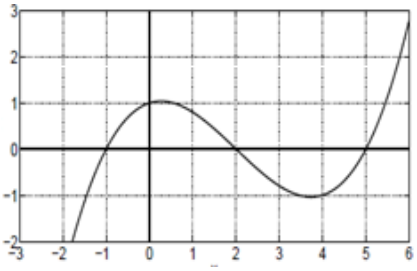



a) 


b) 


c) 

d) 

e) 

f) None of these

6. If a function is always positive, then what must be true about its derivative function?

a) The derivative function is always positive.

b) The derivative function is never negative.

c) The derivative function is increasing.

d) The derivative function is decreasing.

e) You can't conclude anything about the derivative function.

7. The derivative of a function *f(x)* is negative on the interval $x$=2 to $x$ =3. What is true for the function *f(x)*?

   a) The function *f(x)* is positive on this interval.

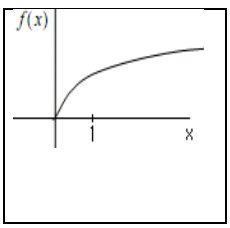
   b) The function *f(x)* is negative on this interval.

   c) The maximum value of the function *f(x)* over the interval occurs at $x$=2.

   d) The maximum value of the function *f(x)* over the interval occurs at $x$=3.

   e) We cannot tell any of the above.

8. Consider the graph below. The tangent line to this graph of *f(x)* at $x$ = 1 is given by $y = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?




   a) $\frac{1}{2}x + \frac{1}{2} = f(x)$ b) $\frac{1}{2}x + \frac{1}{2} \le f(x)$ c) $\frac{1}{2}x + \frac{1}{2} \ge f(x)$ d) $\frac{1}{2}x = \frac{1}{2}f(x)$

   e) None of these

9. The derivative of a function, *f,* is $f'(x) = ax^2 + b$. What is required of the values of $a$ and $b$ so that the slope of the tangent line to the function *f* will be positive at $x$ = 0.

   a) $a$ and $b$ must both be positive numbers.

   b) $a$ must be positive, while $b$ can be any real number.

   c) $a$ can be any real number, while $b$ must be positive.

   d) $a$ and $b$ can be any real numbers.

   e) None of these

   Why?

## Appendix 2: Warm-up Questions

Q1. What is the derivative? Can you make a sentence with the word, “derivative”?

Q2. What is the derivative of a function?

Q3. What is the derivative at a point?

Q4. Is there any relationship between the last two terms?

Q5. Is a function related to the derivative of a function or derivative at a point?

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# Teaching eigenvalues and eigenvectors with a modeling approach

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*This investigation reports a classroom experience in which eigenvalues, eigenvectors, and eigenspaces were taught using Models and Modeling and APOS Theory. Models and Modeling was used to design a problem in a realistic context and, with the genetic decomposition of APOS Theory, activities were designed to help students make the mental constructions needed to have a better understanding.*

*The research question is: Is it possible for students to construct an object conception of eigenvalues, eigenvectors, and eigenspaces when they are taught using a didactical design based on Models and Modeling and APOS Theory? Using one team as a case study, team A, we describe work done by students and show that two students were able to construct an object conception of these concepts. We also show that both theoretical frameworks can be used in an integrated way and students learned the mathematical concepts in a more meaningful way.*

*Keywords*: eigenvalues, eigenvectors, case study, APOS Theory, Models

Linear Algebra has become an important subject at university level because of its many applications. However, as researched by Larson, Rasmussen, Zandieh, Smith, Nelipovich (2007), Sierpinska (2000), Dorier (2002), Possani, Trigueros, Preciado y Lozano (2010), Tucker (1993), among others, students find it a subject hard to understand and likewise, many teachers report frustration when they find out these same problems in their students.

In order to contribute to the understanding of the teaching and learning of Linear Algebra, we are involved in a project where we use models to introduce students to Linear Algebra concepts. Our aim of the part of the project we report here is to study the possibility of introducing eigenvalues and eigenvectors in a Linear Algebra course through the use of models and activities designed using APOS (Action, Process, Object, Schema) Theory. The research question we discuss here is: Is it possible for students to construct an object conception of eigenvalues, eigenvectors, and eigenspaces when they are taught using a didactical design based on Models and Modeling and APOS Theory?

## *Theoretical framework*

The theoretical framework used in this study involves two complementary theories: Models and Modeling and APOS Theory.

Models and Modeling is a theoretical approach that encourages students to develop a model from a real situation. This model helps them to construct mathematical ideas that can be used to introduce new concepts (Lesh and Doerr, 2003). APOS Theory was developed to understand how mathematics can be learned by explaining phenomena that can be observed on students who are trying to construct mathematical concepts and by suggesting pedagogical activities that can help in this learning process. This theory (Dubinsky and McDonald, 2001) proposes a set of mental constructions that a student might make in order to understand mathematical concepts. This detailed description is called genetic decomposition. In Figure 1 we show a diagram of the genetic decomposition we designed for the construction of the concepts of eigenvalues, eigenvectors, and eigenspaces.

Following, we give a brief description:

Previous knowledge required: operations with vectors and vectors as objects, matrices as objects, their operations and properties, systems of equations as process, determinant as process.

- Construction of a process that interiorizes the actions needed to construct the equation $\boldsymbol{A\bar{v} = \lambda\bar{v}}$.
- Coordination of that process with the processes that result in an algebraic and geometric representation of the equation.
- Construction of processes to find eigenvalues and eigenvectors of a given matrix.
- Coordination of all the previous processes in a new process to find eigenspaces.
- Encapsulation of eigenvalues, eigenvectors, and eigenspaces into objects.

We used Models and Modeling to design a problem in a realistic context. Students, using their previous knowledge, worked on the problem to find a model that represents the situation. When they worked with the model, they needed new concepts. Using APOS Theory, we designed activities based on the genetic decomposition to guide them in the construction of the concepts.

Models and modeling is used to study the activity generated by the model in the classroom and the viability of the problem itself. APOS theory is used to study the mental constructions students make in this process. In this way, the two theoretical frameworks complemented each other.

***Methodology***

One of the researchers played the role of the teacher since we considered it would be difficult for another teacher to test the model and the activities in its first use and, if someone else taught the course, that person would need a lot of preparation to get to know the model and the activities. In this report, we present the results of the research experience describing one case study. The case describes the analysis of the work of team A, one of the three teams that successfully solved the model and whose participants worked on the proposed activities, and in a follow up interview, two of the members of team A demonstrated an object conception of eigenvalues, eigenvectors, and eigenspaces. Our purpose is to show what can be accomplished using a didactical design as the one described before. We also describe, more generally, some of the difficulties faced by some of the teams.

The students involved in the research are studying Linear Algebra as part of an Economics degree at a private university in Mexico. The 30 students were divided into 8 teams of 3 or 4 students each. They worked collaboratively on the modeling and conceptual activities. Students were given the following problem.

> *In the Treasury Department, where you work, they ask you to make a model to explain how many employed and unemployed people exist in certain period. Let* ***p*** *represent the probability that an unemployed person finds a job in any given period and* ***q*** *the probability that an employed person continues to be employed. Suppose you have the data of the number of employed and unemployed in the last months and the probabilities* ***p*** *and* ***q*** *have been estimated.*

Students first read the case and could ask questions to the teacher to ensure an understanding of the problem. The teacher led a discussion to answer any doubts. Students started working with the situation. About twenty minutes later, three or four teams explained their model and progress to the whole group, and the teacher led a whole class discussion, where the teacher and the other students asked questions. After listening to comments from others, the teams

continued working on their own models. This cycle was repeated twice, after which, the class decided to use one of the proposed mathematical models that seemed appropriate for the situation.

Students worked on a solution for the model. The teacher helped the students by providing some values for the parameters of the model to make it more specific and easier to solve. Students used their previous knowledge when they faced difficulties and, the teacher, when she considered it necessary, provided activities designed using the genetic decomposition to guide the construction of the concepts of eigenvalue, eigenvector, and eigenspace, and their relation with the matrix $\boldsymbol{A}$. Some conceptual activities helped them build the concept of eigenvalue and eigenvector, $\boldsymbol{A\bar{v} = \lambda\bar{v}}$. Others had the purpose of building a relationship between the geometric and the algebraic interpretation. After working with the activities another whole class discussion was conducted to discuss the activities and to introduce related definitions and theorems. Then, students went back to work on the modeling situation. The researcher's hypothesis was that students would use the newly constructed actions, processes and objects to finish their work on the model. The teacher then led a final discussion with the whole class on the solution process and gave the students similar models as homework.

All the productions of the teams were collected and their work was videotaped and transcribed. The two researchers then analyzed the data independently and results were negotiated between them. After this class experience, interviews were conducted, videotaped and transcribed. This data was also reviewed by the two researchers to study students' constructions.

***Results***

Several teams needed ideas from the teacher to start working on the problem and explore possibilities. The team we followed in this case study, team A, in contrast suggested, not without hesitation, a model of the form:

$$x_{t+1} = qx_t + py_t$$
$$y_{t+1} = (1-q)x_t + (1-p)y_t$$

with $x_t$ representing the employed and $y_t$ the unemployed in period $t$. Team A was able to explain the model to the whole class and, actually convinced them to use it as "the model".

When solving the mathematical model, team A, and another two teams generalized the solution they had previously found for a population model using vectors and matrices: "*something like* $\bar{x}_t = \lambda^t\bar{v}_0$, *the one we used in the population growth model."* The teacher asked them to verify this solution. By doing actions on this solution, they arrived at the equation $\boldsymbol{A\bar{v} = \lambda\bar{v}}$ and wrote the system of equations equivalent to $(\boldsymbol{A - \lambda I})\bar{v} = \bar{0}$, demonstrating a process conception for systems of equations. At this moment, the teacher gave them values for the parameters and they continued working. They spent time arguing that the solution of the system was $\bar{0}$, but one member of team A said: *"the solution depends on* $\boldsymbol{\lambda}$*, and it makes no sense to have a zero vector as solution; the system can have multiple solutions. Why don't we solve it and see?*" Then they used the condition $|\boldsymbol{A - \lambda I}| = \boldsymbol{0}$, showing an object conception of the solution set of a system and said *"for* $\lambda_1 = 1, \bar{v} = (2x_2/3, x_2)$ *and for* $\lambda_2 = 1/6$, $\bar{v} = (-x_2, x_2)$*."* Besides, *"we found a particular case for* $\lambda_1 = 1$, $\bar{v}_1 = (2, 3)$ *and for* $\lambda_2 = 1/6$, $\bar{v}_2 = (-1, 1)$*."* With these sets, they found the vector space spanned by these vectors, *"*$gen\{\bar{v}_1\} \Longrightarrow$ *line and* $gen\{\bar{v}_2\} \Longrightarrow$ *line"*, which shows they were able to coordinate the process of span with its geometrical representation. Then one student in team A said: *"…this is great, but we have different solutions, and which one is the correct one?"* As many of them had the same question, the issue was discussed with the whole class. During discussion, several students

remembered another model used in class, "the magic carpet model" (Wawro, Zandieh, Sweeney, Larson, and Rasmussen 2011), and one of them exclaimed: *"We could then take one vector from each family, and form a linear combination that would span a vector space,* $\mathbb{R}^2$*, perhaps?"*

The teacher then asked the students if the linear combination was also a solution and worked with them to prove it. She used a diagram (Figure 2) to explain the relation of the solutions of the system to their geometric representation. Although the diagram is not discrete as the model being worked, it was useful. Then, she explained them that the vector space generated was a space of solutions of the difference equations. The students started working with the activities designed using APOS Theory at this moment and, later on, they discussed and formalized what they had found in terms of eigenvalues, eigenvectors, and eigenspaces.

New activities designed with APOS Theory were used so students could relate the new concept, eigenvalue and eigenvector, to a graphical representation where they could observe that the definition of eigenvalue and eigenvector, $\boldsymbol{A\bar{v} = \lambda\bar{v}}$, means that the product $\boldsymbol{A\bar{v}}$ is a parallel vector of $\boldsymbol{\bar{v}}$, $\boldsymbol{\lambda\bar{v}}$. Many approaches to these concepts ignore the geometric aspects. One student in team A said: *"when you multiply* $\boldsymbol{A\bar{v}}$ *you don't change the direction, it is a parallel vector to* $\boldsymbol{\bar{v}}$*"*, showing he was able to coordinate the algebraic equation with its geometric representation (process).

In the conceptual activities, there were some where new concepts were related to previous ones. After working on them, students of team A discussed *"*$A_1$*: so* $\boldsymbol{\bar{v}_1}$ *and* $\boldsymbol{\bar{v}_2}$ *form a basis?* $A_2$*: yes and so the linear combination generates many solutions."* Then they asked the teacher "*which is the correct solution to give to the treasury?"* The teacher explained the role of initial conditions using the same diagram (figure 2).

Finally, they figured out what happened in the long term, and concluded: "*when* $\boldsymbol{t} \to \infty$, $\boldsymbol{(1/6)^t \to 0}$ *and we will have that* $\boldsymbol{\bar{x}_{t+1}}$ *is a multiple of* $\boldsymbol{\bar{v}_1 = (2, 3)}$*, we have a ratio of 2 employed and 3 unemployed."*

After some weeks, the researchers interviewed nine students and asked them questions about eigenvalues, eigenvectors, eigenspaces, and their geometric interpretation. Those in team A gave all the correct answers for traditional questions, while others showed difficulties with some questions. Throughout the interviews, two students in team A showed an object conception of these concepts. For example, when facing a question where there was only one eigenvalue with multiplicity 2, $A_2$ said: *"there is only one eigenvector … no! because you really have two linear independent vectors for that* $\boldsymbol{\lambda}$*."* In the question that most students had problems with, which involved matrix A

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{5} & \mathbf{5} & \mathbf{5} \\ \mathbf{5} & \mathbf{5} & \mathbf{5} \\ \mathbf{5} & \mathbf{5} & \mathbf{5} \end{pmatrix},$$

and where students were asked about one eigenvalue and its eigenvectors without doing operations, student $A_4$ demonstrated he had constructed an object conception by answering "$\boldsymbol{\lambda = 0}$*, because the vectors are linearly dependent*." Later, when working with a question related to a Markov Process $A_4$ said: *"the steady state vector is an eigenvector, and* $\boldsymbol{\lambda}$ *needs to be* $\boldsymbol{1}$ *since* $\boldsymbol{P\bar{w} = 1\bar{w}}$*."*

This interview showed evidence that two of the students of the case study, team A, had constructed an object conception of these concepts through their work with the model and the conceptual activities. The other members of team A showed evidence of having constructed an object conception of eigenvalues and eigenvectors, but a process conception of eigenspaces.

***Conclusions***

The learning and teaching of eigenvalues, eigenvectors, and eigenspaces is challenging for both students and teachers – in fact appearing "magically" from definitions and theorems. We propose an approach based on Models and Modeling and APOS Theory that appears to be very promising. The model was suitable to give sense in context to eigenvalues, eigenvectors, and eigenspaces. The use of a previous model, a population model, related to other situation helped students in finding an analogy which was very useful to find a suitable model and a possible solution.

Results showed that students who constructed an object conception could relate the algebraic and geometric aspects of eigenvalues, eigenvectors, and eigenspaces. The activities designed for this experience stressed them.

Three teams, like team A reported here, followed, with certain difficulties, all the teaching process associated with the model, which makes us think that the design of the model and the activities helped them to construct the concepts of eigenvalues, eigenvectors, and eigenspaces. As demonstrated in the interviews, some students still had problems with these concepts. Some constructed a process conception and a few of them an action conception.

The definition of eigenvalue and eigenvector $\boldsymbol{A\overline{v} = \lambda\overline{v}}$ is very difficult for students since the two sides of the equation represent different mathematical processes, and at the same time both sides represent the same vector. Stewart and Thomas (2007) talk about the importance of using algebraic and geometric representations of eigenvalues and eigenvectors to teach these concepts and conclude that eigenvalues and eigenvectors are probably the most difficult part of a Linear Algebra course. Our results agree with this conclusion although, we found that the modeling experience together with the designed activities, helped most of the students in the group to coordinate the processes involved in the equation and give meaning to the intended concepts.

Larson, Zandieh, and Rasmussen (2008) comment on the difficulty involved in the transformation from $\boldsymbol{A\overline{v} = \lambda\overline{v}}$ to $\boldsymbol{(A - \lambda I)\overline{v} = \overline{0}}$ and then to $\boldsymbol{|A - \lambda I| = 0.}$ We also found this difficulty in some students, but the case study members of team A and other students, did the transformation and used determinants by themselves as a means to verify the solution to the mathematical model. Work on the model seemed to focus their attention on properties of systems of equations they had already studied.

We can conclude that this experience was useful to help some students develop an object conception of eigenvalue, eigenvector, and eigenspace concepts. Although not all the students developed such conception, most of the interviewed students showed a better understanding of these concepts than students in previous groups we had taught before. The use of the model together with the activities designed using APOS theory contributed to this understanding.

Acknowledgement.
This project was funded by CONACyT Grant No. 62375. We would also want to thank Asociación Mexicana de Cultura A.C. and ITAM for support.

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***Figures***

*Figure 1. Genetic decomposition of eigenvalues, eigenvectors, and eigenspaces.*

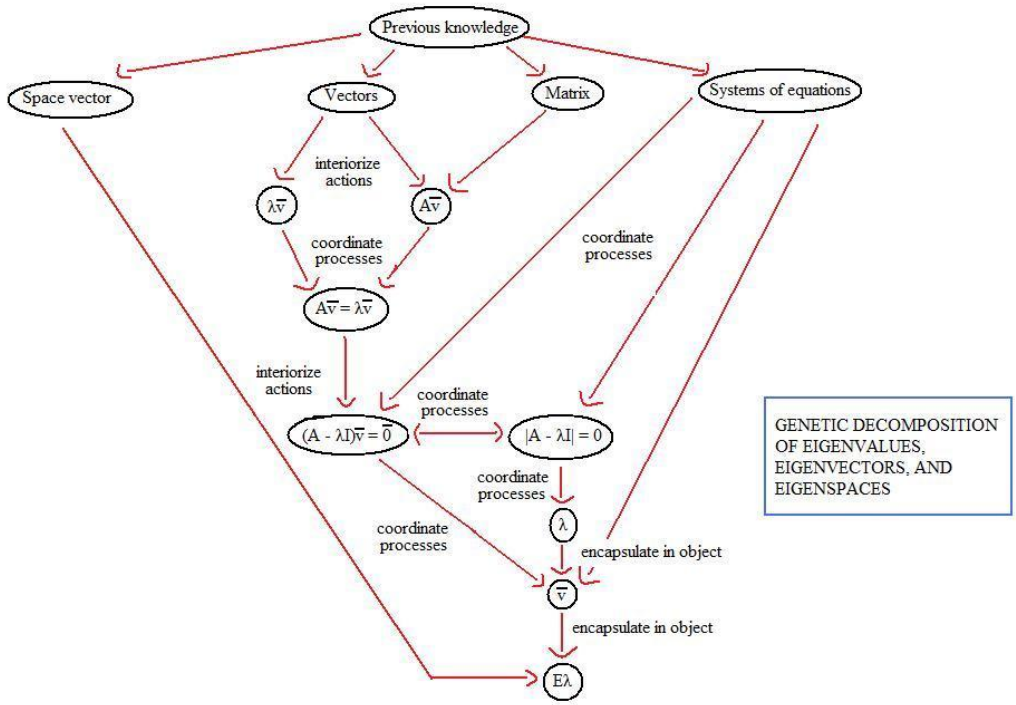
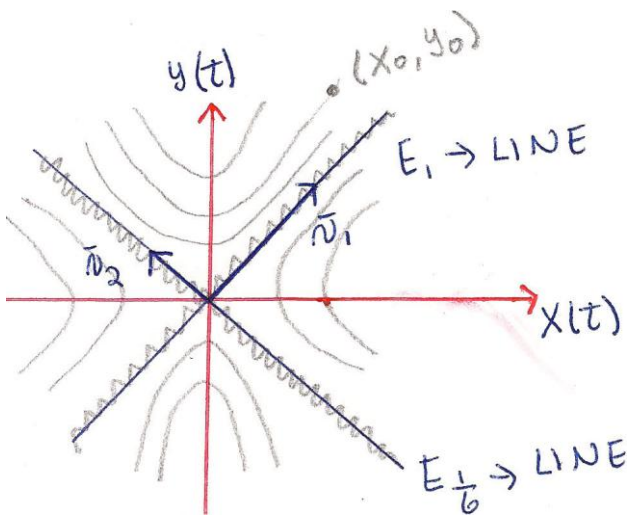



*Figure 2. An explanation for solution families.*



# The Effectiveness of Local Linearity as a Cognitive Root for the Derivative in a Redesigned First-Semester Calculus Course

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Abstract

In this report we investigate an innovative, reorganized curriculum for first-semester calculus which emphasizes local linearity and uses it as the fundamental principle on which the rest of the curriculum is based. Technology and visualization are used as tools for guided discovery of local linearity and other aspects of calculus. How students used local linearity as a cognitive root for the derivative will be discussed. Student learning outcomes will also be presented, with some examples of student work demonstrating the results of the approach.

Keywords: local linearity, cognitive root, calculus, technology, mathlet

## Background and Theoretical Perspective

Previous research into student difficulties in calculus has addressed student conceptions and misconceptions of limits (Oehrtman, 2002; Tall, 1992; Bezuidenhout, 2001a), of the derivative (Ubuz, 2007; Bezuidenhout 2001b), and of the integral (Bezuidenhout & Olivier, 2000; Orton, 1983). The most intransigent of the difficulties occur with limits (Tall & Vinner 1981; Cottrill et al., 1996). Further, many students cannot say why limits are important in calculus (Davis & Vinner, 1986). Students have formed robust concept images (in the sense of Tall & Vinner, 1981) of the derivative which do not feature limits (Frid, 1994). Tall (1991) defined a cognitive root to be a unit of knowledge which is a meaningful part of prior knowledge for the learner and allows for further theoretical development, and he referred to local linearity as the proper cognitive root of the derivative (Tall, 1992). Several studies have observed that students can successfully be introduced to the derivative using local linearity instead of limits (Tall, 1986; Maschietto, 2002).

Mathematical situations may be represented in symbolic, graphical, numerical, and verbal ways (Hughes-Hallett, 1991). It is important for students to be able to use each of these representations, and to translate between them (Tall, 1991). There are specific problem-solving benefits to the graphical representation (Hershkowitz & Kieran, 2001; Larkin & Simon, 1987). In particular, more frequent use of visual methods has been associated with higher performance in calculus (Haciomeroglu et al., 2010).

Java applets and similar computer applications for mathematics instruction, known as mathlets (Roby, 2001), have led to improved learning outcomes in mathematics education (Kidron et al., 2001; Heath, 2002). A mathlet is a small platform-independent application, typically interfaced through a web browser, offering interactive tools to explore a particular mathematics topic. Prior research has identified three main reasons for the potential positive impact of applets throughout mathematics education, and particularly in calculus. First, dynamic interaction is beneficial to the learner because of immediate feedback and the capacity to explore (Arcavi & Hadas, 2000); second, lack of dissemination and other logistical difficulties have, in the past, been obstacles to instructional change, but this is not a problem with applets because they can be accessed for free on the internet (Hohenwarter & Preiner, 2007); and third, mathlets are easy to use (Heath, 2002).

Numerous approaches have been used to assess derivative proficiency. Zandieh (2000)

divided comprehension of the derivative into 3 layers (ratio, limit, and function) and considered multiple representations (graphical, verbal, motion, symbolic) of each. Maschietto (2002) used epistemological, cognitive and didactic considerations. In their Differentiation Competency Framework, Kendal & Stacey (2003) focused on the representation (symbolic, graphical, and numerical) of the question and of the answer, and on the solution process (formulation or interpretation), resulting in 18 categories.

Research Questions

There is a gap in the literature regarding the role of local linearity in student thinking throughout a first-semester calculus course. Little is known about student learning outcomes in a course which is reorganized to use local linearity as the cognitive root for the derivative. This study proposes to fill these gaps by investigating two important research questions about students in a first-semester calculus course reorganized to use local linearity to introduce the derivative. First, did students use local linearity as a cognitive root to understand the derivative? Second, did students achieve high proficiency with the derivative?

Procedures

The subjects were 28 students enrolled in a first-semester calculus course at a large urban community college. The instructor was the researcher, who taught one calculus section.

The curriculum for first-semester calculus was reorganized so that the derivative was introduced primarily graphically through discussion-based lectures and student activities of guided discovery featuring mathlets to explore local linearity. (The mathlet for local linearity was designed by the researcher.) This was followed by instruction in techniques of differentiation. Formal limits were covered near the end of the semester.

The framework for calculus assessment was designed by the researcher. This rubric was influenced by assessment approaches in the literature and adapted to reflect the revised curriculum. Calculus content was categorized according to topic (definition of the derivative, finding the derivative at a point, finding the derivative as a function, non-differentiability, and applications of the derivative) and mathematical representation (symbolic, graphical, numerical, and verbal). Questions in each topic were chosen to ensure that use of multiple representations was assessed. This framework includes both procedural and conceptual knowledge (as defined by Hiebert & Lefevre, 1986).

Data on learning outcomes were collected from exams and other assignments throughout the semester using expert-validated items, as well as from three audio-recorded task-based semi-structured interviews during the semester. Each unit of student work was categorized and graded for proficiency on a scale from 0 to 4 as described in Table 1. A score of 3 or greater was

Table 1. Scoring Rubric for Demonstrated Proficiency

| Score | Description |
|---|---|
| 0 | Blank, or no contribution to a solution/understanding |
| 1 | Initial step only toward a solution/understanding with no coordination of components |
| 2 | Some steps toward a solution, or understanding of simple case only |
| 3 | Nearly complete solution/understanding, or complete with several minor errors |
| 4 | Full solution or understanding with at most one minor error |

categorized as high proficiency. In addition, data were coded using grounded theory (Glaser, 1992) to capture the role of local linearity in student thought processes.

Results

Students were assessed in 15 subtopics of the derivative in first-semester calculus. The learning outcomes are displayed in Table 2. Students on average displayed high proficiency in every topic aside from the limit definition of the derivative.

Table 2. Frequencies for each proficiency score in each derivative topic (n=28)

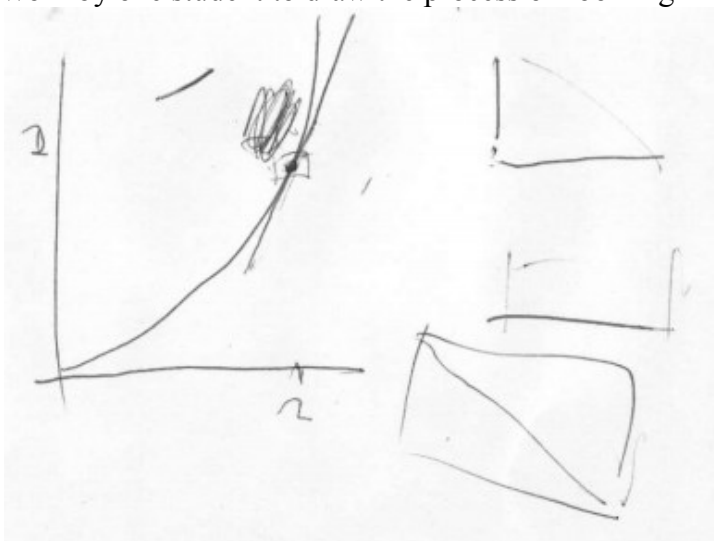
| | | SCORE | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **ASPECT OF DERIVATIVE** | **SUB-TOPIC** | **4** | **3** | **2** | **1** | **0** | **Mean** | **St.Dev.** |
| Definition of the derivative | slope of graph | 28 | 0 | 0 | 0 | 0 | 4.0 | 0.00 |
| | slope of tangent line | 19 | 1 | 3 | 2 | 1 | 3.3 | 0.24 |
| | slope of line after zooming in | 16 | 2 | 4 | 5 | 1 | 3.0 | 0.26 |
| | instantaneous rate | 19 | 5 | 2 | 1 | 1 | 3.4 | 0.20 |
| | using limit | 17 | 1 | 1 | 2 | 6 | 2.8 | 0.34 |
| Derivative at a point | draw tangent line | 23 | 3 | 2 | 0 | 0 | 3.8 | 0.11 |
| | estimate from graph | 26 | 2 | 0 | 0 | 0 | 3.9 | 0.05 |
| | calculate with formula | 20 | 3 | 0 | 1 | 4 | 3.2 | 0.28 |
| | solve verbal problem | 21 | 3 | 1 | 1 | 2 | 3.4 | 0.23 |
| Derivative as a function | given graph of f, graph f' | 15 | 6 | 2 | 2 | 3 | 3.0 | 0.27 |
| | given graph of f, describe f' | 17 | 3 | 4 | 2 | 1 | 3.2 | 0.23 |
| | symbolic techniques | 20 | 4 | 4 | 0 | 0 | 3.6 | 0.14 |
| Non-differentiability | non-differentiability | 22 | 0 | 5 | 1 | 0 | 3.5 | 0.18 |
| Applications of the derivative | find formula of tangent line | 19 | 1 | 2 | 2 | 3 | 3.1 | 0.29 |
| | optimization | 20 | 4 | 2 | 1 | 0 | 3.6 | 0.16 |

The notion of local linearity, which can be understood by zooming in on a graph, was easily adopted by many students and used to discuss their concept image of the derivative. While students in this course were first introduced to the derivative through their use of a mathlet in structured exploration, many students thereafter referred to zooming in as a regular part of their discussion, and performed the act of zooming in even when the technology was unavailable. They described their mental actions, and even provided illustrations. (Except where noted, the following student quotes and descriptions all arose during interviews.)

When asked which of the guided exploration activities were useful, one student cited the work with the local linearity mathlet. He discussed the usefulness of zooming in to see the straight line at a point and determine the derivative there. He supported his description by recreating the actions of the mathlet with drawings, displayed in Figure 1. Another student cited the same mathlet and said, “It was helpful because it made me understand that as you zoom in, you can find the tangent line and approximate what the slope is at that point.”

Another student was asked to draw the graph of the derivative given the graph of the function. When asked to explain her solution, she said that she estimated the slope and to do that “you have to have a computer to zoom in,” yet she did her work and completed her explanation without the computer mathlet. When asked to compare growth rates at different moments given a graph of height vs. time, another student said he “tried to zoom in with my eyes, and it looks more straight up here, and more down a bit [there].”

Figure 1. Work by one student to draw the process of zooming in on a graph



One student demonstrated the ability to distinguish between average and instantaneous slope using the techniques and terminology of local linearity. When asked about the slope of a secant on a graph, he noted that it represented an average rate. When asked for an example of a slope that is a derivative, he said, "Take this point, for example. Zoom in on it, and it becomes like a straight line." This is particularly noteworthy because before the local linearity exploration activities, this student demonstrated repeated confusion between average and instantaneous rates.

When asked which mathlet was helpful, one student mentioned the mathlet to explore the derivative as a function. When asked to explain his feeling, he replied, "It helped me to understand why the function and the derivative [are related], where the power rule comes from, why it's one less. ... It helped me picture the derivative of each function, cos, sin, $x^2$, ln(x), $e^x$. I mean when I visualize something I get a better understanding for when I'm doing problems." Another student described how, in attempting to understand the derivative, the formulas were unhelpful but the presentation using local linearity and that same mathlet was illuminating. "I didn't have a consolidated idea of what a derivative is. I did some research online. I saw a bunch of formulas, it looked complicated... But in class, the graphs, the graph with the sliding thing ... just moving it back and forth, trying to figure out, what is it drawing, what is the green line doing. And I got it with that. It's the value of the slope."

When asked how he would explain the derivative to another student, one student described how he would develop the idea by using the mathlets to demonstrate local linearity. Later in the semester he reported that he actually used the mathlets to teach his friends calculus.

Local linearity (and the activities to explore it) led directly to the topic of left- and right-handed derivative. This notion arose before any instruction on limits or derivatives; the proper ideas were present, although the terminology was not. When encountering points where a function is continuous and non-differentiable ("corners" such as y=|x| at x=0), several students initially proposed in writing for homework and in class that the function has two slopes at that point. For example, one student, when examining the function f(x) = |2x| and its graph using the local linearity mathlet, was asked for the slope at x=0, and wrote, "-2, or 2. It depends which direction from the y axis you are traveling." Later in the semester, the class discussion on this topic was used as the impetus to formalize handed derivatives using limit notation.

## Conclusions

Previous studies have suggested the potential value of using local linearity as the cognitive root for the derivative in calculus instruction and learning. In the present study, students took a first-semester calculus class with a redesigned instructional sequence in which local linearity was used to introduce the derivative. This introduction was achieved with mathlets and visualization playing large roles, and formal limits were delayed until near the end of the semester. We sought to investigate whether students would use local linearity as the central organizing concept for their concept image of the derivative. Further, we assessed the student learning outcomes in this approach.

We found that many students used local linearity as a cognitive root. They used the notion in order to explain what the derivative was. They cited the useful role the mathlets played as they achieved understanding of the derivative. They mimicked the actions of the local linearity mathlet in their verbal descriptions and responses. Further, local linearity was specifically referred to by students in their explanations of related ideas such as the tangent line, left- and right- handed derivatives, and it was even used to make sense of symbolic rules for differentiation. As a cognitive root, local linearity was powerful enough to help students revisit incorrect prior knowledge and correct it.

Most students achieved high proficiency with the derivative as measured by the assessment framework. The cohort, on average, demonstrated high proficiency on all 15 derivative topics (except for the one involving limits). This included topics of a more conceptual nature, such as describing properties of the derivative given the graph of the function, and topics of a more procedural nature, such as optimizing f(x).

This report suggests the tremendous potential impact of using local linearity as the cognitive root for the derivative. One potential area for future research is a teaching experiment to examine the genetic decomposition for students as they build their conception of the derivative in this curriculum. This may lead to further refinements of as well as greater appreciation of this innovative approach to calculus instruction.

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# STUDENT LEARNING OF KEY CONCEPTS IN SEQUENCES AND SERIES

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*This study examines students' foundational understanding and misconceptions of infinite repeating decimals in a variety of calculus courses and the roles of instruction in improving knowledge. We discuss Tzur and Simon's (2004) Reflection on Activity-Effect Relationships Model for student understanding and its application to the survey used in the current study. We examine a variety of matched questions concerning .777… and .999… and use these results for both performance measures and indicators of higher levels of student cognition. We look at the influence of different instructional practices, which include conceptual discussions and related group works, and their impact on student understanding. Our results indicate that these instructional practices can significantly improve student performance and understanding, even among mathematically talented students.*

*Key words:* [calculus, misconceptions, infinite decimals, sequences and series]

The question of whether .999… equals 1 has historically been a source of confusion among calculus students (Schwarzenberger and Tall, 1978). This question challenges students' conception of infinite processes and their belief a unique decimal representation for real numbers. Tall and Vinner (1981) considered this question as part of their study on student understanding of limits of sequences and functions. They identified several common misconceptions students held regarding the limiting process and noted the cognitive conflict students experience when believing that .333…=1/3 but .999…<1. In the same paper, Tall and Vinner outlined their theory of student learning which has since contributed to the development of other theories of learning including the Action-Process-Object-Schema (APOS) framework for student thinking. Many related studies on limits followed (Sierpinska, 1987; Tall 1990, 2000; Cottrill et al 1996; Dubinsky et al 2005a, b; Alcock & Simpson, 2004). These studies further developed theories of student learning and sought to find approaches to help students overcome common misconceptions. Strikingly, Cottrill et al (1996) reported, "We have not, however, found any reports of success in helping students overcome these difficulties [of the concept of limit]."

In this paper, we expand and extend our initial analysis in Keynes et al (2009) of student understanding and misconceptions regarding foundational concepts of sequences and series. We develop a deeper data-driven study involving large populations of diverse students and examine both similarities and differences among these different populations. We discuss the specific applications of the general theoretical background presented in Keynes et al (2009) to our analysis and the extent to which our data supports this model of student development of understanding sequences and series. We analyze these results and the questions that they raise, identifying some interesting directions for future study.

The major findings suggested by the initial analysis in Keynes et al (2009) were the following:

- All students, including mathematically talented students, have levels of misconceptions that need to be addressed via instruction.
- All students are capable of making gains in understanding with appropriate instruction and the gains could be significant.
- Students in our initial survey showed different levels of understanding embedded in the Tzur and Simon model (see Theoretical Background for more details).

In this paper, we examine these findings in our broader student population. We use a targeted emphasis and focus on the understanding of .999… since this representation seems to be a source of cognitive conflict for all student populations. As a general observation, the data presented here supports and even strengthens the major findings from our initial analysis (Keynes et al, 2009). For example, our analysis of matched pairs of questions about .777… and .999… shows improved post-instructional understanding, especially among talented students, of the cognitive links and similarities, despite the frequent confusion from the surface distinction of their actual values. This suggests that even in advanced undergraduate courses for math majors, instruction based on conceptual presentations addressing misconceptions, incorporating group works to supplement and enhance lectures, may improve student understanding and retention.

## Theoretical Background

We refer to the reader to Keynes et al (2009) for a more complete development of the theory and focus here on how our study fits Tzur and Simon's (2004) model for conceptual learning. Tzur and Simon's model is a refinement of the Action-Process-Object-Schema (APOS) framework that has often been used to explore student thinking about many calculus concepts, including limits and derivatives. Their model looks at the transition from process to object levels of understanding and breaks it down into two distinct stages: participatory and anticipatory. The process by which students make this transition is by reflection on an activity-effect relationship. When reflecting on the goal of an activity and on the effects of that activity towards reaching that goal, students observe patterns and can then abstract the results.

In the participatory stage, knowledge is context dependent; in the anticipatory stage, knowledge is independent of context. For instance, Tzur and Simon (2004) describe an activity where students learn to compare unit fractions of different sizes, for instance, 1/3 and 1/5. Students divided strips of paper into the correct number of pieces and compared the relative sizes of the pieces, learning that 1/3>1/5. The next day, students were unable to make similar comparisons. However, when prompted to think back to that activity and reflect on its outcomes, students could anticipate the results of the activity without redoing it (activity-effect relationship) and then make the comparisons. Students are in the participatory stage because they can use the knowledge they gained from the activity, but only within the context of that activity. At the anticipatory stage, students can call upon this knowledge in a different context.

For the current study, students were in the participatory stage when they could not call upon their knowledge of geometric series within the setting of infinite repeating decimals. Students able to apply their knowledge of geometric series in this setting were considered to be in the anticipatory stage. This was demonstrated most clearly when students are able to understand why .999…=1.

## Data Collection

The development and history of the survey instrument is described in Keynes et al (2009). The survey was designed to determine each student's stages of understanding, to identify

some common misconceptions of students, and to determine whether students were able to correctly put meaning on different representations of infinite repeating decimals. The survey focuses on infinite repeating decimals and does not provide information about a student's broader understanding of sequences and series. One important reason for such a narrowly focused survey was to create an instrument that could be easily and quickly given in many different classroom settings without using up valuable class time. By starting the survey during the passing time between classes, students were usually able to complete the surveys within the first five minutes of class. This aided in recruiting instructors from multiple courses. The authors came to each of the classes, introduced themselves and the study, and distributed the assessments, giving students approximately 10 minutes to complete the survey. Compliance was very high, at over 95% of students in attendance.

The survey was given to over 1000 students during 2008 - 2010. Pre- and post-data was typically collected in most semesters. Surveys were given before and after instruction on sequences and series. The survey itself consists of 4 questions each with multiple parts, with a total of 20 parts. The surveys were scored and students received one point per correct answer, for total of 20 possible points. See Figure 1 for the two questions we will focus on here. In addition to the mathematics content, the survey included several demographic questions, such as year in school, AP calculus experience, and other previous calculus experience. Unfortunately we do not have ACT or SAT scores for our populations. In a future paper, we plan to look at the influences of these other factors on student performance.

The survey was given to several different calculus classes, all of which currently cover sequences and series, as well as to several classes whose students study these topics in prior classes. We will focus on the calculus classes here. These groups, with the number of pre- and post-surveys collected, are:

- MC1: main stream calculus for non-math and science majors (288 pre, 247 post)
- MC2: main stream calculus for math and science majors (257 pre, 222 post)
- HC: university-wide honors calculus (106 pre, 33 post)
- MTS: honors calculus for mathematically talented middle and high school students. (113 pre, 156 post)

In addition to each of the different target audiences, there are also instructional differences between each of the four groups. MC1 is a traditional calculus class with three hours of lecture and two hours of recitation per week. MC2 has only two hours of lecture per week and supplements lecture with three hours of group work activities in recitation. Group works tend to be computational in nature and help students practice the mechanics of solving problems. HC is a standard lecture-recitation course that uses a more theoretical approach than MC1 or MC2 and does not incorporate group work activities. Finally, MTS combines both a theoretical approach in lecture and the use of carefully designed group works that enhance the content of lecture in recitation.

## Discussion

We first look at overall performances between the four groups (see Figure 3) and use Student's *t*-test for equality of means to make comparisons. We compared group means on the pre-survey and found the only statistically significant match in performance on the survey was the expected one between HC and MTS ($p=0.326$). On the post-survey, this match was no longer significant ($p=0.014$). There were no other statistically significant comparisons between groups on the pre- or post-surveys ($p=0.000$). It is also important to note that while every group had overall statistically significant gains with instruction (see Figure 2), statistically significant gains were not always seen on each of the four questions that comprised the survey. In fact, the only group to see statistically significant gains on each of

the four questions was MTS. This evidence helps to support our claim that all students are capable of making gains with instruction and that instruction matters even for bright students.

On question 1, there were no statistically significant gains with instruction in MC1; however significant gains with instruction were found with the other three groups (see Figure 4). It is worthwhile to notice that while the numerical gains with instruction in MC1 (from 3.90 to 4.18) and HC (from 5.52 to 5.82) look very similar, the gains seen with HC are statistically significant in part because of their higher means and different sample size. On question 3, statistically significant gains with instruction were found with MC2 and MTS, but not with MC1 and HC (see Figure 5). For completeness, we have included the performances of the groups on each of the parts of questions 1 and 3 (see Figures 4 and 5), where we put the columns for question 3 in the order that matches the parts from question 1. In particular, note that students in MC2 and MTS improved significantly on almost every part of question 3. In fact, after instruction, students in MC2 performed at about the same level as students in HC on question 3. Since students in both MC2 and MTS extensively use group work as part of instruction, this evidence supports the claim that using group work to support lecture can lead to greater understanding for all students.

For the current study we were interested in seeing how students performed on the matched parts of questions 1 and 3 and identifying whether or not students were able to perform similarly on both parts of the matched pair. We hypothesize that if students perform similarly on both parts, then they are able to recognize that the questions being asked are the same, regardless of numbers, and they are able to apply their knowledge of sequences and series in either context. In the context of our theoretical framework, this would suggest that students might be in the anticipatory stage. Likewise, students who are unable to perform similarly on both questions might be in the participatory stage.

First we will consider the performances on the pre-survey (see Figure 6). In MC1 students performed differently on each item of a matched pair except for the pairing of 1a and 3e, where their performance was matched but also low. In MC2 there was no statistical significance in performance on any of the matched pairs. HC had statistical significance on the pairings of 1c and 3a as well as the pairing of 1b and 3b. MTS also had significance on the pairing of 1c and 3a.

In the post-survey after instruction (see Figure 7), students in MC1 still had statistical significance again only on the matched pair of 1a and 3e but again with low performance. HC also still only had statistical significance on the pairings of 1c and 3a as well as 1b and 3b. In areas of initial low performances (3e, 3d, 3c), post-instructional performance, while improved, remain low. MC2 gained statistical significance in the pairing of 1c and 3a but again with lower scores for 3e, 3d, and 3c. In contrast, MTS had statistical significance in four of the five pairings and all low or moderate performance scores improved to be strong performances. As a result of this, we posit that after instruction students in MTS are operating in the anticipatory stage; students in MC1 are operating in the participatory stage; and students in HC and MC2 are somewhere between the two stages. Instructional practices could be partially responsible for the differences seen in gains with instruction and we plan to explore this avenue of research in a future study. In particular, it is of great interest to look at how instructional differences impact students from a group of students with similar backgrounds and abilities.

It is clear from our data that many students, even after instruction on sequences and series, see .777…=7/9 and .999… =1 as two distinct concepts. One possible explanation for this discrepancy is that the equality of .999… and 1 conflicts with students' belief of the uniqueness of decimal representations of numbers. Another is that the number 7/9 indicates an operation (division) which helps students recall the knowledge learned in that context

whereas the number 1 does not prompt students to consider applying their knowledge from division. In addition, many students have difficulty understanding the "…" notation. In an interview, one MTS student, arguably the brightest student in her class, explained that she found the "…" notation confusing and ambiguous. While she understood that "…" means that the pattern continues, she was unsure as to whether the pattern ever terminated or if the pattern was allowed to suddenly change (Keynes et al, 2009). The notation "…" is learned years before students ever encounter an infinite geometric series or learn about sequences and as a result, students often have difficulty putting the proper meaning on the notation. This highlights the need for calculus instructors to explicitly discuss the meaning of mathematical symbols with their students.

Another relevant aspect is prior student instruction of incorrect "facts" about infinite repeating decimals as early as elementary school. It can be difficult for current instruction to overcome such ingrained beliefs. When discussing the question of whether .999… equals 1 with an elementary school teacher, she explained how she presents these topics to her own students. First, she explains that 1/9 =.111…, 2/9=.222…, all the way up through 8/9=.888… Then, she says, *mathematics breaks down*, this pattern no longer holds, and so 9/9=1. Her belief in the inconsistency of mathematics is based in her belief of the uniqueness of decimal representations of numbers. Another elementary school teacher, with a mathematics major from the University of Minnesota, finally agreed with us that .999…=1, not because we were able to convince her with any mathematical argument, but because she believed in our authority and since we said it was so, it must be. A more extensive study of deeply ingrained student beliefs in calculus instruction could provide more useful information.

## Final Conclusions

The question of whether .999… equals 1 is one that students and teachers alike struggle with. It incorporates deep ideas (the concepts of infinity and limit) and challenges students' beliefs about real numbers. What are the best ways to teach all students these types of conceptual concepts? How can we help them make the transition from the participatory to the anticipatory stages? We believe that our approach of combining more theoretical lectures with targeted group works which serve to both strengthen conceptual knowledge and to address student misconceptions leads to deeper understanding and better retention of conceptual ideas. In a future study, we plan to explore this idea further by looking at the impact of such instructional practices on mainstream calculus courses, gathering more qualitative data. Additionally, we would like to expand the focus of the study to include more typical series questions to see how student understanding of .999…=1 correlates with understanding in the larger context of sequences and series.

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## Figures

| Which of the following are true? Circle your answer(s). | |
|---|---|
| 1a. $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots=\frac{7}{9}$ | 3e. $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots=1$ |
| 1b. .777… is a number less than $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots$ | 3b. .999… is a number less than $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots$ |
| 1c. $.\overline{7}$ is a number less than $\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\ldots$ | 3a. $.\overline{9}$ is a number less than $\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots$ |
| 1d. .777… is the same number as $.\overline{7}$ | 3f. .999…=1 |
| 1e. $.777\ldots=\frac{7}{9}$ | 3c. $.999\ldots=9\cdot\frac{1}{9}$ |
| 1f. $.\overline{7}=\frac{7}{9}$ | 3d. $.\overline{9}=1$ |

Figure 1: Survey instrument, matched questions

| Class | Pre | Post | *p-value* |
|---|---|---|---|
| MC1 | 10.86 | 11.76 | *0.011* |
| MC2 | 12.63 | 14.88 | *0.000* |
| HC | 16.18 | 17.52 | *0.031* |
| MTS | 16.63 | 18.71 | *0.000* |

| Pairing | MC1 & MC2 | MC1 & HC | MC1 & MTS | MC2 & HC | MC2 & MTS | HC & MTS |
|---|---|---|---|---|---|---|
| Pre | *0.000* | *0.000* | *0.000* | *0.000* | *0.000* | *0.326* |
| Post | *0.000* | *0.000* | *0.000* | *0.000* | *0.000* | *0.014* |

Figure 2: Group means (out of 20) Figure 3: Matching groups (p-values)

| | 1a | | 1b | | 1c | | 1d | | 1e | | 1f | | Total (6) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Class | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | |
| | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post |
| MC1 | *0.069* | | *0.245* | | *0.536* | | *0.043* | | *0.111* | | *0.292* | | *0.051* | |
| | 0.37 | 0.45 | 0.84 | 0.88 | 0.90 | 0.88 | 0.78 | 0.85 | 0.47 | 0.53 | 0.54 | 0.59 | 3.90 | 4.18 |
| MC2 | *0.000* | | *0.193* | | *0.391* | | *0.179* | | *0.116* | | *0.478* | | *0.008* | |
| | 0.61 | 0.77 | 0.85 | 0.89 | 0.90 | 0.95 | 0.84 | 0.88 | 0.67 | 0.74 | 0.77 | 0.80 | 4.65 | 5.03 |
| HC | *0.001* | | *0.025* | | *0.083* | | *0.915* | | *0.186* | | *0.127* | | *0.048* | |
| | 0.91 | 1.00 | 0.95 | 1.00 | 0.97 | 1.00 | 0.92 | 0.91 | 0.87 | 0.94 | 0.91 | 0.97 | 5.52 | 5.82 |
| MTS | *0.045* | | *0.015* | | *0.166* | | *0.008* | | *0.018* | | *0.163* | | *0.002* | |
| | 0.86 | 0.94 | 0.88 | 0.96 | 0.94 | 0.97 | 0.88 | 0.97 | 0.84 | 0.94 | 0.89 | 0.94 | 5.28 | 5.72 |

Figure 4: Means on Question 1

| | 3e | | 3b | | 3a | | 3f | | 3c | | 3d | | Total (6) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Class | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | |
| | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post | Pre | Post |
| MC1 | *0.679* | | *0.630* | | *0.468* | | *0.747* | | *0.022* | | *0.718* | | *0.275* | |
| | 0.40 | 0.42 | 0.74 | 0.76 | 0.81 | 0.83 | 0.27 | 0.28 | 0.20 | 0.28 | 0.33 | 0.35 | 2.75 | 2.92 |
| MC2 | *0.000* | | *0.042* | | *0.107* | | *0.000* | | *0.000* | | *0.000* | | *0.000* | |
| | 0.34 | 0.66 | 0.74 | 0.82 | 0.82 | 0.87 | 0.28 | 0.61 | 0.26 | 0.50 | 0.35 | 0.65 | 2.80 | 4.11 |
| HC | *0.571* | | *0.760* | | *0.696* | | *0. 463* | | *0.220* | | *0.346* | | *0.414* | |
| | 0.52 | 0.58 | 0.95 | 0.94 | 0.98 | 0.97 | 0.47 | 0.55 | 0.45 | 0.58 | 0.48 | 0.58 | 3.86 | 4.18 |
| MTS | *0.000* | | *0.001* | | *0.108* | | *0.000* | | *0.000* | | *0.000* | | *0.000* | |
| | 0.75 | 0.92 | 0.80 | 0.94 | 0.88 | 0.94 | 0.66 | 0.92 | 0.67 | 0.88 | 0.72 | 0.94 | 4.49 | 5.54 |

Figure 5: Means on Question 3

| | 1a v 3e | | 1b v 3b | | 1f v 3d | | 1c v 3a | | 1e v 3c | |
|---|---|---|---|---|---|---|---|---|---|---|
| Class | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | |
| | 1a | 3e | 1b | 3b | 1f | 3d | 1c | 3a | 1e | 3c |
| MC1 | *0.462* | | *0.000* | | *0.000* | | *0.000* | | *0.000* | |
| | 0.37 | 0.40 | 0.84 | 0.74 | 0.54 | 0.33 | 0.90 | 0.81 | 0.47 | 0.20 |
| MC2 | *0.000* | | *0.000* | | *0000* | | *0.002* | | *0.000* | |
| | 0.61 | 0.34 | 0.85 | 0.74 | 0.77 | 0.35 | 0.90 | 0.82 | 0.67 | 0.26 |
| HC | *0.000* | | *1.000* | | *0.000* | | *0.320* | | *0.000* | |
| | 0.91 | 0.52 | 0.95 | 0.95 | 0.91 | 0.48 | 0.97 | 0.98 | 0.87 | 0.45 |
| MTS | *0.028* | | *0.012* | | *0.000* | | *0.083* | | *0.001* | |
| | 0.86 | 0.75 | 0.88 | 0.80 | 0.89 | 0.72 | 0.94 | 0.88 | 0.84 | 0.67 |

Figure 6: Matched Pairs, Pre-survey

| | 1a v 3e | | 1b v 3b | | 1f v 3d | | 1c v 3a | | 1e v 3c | |
|---|---|---|---|---|---|---|---|---|---|---|
| Class | *p-value* | | *p-value* | | *p-value* | | *p-value* | | *p-value* | |
| | 1a | 3e | 1b | 3b | 1f | 3d | 1c | 3a | 1e | 3c |
| MC1 | *0.415* | | *0.000* | | *0.000* | | *0.042* | | *0.000* | |
| | 0.45 | 0.42 | 0.88 | 0.76 | 0.59 | 0.35 | 0.88 | 0.83 | 0.53 | 0.28 |
| MC2 | *0.002* | | *0.002* | | *0.000* | | *0.189* | | *0.000* | |
| | 0.77 | 0.66 | 0.89 | 0.82 | 0.80 | 0.65 | 0.95 | 0.87 | 0.74 | 0.50 |
| HC | *0.000* | | *0.160* | | *0.000* | | *0.325* | | *0.000* | |
| | 1.00 | 0.58 | 1.00 | 0.94 | 0.97 | 0.58 | 1.00 | 0.97 | 0.94 | 0.58 |
| MTS | *0.407* | | *0.566* | | *0.132* | | *0.181* | | *0.012* | |
| | 0.94 | 0.91 | 0.97 | 0.96 | 0.95 | 0.91 | 0.97 | 0.95 | 0.94 | 0.88 |

Figure 7: Matched Pairs, Post-survey

# Calculus beyond the Classroom: Application to a Real-Life Problem Simulated in a Virtual Environment

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This study concerns the correlation of mathematical knowledge with a corresponding real life object within the theoretical framework of Realistic Mathematics Education. It shows to what extent students, who had almost completed the AP calculus course, were able to apply their knowledge to a real-life situation. By simulating the interactive milieu in the Second Life Virtual Environment (VE), this study explores how students find a 'real-life' optimal path 'practically', and how they then re-invent the corresponding calculus task. The research revealed that one out of ten participants mathematized the problem horizontally and vertically without any guidance. Another four students demonstrated independent horizontal mathematizing, still others needed guidance. The study instructional design, based on simulation of a real-life situation in VE allowed students to explore mathematical solutions relative to their intuitive findings in VE. By mathematizing their own 'real-life' activities, students connected them with corresponding mathematics at an intuitive level.

**Key Words:** Realistic Mathematics Education, calculus, virtual environment, intuition

## Introduction

A troubling problem with current education is the practical application of knowledge to life. Graduates do not know how to apply knowledge to many problems that arise outside the walls of school (Ilyenkov, 2009). There is a common recognition among mathematics educators that a serious mismatch exists and is growing between the skills obtained at schools and the kind of understanding and abilities that are needed for success beyond school (Lesh, & Zawojewski, 2007).

The attempts of some instructional theories to solve the problem by creating systems of rules of 'how to apply knowledge to life' impede rather than help things (Ilyenkov, 2009). Visual aids provided to students were created independent of their activity. That is, the decisive part of cognition – to go from the object to abstract remains outside of student activity. A special kind of activity of correlating knowledge and its' object should be implemented in contemporary classrooms. "Here, what is needed is activity of a different order – activity oriented directly at the object. Activity that changes the object, rather than an image of it" (ibid, p. 223).

The problem of 'the practical application of knowledge to life' is especially significant for calculus, which was developed from the real world application and has a real world context. In the late 1980s the 'Calculus Reform Movement' began in the USA. The Calculus Consortium at Harvard (CCH) was funded by the National Science Foundation to redesign the curriculum with

a view of making calculus more applied, relevant, and more understandable for a wider range of students.

The purpose of this study is to set out an instructional design utilizing VE as an alternative to the 'word problem' method of simulating real-life situations. The goal of the instructional design is to bring the reality to the classrooms exploiting such concepts of VE as *simulation, interaction, immersion, and full body immersion* which are described in (Heim, 1993). Specifically, we simulated the interactive milieu in the Second Life VE, which encouraged students to find an optimal path on the basis of their primary intuitions; and then with the help of a specially designed journal, to re-invent the corresponding calculus task. The experimental part of the study aimed to explore students' intuitive solutions of optimal navigation in VE from the prospect of correlation with their mathematical formal solution. Another research question related to whether and to what extent the students who had almost completed the AP calculus course would transfer the simulated in VE real-life situation into mathematical formal task.

## Theoretical Background

More than forty years ago Freudenthal (1968) posed the problem of lack of connection between knowledge and its real-life object. The Freudenthal Institute has developed a theoretical framework now referred to as Realistic Mathematics Education (RME) (Freudenthal, 1968, 1991, 1973; Gravemeijer, 1994).

The RME instructional theory is based on Freudenthal's idea that mathematics must be connected to reality. The use of realistic contexts became one of the determining concepts of RME. The most general characteristic of RME is mathematizing; the realistic contexts must be used as a source for mathematizing. Freudenthal (1968) wrote: "What humans have to learn is not mathematics as a closed system, but rather, as an activity, the process of mathematizing reality..." (p.7). The role of mathematizing in mathematics education was also stressed by a number of authors (De Lange, 1996; Liljedahl, 2007; Mason, 2004; Presmeg 2003; Russmussen, Zandieh, King, Terro, 2005; Treffer, 1986; Wheeler, 1982). Particularly, Treffer (1986) formulated the idea of 'progressive mathematizing' as a sequence of two types of mathematical activity – horizontal mathematizing and vertical mathematizing. He suggests that horizontal mathematizing consists of non-mathematical real world situations and transforming the situations into mathematical problems. Vertical mathematizing is grounded on horizontal mathematizing and includes reasoning about abstracts within the mathematical system itself. The process of extracting the appropriate concept from a concrete situation is stated by De Lange (1996) as *'conceptual mathematization'*. This process forces the students to explore the situation, find and identify the relevant mathematics, schematize, visualize, and develop a corresponding mathematical concept. By reflecting and generalizing the students will be able to apply the mathematical concept to new areas of the real world.

The RME theory has been accepted and adopted by some educational institutions of England, Germany, Denmark, Spain, Portugal, South Africa, Brazil, Japan, and Malaysia (de Lange, 1996). In America, RME was adopted in the "Mathematics in context" project for the U.S. middle schools. In spite of such a wide acceptance and adaptation of RME, the recent research shows that there is still a wide gap between the world of knowledge obtained at school and the world of conceptions found in everyday experience (Ilyenkov, 2009; Lesh, & Zawojewski, 2007).

The major idea of this paper is to point out that the reason why students do not connect the mathematical world with reality is because they continue to mathematize 'word problems' with 'ready-made' images instead of active real-life situations. Moreover, students do not involve their intuitive cognition while mathematizing 'word problems' and 'ready-made' images. Intuition, intuitive cognition, intuitive understanding, and intuitive solutions form one of the basic components of mathematical activity along with formal aspects such as axioms, definitions, and algorithmic component (Fischbein, 1994). Furthermore, intuition gives the behavioral meaningfulness of a mathematical notion (Fischbein 1987). Intuition can also play an essential role in biasing notions. To avoid biasing, students should explore and identify those intuitions which may distort knowledge (ibid), and the connection between an object and knowledge about this object.

Mathematizing of active participation in a real-life situation should connect mathematical concepts with intuitions formed on the basis of corresponding previous real-life activity. Students' exploration of mathematical solutions relative to intuitive findings in VE should bring to light whether there are contradictions between their intuitions and mathematical formal solution. Fischbein (1987) emphasised that the students' awareness about tacit mental conflicts should strengthen the control of the taught conceptual mathematical structures over the primary intuitive ones. Although a number of authors have stressed the important role of intuition in mathematics education (e.g., Burton, 1999; Fischbein, 1987, 1989, 1994; Tall, 1991, 1997, 2000), they did not show its role in RME instructional design theory.

## Methodology: Materials, Methods, and Participants

Second Life VE was used for programming interactive setting for the real-life optimal navigation task. The simulated setting includes a pond with shallow water, surrounded by bushes and trees (Fig. 1).

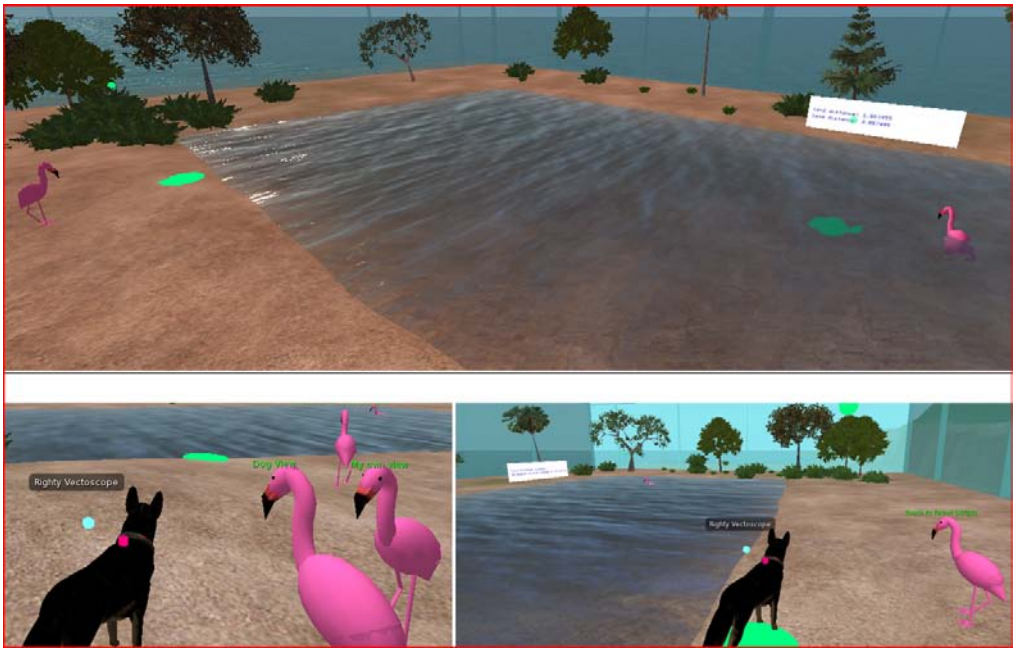


Figure 1: Simulated in the Second Life VE interactive setting for finding optimal path.

The environment was programmed so that walking/running speed on land is twice as fast as walking/running speed in water.

The task for the student in this VE is to travel between the two green platforms (see Fig. 1) trying to minimize the time of travel for every trip. One platform is located on land near the water's edge, another is located in the water. The environment is programmed to record time spent for each trip and its distance traveled by land. After each trip the student must transfer this data into a specially designed guiding–reflecting journal, which is an integral methodological part of the instructional design.

The aim of the guiding–reflecting journal is to connect the student's optimal navigation practice in the VE with the calculus optimal path finding task. The journal contains instructions, tables for transferring data collected from every trip in the VE, reflecting questions, guiding instructions and questions initiating the student's reasoning. It contains areas for independent reasoning as well as schematization of the problem for those students who need detailed guidance. The back page of the journal offers some formula tips and a detailed solution of the calculus problem. The geometrical schematization and solution provided in the journal were adapted from Pennings's (2003) work. Fig. 2 demonstrates a schematization of the task which is integrated with landscape.

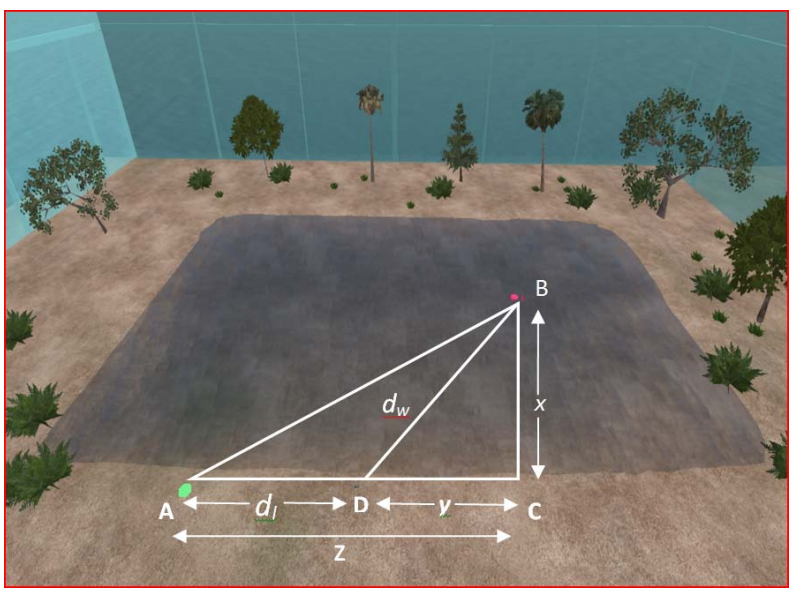


Figure 2: Schematization of some possible paths provided in the guiding-reflecting journal.

According to the schematization, A is a land platform, B is a water platform. The shortest path from A to B is the most direct path AB. Since the speed in water is slower than on land, students can choose the path with the shortest distance traveled in water, path AC and then CB, where ACB is a right angle. Finally, there is the option of using a portion of the land path, up to D, and

then entering into the water at D and moving diagonally to the water platform. In this diagram $x$ represents distance between B and C; $d_l$ and $d_w$ are distances traveled by land and by water respectively. Distance between A and C is $z$, and $y = z - d_l$.
The solution of the minimal path finding includes the following reasoning:

According to the schematization above, $T=T_l+T_w$ . Since $T_l = \frac{d_l}{s_l}$, and $T_w = \frac{d_w}{s_w}$ , then $T = \frac{d_l}{s_l} + \frac{d_w}{s_w}$, which gives $T = \frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w}$, where $s_l$ and $s_w$ are speeds on land and in water respectively; T is the trip time; $T_l$ is time spent for the land portion; and, $T_w$ is time spent for the water portion of the trip.

The condition of minimal time is $T'(y)=0$, or $\left(\frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w}\right)' = 0$

Following the journal instructions, the student obtains the final formula:

$$y = \frac{x}{\sqrt{\frac{s_l}{s_w}+1}\sqrt{\frac{s_l}{s_w}-1}}$$

The journal provides the exact values of virtual distances and speeds in the VE. The student is instructed to use these values to calculate the optimal distance traveled by land with corresponding minimal time using formulas above and to compare the mathematically obtained values with his/her best finding in the VE.

Ten students ranging in age from 17 to 18 years, who had almost completed the AP calculus course at a secondary school, participated in the research study. Each participant provided a signed Parent Consent Form. They also read and signed the Assent Form before participating. The experiments were conducted in the school's Teacher's room, outside of regular calculus class time. Each session of 60-90 minutes included an exploration trial, followed by the main task which consisted of the participant's work with both the computer and a guiding-reflecting journal. The mathematical part was devoted to the participant working solely with the journal. The last portion of the experiment consisted of the completion of a questionnaire at the end of the journal. The participants' exploration of the computer environment was screen recorded by SMR software. Their work with the journals was video-recorded. During one session the computer lost the SMR data while automatically updating its basic software. Therefore, the collected data from 10 sessions included screen recordings of exploration trials and VE tasks from 9 sessions, video-recordings of students' working with journals, and the completed journals from all 10 sessions.

## Results

The first part of the data analysis was devoted to students' finding the optimal path in VE based on their primary intuitions. The students' first trips in the VE demonstrated that students have different life experiences connected with optimal navigation, therefore different intuitive solutions. Fig. 3 shows four different choices of the students' first paths.

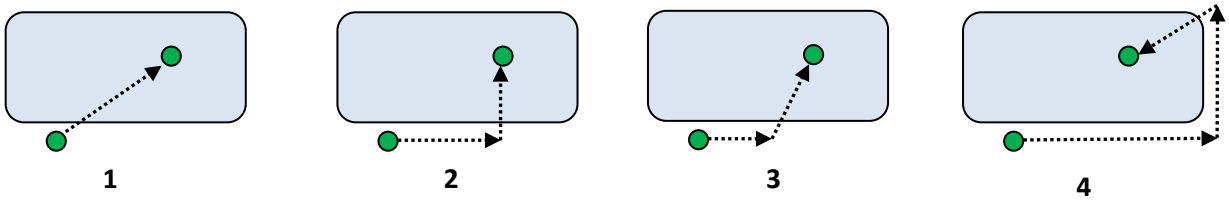


Figure 3: The diagrams of four different choices of first trips with which students started the optimal path finding in the VE.

Analysis of nine computer screen recordings showed that the first trips of four students were the shortest distances as shown on diagram 1. Three students started finding the optimal path trying to minimize the water portion as shown on diagram 2. One participant also tried to minimize water portion in his first trip running around the pool as shown on diagram 4. Only one out of nine students intuitively chose the first path corresponding to exact mathematical solution of the problem (see diagram 3).

Altogether, four students had a trip corresponding to the mathematically calculated optimal path. The remaining six students were not close to the mathematical solution. After completing 10 trips, all students were asked which path they would choose if the platform was located closer to the beach. Six students responded that they would choose the same trip, meaning that their primary intuitions did not agree with the exact mathematical solution. All students became aware of how far their intuitive solutions were from the mathematical one.

The second part of the data analysis was devoted to students' vertical and horizontal mathematizing of their activity. Video-recordings of students working with and completing their guiding-reflecting journals constituted the data source for this part of analysis. One of the ten students mathematized the problem horizontally and vertically without any guidance. He started to mathematize the problem during the completion of the optimal navigation task in the VE. After two trips in the VE this participant asked "Actually can I do math?". After another two trips he started drawing diagrams schematizing his activity in the VE. He then completed two final trips (totalling six out of ten offered in the journal) and switched to developing a mathematical solution of the problem. Figure 4 demonstrates a fragment from his journal with his own schematization and mathematical reasoning.

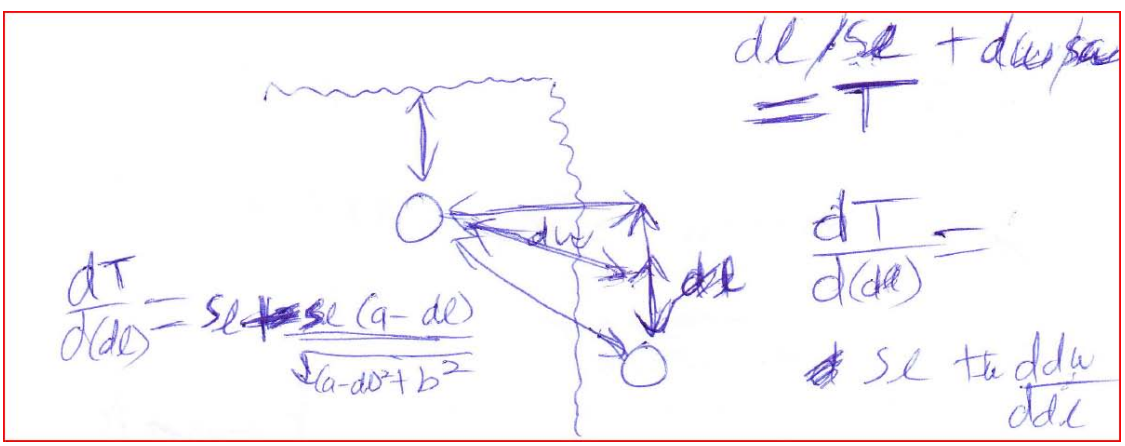

Figure 4: Fragment of a student's mathematizing of the problem.

All in all, five participants demonstrated different cases of their own schematizations of the problem using their own notations which in turn, can be considered as horizontal/vertical mathematizing. The remaining participants referred to journal schematization with corresponding notations.

## Conclusions and Implications for Mathematics Education

The chief outcome of this research is the new approach to RME instructional design. Particularly, we demonstrated that instead of situations described in 'word problems' with ready-made images to be mathematized, the real-life activity can be simulated in VE. We showed the particular example from calculus which allowed students to try to solve the optimal path finding problem 'physically' on the basis of their primary intuitions and then mathematize the problem with guidance of specially designed journal. This 'real life' activity in the simulated VE helps students to become aware of tacit conflicts between their intuitions and the formal mathematical solution. Such awareness helps to shape 'right' intuitions, which in turn gives the behavioral meaningfulness of a mathematical notion (Fischbein, 1987). Practical 'real-life' activity in simulated VE and its further mathematizing connects the particular activity with corresponding mathematical formalities. Implications of the offered instructional design can bring real-life problems from outside the school into the classroom.

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# Embodiment of Struggle in Research Mathematicians: The Case of Proximal Inhibition

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*It's often considered desirable for students to develop ways of engaging with mathematics that mimic the thinking styles of mathematicians. However, there have been very few ethnographic studies of mathematicians to explore how they actually practice mathematical research. This study involves an embodied phenomenological analysis of videos of pairs of mathematicians working together on a current problem in their field. I outline one resulting embodiment of their struggles with the material, which I term "proximal inhibition." The value and implications of this contribution are discussed briefly at the end.*

Keywords: Embodied cognition, mathematicians, microethnography, phenomenology

## Introduction & Background

It's not uncommon in our field to suggest that students should learn to engage with mathematics the way mathematicians do (Brown, Collins, & Duguid, 1989; Cuoco, Goldenberg, & Mark, 1996; Yong & Orrison, 2008). When we ask mathematicians how they think about problems and what general mental strategies they use, it's true that they often report using strategies that we'd be quite happy to see our students using (Burton, 2004; Davis & Hersh, 1981; Hadamard, 1949; Pólya, 1945; Thurston, 1994). Indeed, it seems quite plausible that many of the lists of behaviors meant to capture "what mathematicians do" are derived largely from their authors' reflections on their own experiences with mathematics.

Unfortunately, while this kind of reflection on our own experiences can be and often is valuable, it often isn't as reliable or as detailed as it might subjectively seem to be (Pronin, 2009). Both psychology (Bargh, Chen, & Burrows, 1996) and phenomenology (Gallagher & Zahavi, 2008) have shown that there are subtleties to how and why we structure our experiences the ways we do that simply aren't immediately obvious to our conscious minds. For instance, Merleau-Ponty (1962) pointed out that there's a difference between our experience of our hands as anatomical objects (e.g., when looking carefully at a painful spot on a finger) and that of our hands as lived, "invisible" instruments through which we engage with the world (e.g., when we reach for and lift a glass). It seems likely that there's a similar sort of division in the way mathematicians experience novel mathematical objects on the one hand and mathematical entities they're very familiar with on the other (Nemirovsky, 2005) – and yet we rarely hear mathematicians consciously reporting this difference beyond a sense that concepts and techniques they're quite used to are somehow more ready-at-hand and interconnected for them than ideas they're exploring for the first time (Hadamard, 1949).

If we want to illuminate these more subtle nuances of how mathematicians experience their discipline, we need to examine these individuals as they engage in mathematics, but using some sort of research tool that will allow us to avoid relying on the conscious mind alone. There are a very few studies that have attempted this, such as Weber's (2008) use of a think-aloud protocol as mathematicians examine arguments and

Greiffenhagen's (2008) video analysis of mathematicians lecturing to graduate students. However, none of them attempt to explore how mathematicians experience their attempts to create *novel* math, which seems to be the analog in mathematical research of what many in our field advocate students should be doing (Brown, Collins, & Duguid, 1989; Cuoco, Goldenberg, & Mark, 1996; Yong & Orrison, 2008).

The present study aims to contribute to this apparent gap in the literature. Specifically, since many mathematicians point toward experiences of effort and struggle as being central to their work (Burton, 2004; Hadamard, 1949), I have chosen to focus on the phenomenology of this struggle as they engage in novel research. To be more explicit, the research question from which this study emerged is: *What are some types of experience of struggle mathematicians have while working together in-person on a question from their current research?* I will explain the reasons for the specific choices implied in this research question in the following sections; for now, I would simply like to clarify that for time and space considerations I have chosen to elaborate here on just one of the types that have emerged from this study.

**Theoretical Framework**

From a phenomenological standpoint, it seems as though perceptions, thoughts, and plausible actions are in an important sense inseparable (Gallagher & Zahavi, 2008). For instance, my ability to recognize a pencil as such comes bundled with it a whole realm of possibilities: I could walk around it and look at it from various angles, lift it and move it around, write with it, roll it across the table, and so on. Yet when I recognize the ability to snap it in half and use it as kindling for a fire, something subtle shifts in what that object is for me. Research on patients with lesions in the prefrontal cortex – the region of the brain primarily responsible for impulse control – suggests that this perception of affordance accompanies a neurological impulse to physically initiate the perceived-as-possible task (walking around the object, picking it up, etc.) (Aron et al., 2007). In other words, it seems as though we cannot separate our recognition of, or conception of, an object like a pencil from our body-felt sense of how we could interact with it and how we anticipate those interactions will affect what we experience.

In light of this, I adopt a form of embodied cognition that views cognition, perception, and action as facets of a unified whole, for which I will use the term *perceptuomotor activity* (Nemirovsky & Smith, 2011; Roth & Thom, 2009). So rather than viewing the body as a vehicle for the mind and embodied behavior (gesture, speech, eye gaze, etc.) as indications of some hidden mental activity, I view such embodied behavior as partially constituting the subject's understanding of the situation in question (Radford, 2009). I say "partially" here because in practice, the full range of possible interactions can never actually be enacted. (E.g., I could set down, throw, break, or write with a single physical pencil, but in practice I'm likely to enact only one of these affordances – and the full list of affordances is practically endless.) Indeed, it's possible for someone to engage in perceptuomotor activity without any clearly visible signs of physical movement, such as when holding still while visualizing something.

There are two methodological implications of this framework I'd like to highlight. First, data collection and analysis need to respect multimodality (Williams, 2009): we cannot focus on speech alone, or gesture alone, or inscriptions alone, or any other single facet of embodied behavior in isolation. Doing so would give us too sparse a sense of the

subject's perceptuomotor activity, in much the same way (and from this perspective, for many of the same reasons) as a description of a doorway as a particular visual pattern would seem sorely inadequate. This requires us to record, coordinate, and consider as a whole as many of the various ways in which subjects demonstrate their embodied orientation to the situation being observed as we can.

Second, our sociality is also embodied according to this perspective. This has many important theoretical implications, but for the present study the main point to consider is that subjects don't have privileged access to the nature of their perceptuomotor activity (Pronin, 2009; Scheler, 2007). This makes it plausible for us to understand subjects' phenomenological experiences not as deductions from behavior, but as a kind of perception based on our innate empathy as informed by our immersing ourselves in the context and history in which the subjects reside (Nemirovsky & Smith, 2011; Scheler, 2007). For instance, for most of us, our sense that a friend is happy does not come to us as a deduction based on head position, a smile, etc.; rather, we sense her happiness as a kind of shared and immediate experience to the degree that we understand her situation and empathize with her. There may be subtleties that are obvious to her but not to us (e.g., she might try to hide her happiness, or we might not know why she's happy), and we can never fully capture every nuance of how she experiences her situation, but we sense her state and understand her perspective to the degree that we know her and her circumstances. Thus an important element of analysis in this theoretical framework is immersing ourselves in the context in which subjects dwell and noticing what arises in us as a result.

**Methods**

The subjects of this study were mathematicians working together in pairs on some current mathematical research. One pair consisted of "Joseph," a mathematics professor at a large research university in the southwestern United States; and his doctoral student, "Bill." This pair was exploring an aspect of topology that they hoped would turn into a dissertation topic for Bill. The other pair consisted of two mathematics professors – one, "Matt," from the same campus as Joseph; and the other, "Ballard," visiting from Germany. These latter two had just finished a paper together in algebraic geometry and were exploring ideas for a new paper.

I observed each pair during three of their in-person meetings in which they worked on their research in front of a blackboard. The first pair generally met for an hour at a time, and the second pair for around 2-3 hours. Two cameras were set up on either side of the blackboard, each roughly 45 degrees out from the blackboard's center. The rooms were closed off from outside interference, so the cameras were readily able to pick up the subjects' speech. I asked each pair to do any writing on the blackboard rather than on pieces of paper so that the cameras could capture what they were doing. I operated one camera, and in most cases another person operated the other one. (The intent of this setup was to capture their research "as lived" in a way that would encourage embodied behavior. Mathematicians working while physically alone typically don't seem to move or speak much, and interviews would likely not have captured their embodiment of their work in progress.)

I then synchronized and spliced the cameras' data into a single video for each session (6 videos in all) so both perspectives could be seen side-by-side. I combed

through the videos multiple times to become familiar with the patterns of struggle depicted there and to make note of any particular embodied behaviors that seems especially significant in structuring the mathematicians' efforts. Based on this, I picked out several segments that seemed like they were key either to understanding the mathematics or to the mathematicians' experience of struggle. I then studied these segments carefully and performed unstructured interviews (Bernard, 1988) with each of the four mathematicians in order to develop enough of an understanding of the mathematics and the background for each clip to have a sense that I understood their perspectives in said clips.

In light of this context, I reviewed the videos again to select a few clips that seemed rich in their potential to tell us something about the experience of struggle for these subjects. Each selected clip was subjected to a multimodal microanalysis (Erickson, 2004; Nemirovsky & Smith, 2011), which yielded a movement-by-movement image sequence coordinated with a transcript. I then carefully combed through each sequence repeatedly in an ongoing effort to perceive the subtleties of the mathematicians' encounters with their material. From this, a number of types of embodiment of struggle emerged, which I then articulated and presented to colleagues in order to refine and clarify them.

**Results**

Here I will outline just one of the types that emerged, which I refer to as *proximal inhibition*. Three clips in particular helped to illustrate many of the nuances of this embodied structuring of struggle, although due to space considerations I'll illustrate the key moment from just one of those clips.

One pattern that kept emerging as significant from the data was the importance of proximity: the mathematician who was most animated would often define his relationship with the mathematical objects in question in part by how far away from his torso he would position them and how he put them there. For instance, he might "toss away" the output of a function if the output was not the main focus of his attention, whereas he might gesture as though to bring the output closer if he's mostly concerned with "what this function gives us." This use of proximity turned out to be an incredibly rich area of investigation, from which proximal inhibition emerged as just one of many facets.

Proximal inhibition is a structuring of struggle in which the mathematician seems to be pushed back from the blackboard's diagrams or inhibited in his ability to approach them. The inhibition results not from any physical inability to get closer, but instead comes from a sense that such a physical movement would not afford anything for the mathematics. However, the mathematician *wants* to "zoom in" to the particular aspect that is meaningful. This clash between the desire to modify the key part(s) of the diagrams and the inhibition from doing so due to not perceiving exactly where or how to make the key adjustment creates a palpable tension that an observer can often feel.

For instance, in one clip Joseph and Bill are trying to extend an argument from knot theory. Normally in the branch of topology they're studying (known as "Khovanov homology"), the way in which a knot was put together doesn't matter. But in this case, they're working with a slightly different kind of algebraic structure that makes the way in which a knot was put together highly relevant. The result is that many of the arguments they're familiar with almost, but don't quite, work as-is. After several minutes of silence

punctuated by occasional sentence fragments ("[If we] forget about the grading… well, then, then, um…"), Joseph starts to find the words for what he thinks is part of the confusion. Initially he seems to be holding the algebraic structure they're discussing in front of him as though it's an unruly thing taking up space but that doesn't seem to quite fit together (see Figure 1). But then his attention shifts to the board, where he seems to hope to express his point but is palpably driven back by his sense of uncertainty about where to "zoom in" (see Figure 2). This becomes especially vivid when we see the proximal inhibition lift slightly over one second later, at which point Joseph is abruptly drawn toward the board due to following the impulse that forms his recognition of where he can meaningfully interact with the notation on the board (see Figure 3). He even drops the hand that seemed to be pressing against a resistance (which, for him, was presumably trying to indicate a general region that he knew contained the element he wanted to approach).

It's worth noting that from other clips, we can see proximal inhibition manifesting to structure mathematical struggles we might initially think of as very different. In another segment, Joseph and Bill both seem to be pressed back from the board repeatedly in their effort to develop adequate notation. And in the other pair, Ballard and Matt both seemed proximally inhibited when trying to determine whether something they suspected was true. Furthermore, proximal inhibition doesn't always collapse the way it did in the example given in this discussion, although when it does it seems to result in this same kind of drawing-in-via-impulse that we see here. Yet clearly there is a kind of commonality, too, between these various cases, suggesting that proximal inhibition is a phenomenon we might reasonably expect to see in situations extending well beyond these particular instantiations.

**Implications**

The power of a phenomenological approach lies in its ability to transform how we perceive the world. In Merleau-Ponty's (1962) description of a man who can act through his body to, say, scratch his nose but who cannot reliably move his body in prescribed ways such as touching his nose on command, we can recognize a subtle difference in our and others' bodies when viewed anatomically as opposed to "as lived." This new perception applies to many circumstances beyond the one case study Merleau-Ponty illustrates. In the same way, explorations of the phenomenology of mathematicians' struggles in case studies such as these can teach us to see struggle differently, encouraging us to notice what was once unavailable to us and to ask new questions that would likely not have occurred to us at all otherwise.

Mathematical struggle in particular seems to show great promise as an area of exploration: there's fair reason to suspect that such struggle is a central part of what learning mathematics feels like (Brown, 1993; Hatano, 1988; Hiebert & Grouws, 2007; Schoenfeld, 1988; Wolf et al., 2006). It could be quite powerful, for instance, to explore if and how students demonstrate proximal inhibition, how they react to it, whether and in what ways it carries a different timbre for them as compared to mathematicians, and how such a transformation in the students' relationship to the math (if any) seems to occur over time. A comparison like this could tell us a great deal about what we'd like instructors to learn how to notice and what kind of implicit embodied framing is most likely to help students navigate their encounters with mathematics they find challenging.

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**Figures**


Figure 1: Joseph starts articulating a concern

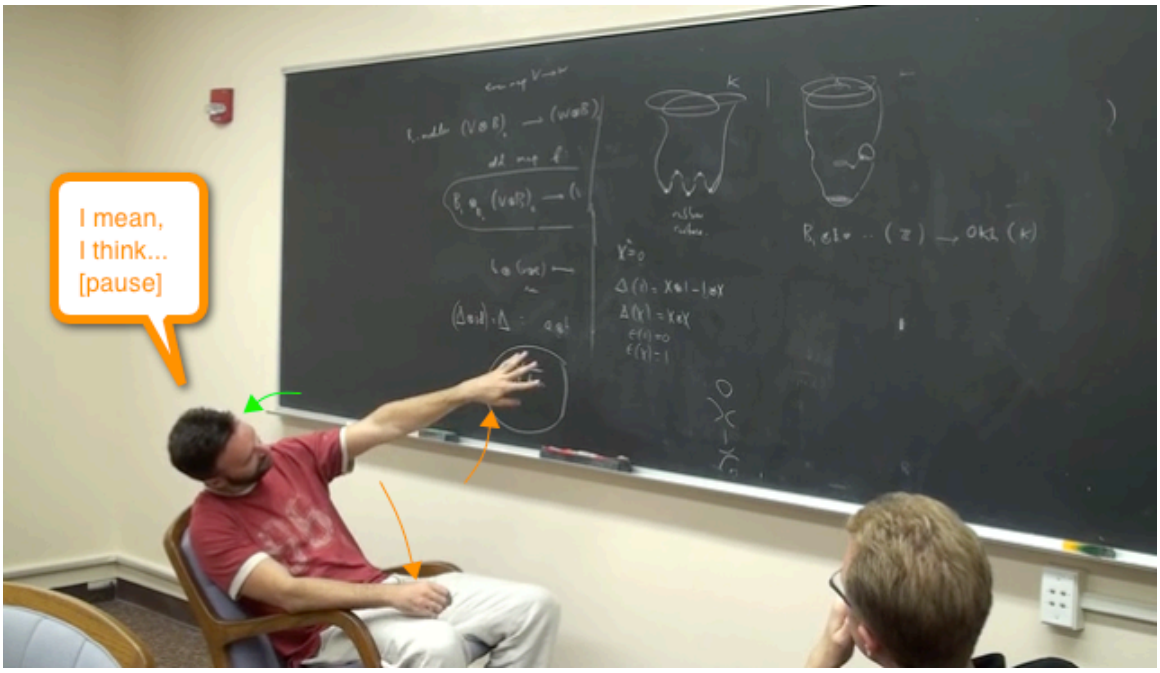

Figure 2: Joseph is “pressed back” from the board

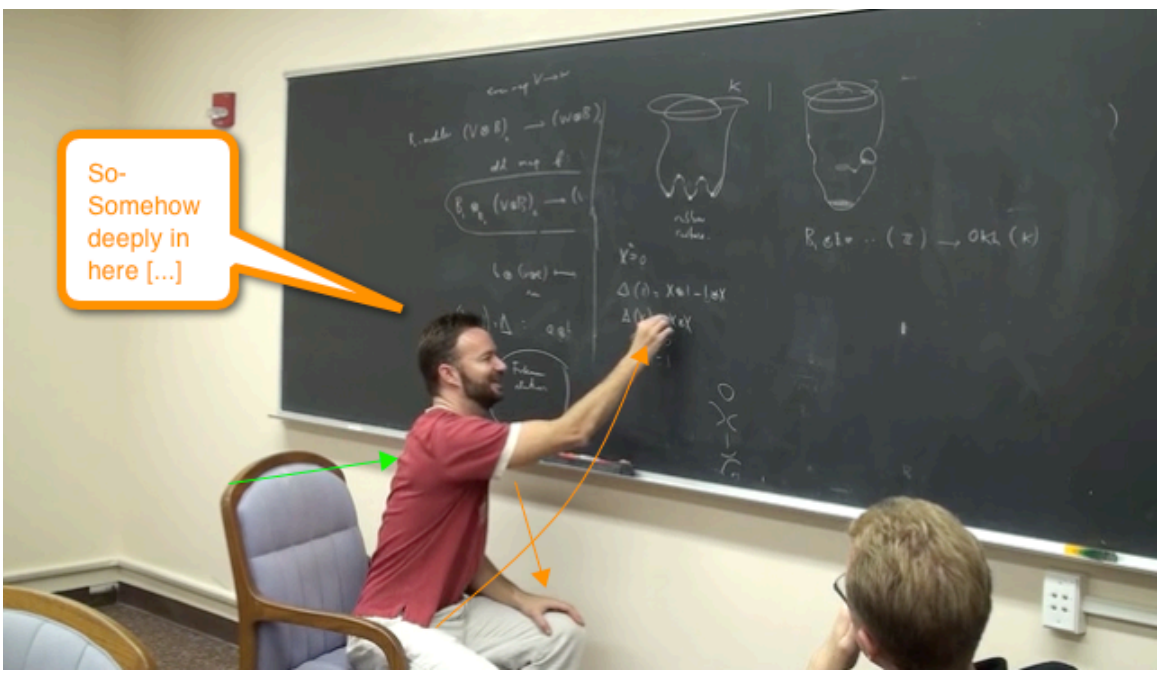

Figure 3: Joseph’s proximal inhibition lifts

# Teacher Change in the Context of a Proof-Centered Professional Development

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*Abstract:* The case study reported here examines the development of proof schemes and teaching practices of one in-service secondary mathematics teacher who participated in an off-site professional development (PD) for two years. Two sources of data were examined: video footage of the teacher doing mathematics at an intensive summer institute and footage of her own classroom teaching. Previously, an analysis of the development of the participant's proof schemes (Harel and Sowder, 1998) was reported, indicating a shift from Empirical to Deductive proof schemes. The current report focuses on the development of one participant's teaching practices during the academic years following each summer institute characterizing the development of: the way the teacher solicited student ideas, handled students' ways of understanding, capitalized on student thinking, and in the type of questions she posed. The report also includes theoretical connections between the development of the teacher's proof schemes, teaching practices and the PD.

*Key words:* Proof schemes, teaching practices, DNR-based instruction, secondary-level algebra, in-service teacher professional development.

Current reform efforts in mathematics education, based on research about learning and teaching mathematics, call for dramatic changes in teaching (Goldsmith & Schifter, 1997, p.20). In order to bring about real change in mathematics education, reform efforts must address, not only revisions of the mathematical content we expect students to learn, but also how teachers view the nature of mathematics and the effects their instruction has on learning. Researchers have identified a need for the creation of models that describe the process of change teachers go through as they attempt to alter their teaching practices in an effort to make meaningful changes in line with current reform efforts advocated by the NCTM (Cooney, 1994; Goldsmith & Schifter, 1993). Schifter (1995a) identified *one* "strand" of teacher change based on her classroom observations, professional development of over 250 teachers, and the reflections of these participants. Using this evidence, she proposed a model for development of teacher change along this one strand. As researchers work to build better models that describe the development of teacher change, generation of other "strands" of teacher change will be necessary.

Although there exists a lack of consensus regarding the role that mathematical content knowledge plays in teachers' practice (cf. Begel, 1979; Monk et al, 1994; Ball, 1991; Hill, Rowan, & Ball, 2005), mathematics educators (cf. Shulman, 1986; Ball,1991; Harel, 1994), have been reluctant to dismiss the intuitive argument that teachers need to know their subject matter well to be effective, arguing that it is important to reframe the debate by rethinking the nature of the variables used to determine content knowledge. A teacher's proof schemes are one important dimension of content knowledge worthy of attention due to the central role of proof in mathematics and a relative lack of knowledge of in-service teachers' proof schemes on the part of professional developers' who might want to know how teachers' dominant sources of conviction can be influenced (Knuth, 2002). The eventual goal of this line of research is to find connections between teachers' proof schemes and teaching practices.

The immediate goal of this case study is to answer the following questions: Given a set of *teaching practices* related to proving, selected from those observed at the PD, grounded in the data, and whose importance is acknowledged in literature, which of these teaching practices are reflected in one participant's teaching? To what extent do her teaching practices reflect those selected? Looking at the dominance of given teaching practices chronologically, can we point to any development? What connections can be made between the changes observed in the participant's proof schemes during the institute and those changes observed in her teaching practices during the academic year?

**Theoretical Perspective: DNR-based instruction in Mathematics**

DNR-based instruction in mathematics (Harel, 2001, in press a, in press b) is a theoretical framework that stipulates conditions for achieving critical goals such as provoking students' intellectual need to learn mathematics, helping them acquire mathematical *ways of understanding* and *ways of thinking*, and assuring that they internalize and retain the mathematics they learn. While Harel and Sowder (1998) clarify the proof schemes construct in DNR, other constructs pertinent to this research have not been explained. Since the goal of the overall case study is to investigate the connection between the development of a participant's proof schemes and *teaching practices*, it is important to clarify what *teaching practices* are in DNR-based instruction. In DNR, "a *teaching action* is a curricular or instructional measure or decision a teacher carries out for the purpose of achieving a cognitive objective, establishing a new didactical contract (Brousseau, 1997), or implementing an existing one." (Harel, in press) Characteristics of teaching actions are called *teaching behaviors*. Teaching actions and teaching behaviors taken together are called teaching practices.

**Research methodology**

Participants attended, and researchers videotaped, an intensive summer institute meeting 6 hours per day, 5 days per week, 4 weeks per summer, for 2 summers taught by a math education researcher (Teacher-Researcher: TR). For two academic years a research team member met with participants on a tri-weekly basis to discuss issues of teaching and learning. Researchers observed the teachers' instruction of students, videotaping and then transcribing the lessons. This study focused on the development of proof production and proof schemes in one participant, Maggie, across two summer institutes, making use of the proof schemes framework outlined by Harel and Sowder (1998). This analysis was qualitative in nature. In order to document the development of Maggie's proof schemes over the two summers, descriptions Maggie's proofs were provided along with characterizations of her sources of conviction whenever conjectures were made.

Development of Maggie's teaching practices was investigated during the two academic years following each summer institute characterizing development of: the way she solicited student ideas, handled students' ways of understanding, capitalized on student thinking, and the type of questions she posed. This analysis made use of a mixture of a priori coding from literature, prior research of the PD, and grounded analysis of the her teaching, to derive categories that could be used to characterize her teaching actions over time. While a quantitative argument is made to document the emergence of certain practices over time, meaningful events were also described to illustrate important ideas and help provide insight into the development of her teaching practices over time.

## Results of the research

In a previous report, it was found that over time in the PD Maggie moved away from primarily authoritative and empirical proof schemes (of the perceptual and inductive sort) toward deductive proof schemes (of the transformational sort). This transition took place in the context of an instructional approach that: made repeated use of inter-related series of tasks which were non-routine in nature, placed within a broader context first and decontextualized later, placed a high emphasis on anticipating and defending attacks to arguments, and provided consistent opportunities to convince others of the truth of assertions. Pattern generalization was also typically present in many of the problems assigned.

In the classroom data, it was found that over the course of the two years Maggie: encouraged more student to student talk, encouraged students to prove their conjectures more frequently, allowed student errors to persist with greater frequency, and asked for alternative solutions in the presence of correct solutions with greater frequency.

The strongest connection observed between the developments in Maggie's teaching practices, the development in her proof schemes, and the Teacher-Researcher's (TR) teaching practices is related to mathematical rigor. In Maggie's teaching it was observed that while she attended to student's mental images throughout the two years, her practice showed development in extending the locus of authority and soliciting alternative solutions in the presence of correct solutions. At the PD, her exposure to these teaching practices began on day 1 when TR said, "Don't look at me. Convince your friends." It was a consistent practice at the PD to solicit multiple correct solutions and for the instructor of the PD to use student solutions to point toward a need for more powerful tools that could solve problems participants could not solve with their existing problem solving approaches.

## Applications to / implications for PD and influencing teaching practice

Descriptions of the process of change teachers go through as they attempt to make lasting and meaningful change to their teaching practice is of paramount importance to mathematics educators advocating reforms in mathematics instruction; as are descriptions of the development of proof schemes of in-service teachers and the interventions that instigated them. Over the past decade we have seen many examples of professional development programs, but there is still much work to be done if we hope to propose a coherent explanation of the mechanisms of change PD programs strive to bring about. This case study helps inform professional developers about possible developmental paths participants can take given a particular PD effort. Furthermore, exploring the connection between proof schemes and teaching practice is a worthwhile and difficult endeavor which will require more targeted study.

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# Experts' Reification of Complex Variables Concepts: The Role of Metaphor

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Abstract

Using a theoretical perspective of embodied cognition, we explored how six experts integrated metaphors to reason and communicate about arithmetic and analytic complex variables concepts. We found that experts who displayed evidence of reification of a complex variables concept or had a need to use a concept imparted their sense of understanding through dynamic representations blended with metaphors. These metaphors were often invented or reinterpreted, based on personal experiences and created to convey nuances of the experts' understanding to students. The experts appeared conscientious of using metaphors relevant to their own students. This research may support practitioners' efforts to create opportunities for students to reinterpret experts' metaphors into personally meaningful metaphors that both capture important mathematical concepts accurately and align with their own understandings, experiences, and culture. Further research may investigate how technology may serve as a tool for such an endeavor.

Keywords: Complex variables, Embodied cognition, Mathematicians, Metaphor

## Introduction and Literature Review

One of the undergraduate mathematical domains that has not received much attention from education researchers is complex numbers and variables. Given there is an immense amount of literature investigating students' understanding of real numbers, ranging from the meaning behind arithmetic operations (Sowder, 1992) through analysis of real-valued functions (Alcock & Simpson, 2004), it is natural to extend these studies to complex numbers, their operations, and functions. Such studies may provide insight into ways to strengthen students' understanding of, representations of, and fluency with operations on intimately related concepts involving vectors, matrices, and transformations all deemed as goals for the 9-12 curriculum. Our research is designed to contribute to the literature on teaching, learning, and understanding undergraduate mathematics. This report is part of a larger exploratory study in which we investigate experts' geometric reasoning about complex variables in an effort to create a framework based on empirical evidence that describes how one perceives and reasons with central ideas from complex variables. In this paper we address the research question: What is the nature of experts' use of metaphor in conveying their perceptions of the arithmetic and analysis of complex variables concepts?

There is limited research investigating the understanding of complex variables, but there are a handful of empirical studies that have begun to pave the road in this domain. In their work on embodied cognition, which we discuss in more detail in the following section, Lakoff and Núñez (2000) presented a framework for the conceptual development of complex numbers. Their framework blends the real number line, the Cartesian plane, and rotations with the use of metaphor for number and number operations. They began by imparting physical meaning to the product of a real number $x$ with $-1$, as a rotation of $180^\circ$ to obtain $-x$. Similarly they depicted multiplication by $i$ as a clockwise rotation of $90^\circ$. Lakoff and Núñez' perception of these numbers as operators that transform an object might suggest that if students perceive

multiplication by –1 as a rotation of $180^\circ$, then they might easily recognize that multiplication by $i$ results in a clockwise rotation of $90^\circ$. Contrary to this framework, Conner, et al. (2007) found that prospective secondary teachers viewed multiplication by –1 in the complex plane as a reflection rather than a rotation. This result could be attributed to the fact that the students focused on the real number line rather than the entire complex plane.

In another study related to complex numbers, Danenhower (2006) examined undergraduate's ability to convert instantiations of the fraction $\frac{a+ib}{c+id}$ to either Cartesian ($x + iy$) or polar form ($re^{i\theta}$). The varied representations included taking the modulus of the numerator, raising the factors in the numerator and denominator to a power, expressing the denominator in terms of sine and cosine, and combinations of these forms. The undergraduates worked flexibly with complex numbers when represented in Cartesian form, but this was not the case with polar representations because the participants were not comfortable with trigonometry. One of the most significant contributions of Danenhower's work was his observation of a phenomenon, which he referred to as "thinking real-doing complex." In his dissertation, Danenhower (2000) briefly explained how this phenomenon emerged when students applied their understanding of $R^2$ while working with complex valued expressions and functions. For example one student attempted to determine if a complex valued function was differentiable by inspecting the mapping. Students' comfort level with $\mathbf{R}$ and $\mathbf{R}^2$ could attribute to the students' preference of the Cartesian form over polar form. Danenhower's findings also suggested that the undergraduates did not attend to geometric representations of the complex number, which could have alleviated much of the computational effort. His work suggests that his participants were limited to viewing $i$ as a static object, and did not possess a dynamic view of multiplication by $i$ as an operation, which acts on other objects. Nemirovsky et al. (in press) presented promising results based on a teaching experiment with preservice secondary teachers. The goal of their teaching experiment was to provide students with an instructional sequence where they created conceptual meaning for adding and multiplying complex numbers. Using methods from microethnography, the researchers generated detailed characterizations of students' gestures during short episodes of the teaching experiment. They found perceptuo-motor activity was central in (1) conceptualizing, (2) communicating geometric representations, and (3) creating a learning environment that influenced the development of structural components behind adding and multiplying complex numbers.

## Theoretical Perspective

Embodied cognition serves as our theoretical perspective and stems from the theory of enactivism. This theory asserts that "the individual knower is not simply an observer of the world but is bodily embedded in the world and is shaped both cognitively and as a whole physical organism by her interaction with the world" (Ernest, 2010, p.42). Metaphor plays a central role in the philosophy behind embodied cognition. Sfard (1994) connected her work on reification with metaphor. She defined reification as a capstone of the development from operational to structural reasoning and focused on the particular metaphor of mathematical constructs as physical objects. Thus for her, reification is the creation of metaphor. Sfard also paraphrased Lakoff and Johnson's definition of metaphor as a "mental construction, which plays a constitutive role in structuring our experience and in shaping our imagination and reasoning" (p. 46). According to Lakoff and Johnson (1980) embodied schema also known as image schemas are the mechanism for creating metaphors. They are structures of an activity by which we organize our experiences in order to create meaning. These image schemas may not be rich in

detail, but they are embodied and can encompass multiple and diverse experiences. Lakoff and Núñez (2000) extended the idea of embodied metaphors to the discipline of advanced mathematics topics. They advocated that "general cognitive mechanisms used in everyday nonmathematical thought can create mathematical understanding and structure mathematical ideas" (p. 29). Furthermore, they described image schemas as the link between language and reasoning and vision.

## Research Methodology

In an effort to obtain rich data, we selected a purposeful sample of six expert participants. The participants Ricardo, Anton, Mark, and Beth were selected based on our personal interactions with them or based on student comments. Upon interviewing Beth, she suggested we interview her colleague Luke and Jane with whom she collaborates on complex analysis research. Ricardo and Anton are also colleagues at the same institution. All the participants are PhD mathematicians except for Mark, who is a PhD physicist. The experts participated in a 90-minute video-taped interview, where two researchers posed questions aimed to reveal the participants' physical interpretation of arithmetic and analytic concepts related to complex variables. We informed the participants that we were investigating their geometrical interpretation of complex numbers and complex variable topics. We conveyed our interest in their use of gestures, diagrams, illustrations, and facial expressions, but we did not use the word metaphor. The participants described their connections between algebraic and geometric representations of addition, multiplication, division, and exponentiation of complex numbers. They also conveyed their geometric perceptions of continuity, the Cauchy-Riemann equations, differentiation, and line integration of complex-valued functions. Probing was used throughout the interview in order to elicit ways in which our participants might incorporate geometric or visual interpretations in explaining ideas to novices such as undergraduates. Four members of the research team each transcribed and conducted an initial analysis documenting where and how a participant conveyed her or his perceptions using geometric methods. After this individual analysis, as a team we watched every interview in its entirety multiple times to determine common themes among the participants' responses. It was during this time, that we noticed the experts' repeated use of metaphor, which led to a more focused analysis of the data. During this phase of the analysis, we attempted to find and describe common uses of metaphors blended with bodily enactments.

## Results

Our results suggest that participants who displayed evidence of reification of a complex variables concept imparted their sense of understanding through dynamic representations and gestures blended with metaphors. These metaphors were often invented or reinterpreted, based on personal experiences, and created to convey nuances of the experts' understanding to students. While not all of our interview questions elicited enactments combined with metaphors from every participant, the items regarding the arithmetic operations of complex numbers and the continuity and differentiation of complex valued functions produced similar actions and metaphors. Most of the experts found the questions about exponentiation of complex numbers, the Cauchy-Riemann equations, and line integration of complex valued functions novel and hesitated to create meaning of these situations. In this proposal, we briefly describe the responses to the addition and continuity interview items in an attempt to give the reader a taste of prototypical responses.

For the arithmetic questions we provided a drawing of the Argand plane with two complex numbers $z$ and $w$ and asked the participants to determine where $z+w$, $zw$, $\frac{1}{z}$ and $z^w$ were located

on the Argand plane. We also asked them to make connections between the algebraic and geometric representation. It may not be surprising that all the experts described addition of complex numbers in terms of vector addition and illustrated a parallelogram created by the two vectors corresponding to the two complex numbers. For example, Beth commented, "*You can think of them as vectors. So I looked at what vector z looked like."* It is interesting that Beth no longer referred to the complex number $z$, but rather the vector $z$, which suggests she viewed complex numbers as physical objects. Accompanying gestures to the parallelogram model included starting at a point $z$ and then sweeping an index finger in the horizontal direction followed by a sweep in the vertical direction to indicate adding the complex number $w$. Some participants used pincher fingers, formed with their thumb and index finger, or their two index fingers to denote the length from the origin to the real component of $w$ and used their pincher fingers as a measuring tool to measure off the distance from the real component of $z$. Similar actions were used for the imaginary component of the complex numbers, which allowed the experts to communicate their understanding between the algebraic and geometric connections of adding complex numbers. Using the sweeping motion, Ricardo mentioned that each vector has a "*motion*" in the horizontal and vertical direction. Both Luke and Ricardo stressed the facility of thinking about addition in terms of rectangular form and then simply adding component wise. Luke remarked, "*...you can see in the picture that if you just look at the x-components of the two complex numbers that you get the x-component of the sum that I've drawn, and similarly for the y-components.*"

Anton hesitated to connect the geometry and algebra of the addition of two complex numbers because there was not a scale on the real and imaginary axis. Anton commented, "*In this particular example, geometry is pretty much the way to go. And since it's the natural way to go, there is no reason to go for algebra*." For this same item, Jane commented, that she would have to draw "*randomly"* because she didn't know the location of the unit circle, which was important for her research. These comments were surprising given one does not need the unit circle to construct the desired parallelogram. Similar remarks were made about the items regarding the multiplication and division of complex numbers, where the unit circle does play a prominent role. In our presentation, we will elaborate on how our experts made use of the unit circle, focused on polar representation, viewed complex numbers as operators, and used hand gestures to demonstrate rotations, expansions, and reflections while responding to the multiplication and division items. Figure 1 illustrates two ways in which Ricardo connected and illustrated the rotation resulting from the multiplication of two complex numbers. We will also share how these bodily movements were merged with metaphors such as a bicycle wheel, turn-tables, spinners, etc. were prominent in the differentiation question.

In the above item, all the experts clarified that using vectors to illustrate addition of complex numbers is natural because students are familiar with vector addition. Since continuity of complex-valued and multi-variable real-valued functions is the same, it did not seem unusual for some experts to use this concept to provide a geometric representation or explanation to convey their understanding of continuity of complex valued functions. Several participants commented that such an explanation allowed them to make connections to students' prior knowledge. Our participants also presented metaphors, which they believed would be relevant to their own students. The images of these metaphors including parking lots, bombs on target, painter's palette, elastic bands, archery competition, and hiking, were combined with drawings, enactments of the metaphor, and gestures from the experts. For example in her hiking metaphor, Jane explained, "… *so then if I'm thinking of the pen off the page,* [used marker positioned out

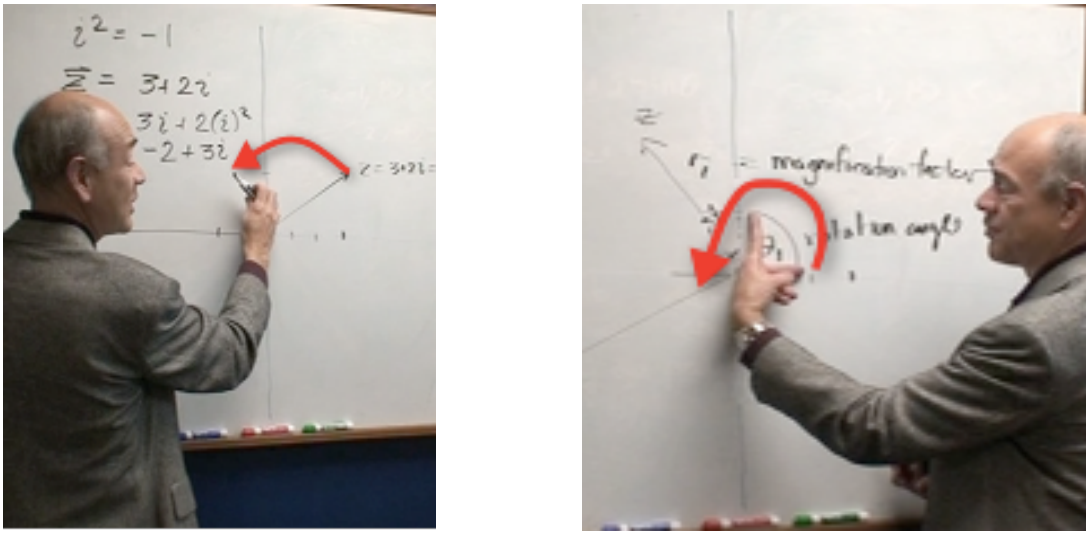


Fig. 1. Ricardo illustrating a rotation as a result of multiplying two complex numbers

from the board to trace the height values] *I might be trying to trace out on the surface, trace out those height values and see if I can draw them, I'm not taking my pen off the page now, but I, I don't want to jump my pen anywhere* [traced heights with marker, then pulled marker out from board to illustrate a jump]. *That's my analogy of- I don't want my pen to make any sudden precipitous drops. So, I often use a hiking analogy, especially in the classroom setting, because the students are familiar with contour maps and falling off cliffs or not falling off cliffs…. So if I can draw this without sudden change of altitude* [traced and pushed pen into the board], *then I'm continuous.* An interesting aspect of this 3-D metaphor was that Jane's description fit with what one would observe with two-variable functions – again because the complex valued and multi-variable real-valued functions are the same.

The intriguing aspect of using multi-variable real-valued function to describe continuity of complex-valued function was that the participants were thinking real while doing complex (we also witnessed this with the differentiation and integration items), which is in line with Danenhower's research (2006). They thought in terms of functions that map from $\mathbf{R}^2$ to $\mathbf{R}$ rather than from $\mathbf{R}^2$ to $\mathbf{R}^2$. We also observed this phenomenon with participants who chose to convey their understanding of continuity by discussing discontinuity though the use of metaphor. For example, Ricardo and Beth used a tearing paper and a silly putty metaphor respectively. Essentially, they both described discontinuity as things that start close together ending up far apart. In Beth's description she commented, "*… in the analogy of not lifting up the pen is if you made a region out of silly putty, and you applied the function to every point in that region, what would that shape look like.*" As she made this statement she clasped her hands together in a horizontal position, rubbed them together as if rolling silly putty, stopped and arced her arms with hands together to indicate the mapping, then she separated her hands. She further remarked, "*Would you have to rip the silly putty to get there?*" as she put her fists together followed by pulling her hands apart in opposite directions. She completed with the statement, "*An analog to not lifting your pencil, where we usually think of discontinuity as having a break in the graph, in complex we think of there being a tear in the image.*" This sequencing is illustrated in Figure 2. Beth effortlessly switched from an image mapping from $\mathbf{R}^2$ to $\mathbf{R}$ (tracing a curve on a surface) to an image mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$ (separating the silly putty).

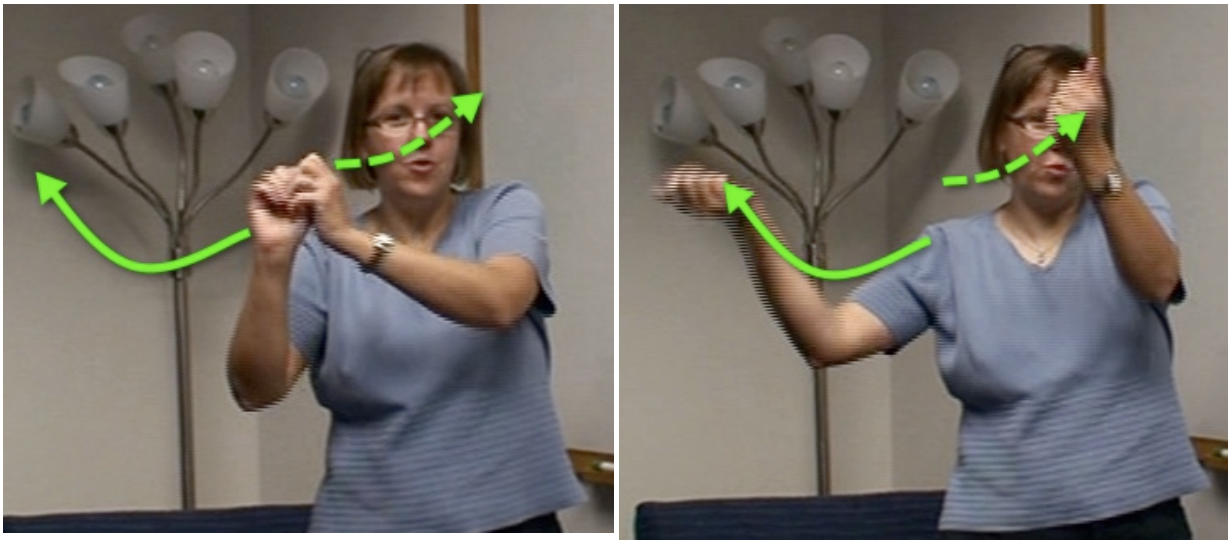

Fig. 2. Beth illustrating discontinuity of complex valued functions through silly putty

Mark provided a different perspective, in terms of how discontinuity is very problematic in the area of physics. He used gestures to explain that, *"... you need to be very careful about your models ... to make sure that you're staying physical ... that you know how to interpret the particular model. You imagine a charge at rest* [put one palm behind the other, with thumbs up, while his hands were facing him as if making a vertical plane] *that you suddenly exert a constant force on* [pushed hands forwards]. *There's a discontinuity there, the force went from 0 to some finite value for a certain amount of time and then drops to zero suddenly*." He elaborated on the fact that sometimes models need to be refined by determining what it would take to make a function smooth. As he made this statement he made a motion with cupped hands as if running his hands over a bell-shaped curve.

Anton was the only participant to not provide a metaphor for the continuity item. This might be attributed to the fact that he saw no need for a metaphor because as he pointed out, "… *in complex analysis continuity is not the most important one. The most important one is the idea of analyticity. So you don't really think about the continuity*." Similar comments were made about the question regarding the exponentiation of complex numbers. Everyone except for Ricardo explained that he/she had no need to think about raising a complex number to a complex number but expressed a need to consider exponents of the form $\frac{1}{n}$ where $n$ is a whole number, for research purposes.

## Discussion

Our findings suggest that experts tended to create metaphors relevant to their own students' experiences and did not bring objects into being if they did not see a need to work with a particular concept. Danenhower's (2006) phenomena of "thinking real-doing complex" was also prominent in the experts responses. Our research suggests that investigating experts' use of dynamic imagery and metaphor may allow researchers to gain insight into the development of systematically structured conceptual understanding. This insight may support practitioners' efforts to create opportunities for students to reinterpret experts' metaphors into personally meaningful metaphors that both capture important mathematical concepts accurately and align within their own understandings, experiences, and culture. As Sfard (1994) pointed out, "*Because of the tight relationship between the metaphor of an ontological object and the issue of*

*visualization it seems that today's wide accessibility of computer graphs opens promising didactic possibilities*" (p. 54). This is in our radar for future research.

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# Notion of Reducing Abstraction in Teaching: The Case of Mathematics Instruction

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*Mathematics is an abstract subject. When teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. Reducing Abstraction in Teaching (RAT) is one of the theoretical frameworks that provides a window for looking at how teachers deal with abstraction in teaching. By analysing teaching practices of two of the mathematics teachers in college preparatory course, this paper illustrates various tendencies of teachers dealing with mathematical abstraction. It also exemplifies some instances where 'reducing abstraction' seems to be an effective teaching strategy while in other cases it may go unsupportive for the development of students' mathematical understanding.*

## Introduction

Abstraction is often seen as the fundamental characteristic of mathematics; and it "has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (Ferrari, 2003, p. 1225). As such, in the recent years, abstraction has received a growing interest in research community among psychologists and mathematics educators. In fact, when teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. Hence, my aim in this paper is to explore the notion of abstraction from teaching view point, particularly in the context of mathematics instruction. More specifically, I will attempt to answer the following questions: 1) how teachers deal with abstraction in mathematics teaching? 2) Can the Reducing Abstraction in Teaching (RAT) framework suggest a plausible explanation for the action of teachers and sources of teaching activities? Here is the brief itinerary for the rest of my paper. First, building on the Hazzan's (1999) work, I briefly attend to the notion of reducing abstraction in teaching (RAT) framework. Second, I provide an overview of the methodology followed by the results and discussion. Finally, some concluding remarks will follow.

## Theoretical Framework

Hazzan's ( 1999) research on how undergraduate students learn abstract algebra is an important work that provides a window to look at the mental process of students while learning new mathematical concept. Her finding is that students usually do not have the mental construct or resources 'to hang on to' to cope up with the same abstraction level of the concept as introduced by the authorities (teacher, textbook etc.). Consequently, they tend to reduce the level of abstraction in order to make the concept mentally accessible. In other words, when a student sees a mathematical object, he or she will try to make sense of it based on his or her past experiences with other mathematical objects.

From teaching view point, this idea suggests that while introducing new mathematical concept, teachers should make an effort to use students' previously acquired knowledge, experience and level of thinking as well as their familiar contexts. Safuanov (2004) suggests:

> "Strict and abstract reasoning should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient supply of examples and facts at a lower level of abstraction" (p.154).

This idea is in line with many other psychologists and educators (see Hershkowitz, Schwarz, & Dreyfus, 2001; Piaget, 1970). For example, Piaget's idea of developmental psychology and genetic epistemology tells that children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. From this perspective, effective teaching should involve with the process of introducing new abstractions; concretising or semi-concretising them; then repeating at a slightly higher level. That is, the concept are concretized and presented to the students in a lower level of abstraction temporarily. The goal is however to go to the higher level of abstraction using the lower level as stepping stone. This activity certainly is an attempt to reduce level of abstraction of the concept on teacher's part in order to make the concept mentally accessible to the students. Hence, the notion of Reducing Abstraction in Teaching (RAT) comes into play. Because of the space limitation, detailed discussion of the framework is not possible here. I, however, provide a brief overview of the RAT framework.

Building on the work of Hazzan (1999), Wilensky (1991) and Sfard (1991), RAT framework provides three interpretations for abstraction level, all of which interpret teacher's action as some way of reducing abstraction of the concept. These three categories have been further divided into subcategories in order to incorporate different nature of the teachers' act of reducing abstraction:

***Category 1: Abstraction Level as the Quality of the Relationships between the Mathematical Concept and the Learner***

It is based on the Wilensky's (1991) assertion that whether something is abstract or concrete is not an inherent property of the thing, "but rather a property of a person's relationship to an object "(p.198). On the basis of this perspective, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). Reducing abstraction in this category refers to the situation where an attempt has been made to make unfamiliar (therefore abstract) concept more familiar (therefore concrete) to the students by any of the following ways:

**1.1 FamRw** : Reducing abstraction by connecting mathematical concept to real-world situations

**1.2 FamLang**: Reducing abstraction by using familiar but informal language rather than formal mathematical language

**1.3 FamRep**: Reducing abstraction by connecting new mathematical concept to familiar representations (that includes use of pedagogical tools such as graphs, diagrams, tables, metaphors, gestures, manipulative etc.)

***Category 2: Abstraction Level as Reflection of the Process-Object Duality***

Reducing abstraction in this category is based on Sfard (1991) theory of 'process-object duality' which states, "abstract notation such as a number, function etc. can be conceived in two fundamentally different ways: structurally- as objects and operationally- as processes" (p 1). According to this theory, the process conception is less abstract than an object conception. This category involves the following tendencies:

**2.1. DuProc**: Teacher reducing abstraction by shifting the focus on procedure even though the problem or discussion implies a focus on concepts, meaning, or understanding

**2.2 DuAns**: Reducing abstraction by shifting the focus on answer even though the problem or discussion implies a focus on concepts, meaning, or understanding

***Category 3: Degree of Complexity of Mathematical Concepts***
In this category, abstraction level is determined by the degree of complexity. The working assumption here is that the more complex a problem or concept is the more abstract it is (Hazzan, 1999). Reducing abstraction in this category involves the following situations.
**3.1. CompxPG**: Reducing abstraction by shifting focus on particular rather than general (thus making the problem less complex which often results with having a partial picture of the concept rather than the complete one.)
**3.2. CompxRO**: Reducing abstraction by routinizing the problems (by taking over the challenging aspects of the problems either by telling student how to solve or by solving the problem for the students which often results with reduction in complexity, but takes away the opportunities for students to do mathematics on their own.)

**3.3. CompxSC**: Reducing abstraction by stating the concepts rather than developing it.

**3.4. CompxGA**: Reducing abstraction by giving away the answer in the question or provide more hints than necessary (Topaze effect- See Brousseau, 1987)

I want the reader to note that these three categories of abstraction should not be thought of as hierarchical or disjoint; they are rather intersecting, or one may even emerge from the other. So, based on the perspectives one takes, one category of reducing abstraction can be thought of as reducing abstraction in the other category. I, therefore, assign the teacher's act of reducing abstraction to the categories that I deem they fit best.

## Methodology

The research questions that guided this work are: 1) how do teachers deal with abstraction in mathematics teaching? 2) Can the Reducing Abstraction in Teaching (RAT) framework suggest a plausible explanation for the action of teachers and sources of teaching activities? The strategy for gathering data consisted of an observation of two university preparatory mathematics classes at a university taught by two different teachers, who are well experienced and professionally trained mathematics educators. This course is offered to those students who were identified as having some kind of deficiency to enroll to first year regular university course such as calculus I. Each lesson lasted about an hour and half. All data were collected by the author, who attended the lecture and took extensive field notes. As much as possible, the phrases, statements or sentences the instructors used to explain the concepts or solve the problems including some observable behaviour such as 'gestures' as well as students response that the observer found relevant for the study were noted down. An audio or video recording was avoided due to the risk of influencing the natural classroom settings.

## Results and discussion

Due to the space limitation, only one example has been selected, analysed and presented here.

### Example:

Teacher posed the question: *Given $2x + 4y = 16$; $4x - 3y = 6$ and $3x + y = -2$. Graph the lines and label them. Do they form a triangle?*

*(T= teacher, G= a group of student, S= an individual student):*

T: How can you graph these equations? (Paused about 4 seconds) Let me show you how. I choose the first equation first and show you how to graph it, ok…? Use the cover up method. I cover 4y (she covers 4y with her hand and completely hide it from the scene). Now tell me what is the value of x?

G: 8 (group response)

T: So, We have one point (8, 0).

T: Now if I cover 2x. (She covers 2x with her hand). What is the value of y?

G: 4 (group response).

T: So, the other point is (0, 4). Now we have two points (8, 0) and (0, 4). Let me plot these points on the graph and draw the straight line.

S: Oh, I see. That's easy!

Understanding the relationship between the graphical representation and algebraic representation of a linear function is one of the most important concepts in this level. These concepts are introduced in the textbooks by three methods: 1) Slope – intercept form ($y = mx + b$); 2) Table method (by randomly plugging in few values for independent variable and calculating corresponding values for dependent variable); 3) Intercept method (finding x and y- intercept, in which case student should know that on the x-axis, y-coordinate is zero and vice versa).

The 'cover up' method as used by the teacher is not fundamentally different from the method (3) above. On the x- intercept, y- coordinate is zero. Hiding 4y with her hand (gesture) to find the value of x, teacher is using "Zero is the lack of an object" metaphor (Lakoff & Nunez, 2001, p.372). Her gesture and the use of metaphor significantly reduce the level of abstraction for the students. This is an attempt from the teacher's part to make the unfamiliar 'intercepts' concept more familiar with the use of gesture and "zero is the lack of an object' metaphor. From this perspective, this act can be interpreted as reducing abstraction in the first category *1.3(FamRep*).

Viewed from the other perspective, it can be put in the process-object duality because the cover up method emphasizes the process conception - how to do it but not what it means. This act can be interpreted as reducing abstraction in category *2.1( DuProc).*

It should be noted however that the teacher's intention to use this method might be to make the process easier while keeping the concept meaningful to the student. But, student's response in the second question below reveals that student did not understand the concept as expected by the teacher.

T: To draw the line for the second equation (pointing to $4x – 3y = 6$), we need to find any two points, yeah! Let's find them. (After an instance of mental calculation, teacher writes (0, -2) and (3, 2) as two points).

S: How did you get (3, 2)? It has to be (1.5, 0). That's what I got.

At this moment, there was confusion among most of the students as to how the teacher found the points (0, -2) and (3, 2). It is evident from the dialogues that the students could find the correct points on the line mechanically but with no meaningful understanding. For them, (3, 2) could not be the point on the line.

T: Oh, I see what you are talking about. Um... cover up the -3y (she covers -3y). Now tell me what is x?

S: 1.5.

T: I don't like that number. So I choose x= 1. Then y= 0.66666…, right? Umm... still I don't get a 'nice' number. If I choose x= 2 then... still I get 'ugly' number. For x=3, what do I get for y?

G: 2

T: So, (3, 2) is one of the points in the line.

S: So, we have to get the whole number, always …?

T: No, no, no. You see, working with whole number is easier than fractions, right? I am making your life easy. If you like, you can have (1.5, 0) as one of the points. The dialogue continues…

The teacher refers to the fractions as 'ugly numbers' and prefers to use whole number (nice number). Students often find fractional numbers difficult to work with. And so is the case with plotting them in a graph paper. The choice of the whole number reduces the difficulty of the situation. However, one of the student's responses "So, we have to get whole number, always…?" reflects the fact that teacher's choice of the whole number provided only a partial picture of the concept. This illustrates the reducing abstraction in category 3.1 (*CompxPG).*

When this situation comes to the notice of the teacher, in the later part of the dialogue, she makes it clear that there are, in fact, infinitely many points in a line but for the sake of simplicity, a 'nice' number (non-fractional number) was chosen.

## Conclusion

In this paper, my aim was to explore how teachers deal with mathematical abstraction in teaching. In order to answer this question, I found it helpful to use RAT framework to explore the actions of teachers and sources of teaching activities in regard to dealing with mathematical abstraction. As has been exemplified, reducing abstraction in teaching, in some cases, proved to be an effective teaching strategy. However, in other cases, it may not be supportive of the development of mathematical knowledge for the students. Finally, the results emphasize the importance of paying attention to the nature of students' understandings that may arise as a consequence of reducing abstraction of the concepts in teaching.

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# Prospective Elementary Teachers' Evolving Meanings for Generalizing, Doing Mathematics and Justifying

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Abstract

We viewed the classroom as a culture of mathematizing (Bauersfeld, 1993) and documented evolving meanings that prospective elementary teachers gave to their instructor's expectation that they "find general solutions," "do mathematics" and "justify solutions" during a semester-long inquiry-based course. Classroom observations and interviews with student informants suggest that almost three weeks passed before the students in the class gave normative meanings to their instructor's request for general solutions and to her expectation that they do mathematics, and it was not until the eleventh week that students understood that justifying a solution meant providing a mathematical argument that explained why the solution was valid. Furthermore, the data suggest that giving normative meaning to these mathematical activities is prerequisite to success, and that as students came to make sense of generalizing, doing mathematics and justifying, they improved in their abilities to do these activities and they began to see them as valuable.

Key Words: sociomathematical norms, classroom culture, justification

## Introduction

In this paper, we document evolving meanings that prospective elementary teachers gave to their instructor's expectation that they "find general solutions," "do mathematics" and "justify solutions" during a semester-long inquiry-based mathematics course. We assumed that these meanings were conceived through primarily cultural and social processes (Cobb & Bauersfeld, 1995), and thus best observed through the lens of the classroom as a culture: "The understanding of learning and teaching mathematics … support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using mathematics, or better: a culture of mathematizing as a practice" (Bauersfeld, 1993, p. 4).

Yackel and Cobb (1996) suggest interpreting mathematics classroom cultures based on both social and sociomathematical norms. For example, it is a social norm that a student should share an idea if it is different from that which has been previously shared. What counts as mathematically different in a particular classroom is a sociomathematical norm. Likewise, what counts as a mathematical explanation or a convincing justification are sociomathematical norms (Yackel & Cobb, 1996). They define a *sociomathematical norm* as one that gives rise to a mathematical distinction. These norms tell the participants of the classroom culture *when* and *how* they should participate, and the taken-as-shared understanding of these norms constitutes the students' understanding about the very nature of mathematical behavior. A student who knows that her solution is mathematically different has made a mathematical distinction. A student who creates a convincing argument understands something about what it means to do mathematics in the class. Indeed, in a study of four elementary school classrooms, Kazemi & Stipek (2001) argued that differences in classroom sociomathematical norms accounted for differences in student performance on even traditional measures of mathematical understanding.

There is a strong consensus among mathematics educators on the broad norms that support learning and autonomous behaviors. More than two decades ago the *Professional Standards for Teaching Mathematics* (NCTM, 1991) called for mathematics teachers to focus on logic and mathematical evidence for verification; mathematical reasoning as opposed to memorization; on conjecturing, inventing and problem solving; and on connections among mathematical ideas. These same general mathematical practices are advocated today in the Standards for Mathematical Practice from the Common Core State Standards initiatives (http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf, 2011, p. 6-7).

The instructor structured the course to be aligned with these norms. Students worked daily in autonomous small groups on challenging problems and then discussed their mathematical work as a class. The instructor saw her role as to embody and make transparent specific mathematical values and behaviors regarding seeking patterns and underlying structures, conjecturing, building models, making sense of definitions, valuing understanding relationships, making arguments, and using precise language and notations. In a previous paper (Seaman & Szydlik, 2007), we describe one who embodies these traits as *mathematically sophisticated*.

In this paper, we provide evidence that the instructor was successful in her attempt to embody these norms; however, our primary goal is to make sense of how the students developed in terms of their own mathematical sophistication as they participated in, and influenced, the classroom culture. Specifically, we explore the developing meanings that students gave to three sociomathematical norms of the classroom, and we draw connections among their participation, their verbal interpretations of the classroom culture, and their mathematical sophistication as observed through their evolving abilities to solve problems and justify mathematical ideas.

## Methodology

In order to represent a variety of viewpoints and interpretations of classroom culture, the research team consisted of a mathematics education researcher, the classroom instructor who is both a mathematician and a mathematics educator, and an undergraduate mathematics student. The observed class was the first in a sequence of three mathematics content courses for prospective elementary teachers; there were 32 students in the class. The course content focused on number theory and arithmetic processes involving natural numbers, integers and rational numbers. Mathematical meanings were constructed through daily student work on, and discussion of, carefully designed problems and activities, and discussion of videotapes of elementary school mathematics classrooms.

The class was videotaped six times: twice at the start of the semester and subsequently at approximately three-week intervals. The research team was present at each hour-long meeting of the course and kept daily field notes. Semi-formal interviews were conducted with four student informants (pseudonyms Lisa, Beth, John and Andy) four times during the fourteen-week course, and informal interviews were conducted throughout the term. The team met weekly to discuss interpretations of classroom mathematical events and to design interview protocols. Written work from all the students was collected throughout the term, and student informants were videotaped solving problems aloud at both the start and end of the semester.

Classroom videotapes were assessed using the Reformed Teaching Observation Protocol (RTOP) (Sawada & Pibum et al, 2000) by both daily classroom observers and by trained RTOP collaborative team members. RTOP scores indicated that the instructor's teaching practice was highly reformed (consistent with the mathematics education community's calls for reform – e.g., NCTM), and observations by the research team confirmed that all nine mathematical sophistication traits were consistently expressed and advocated by the instructor.

## Abbreviated Results

At the start of the course, Lisa, Beth, John and Andy were typical of elementary education majors at our comprehensive university. They described their previous experience with mathematics as primarily memorization of procedures presented by their teachers; they exhibited weak content knowledge of arithmetic, number theory and number systems, they were hard-working and serious about their education; and they were attentive and willing to participate in class activities. In the first interview, conducted after two class meetings, Beth, John and Lisa observed that the social norms of the class were different from that of past mathematics classes. John's statement was typical: "I thought we would be mostly sitting there and listening to her lecture, and not, you know, so much participation … [Now] I think you are going to learn more things about why you're doing what you're doing..."

While students were typically comfortable with the social norms of working on problems in small groups, sharing their mathematical work, and entertaining the mathematical ideas of others almost from the first day, it took them several weeks to recognize that something different was expected of them *mathematically*, and they were struggling to give meaning to the mathematical expectations of the instructor. By the time of the second interview (week four) all four informants emphatically asserted that they had never experienced a mathematics course like this, and all four now attributed the primary differences to sociomathematical norms.

In the following discussion we focus on the evolving meanings that the students ascribed to three specific classroom sociomathematical norms: a) the expectation that they *find a general solution* to problems; b) the expectation that they engage in *doing mathematics*; and c) the

expectation that they *justify solutions mathematically*. Throughout the discussion, we refer to five specific problems (Figures 1 and 2). They are typical of those explored daily in class.

What does it mean to find a *general solution*? In the class, the expected solution to a posed problem was often a generalization that held for all natural numbers. The instructor explained her expectation regarding this type of solution to the class on the second day: "[Penny of Death (Figure 1)] is a game played with some number of pennies. It could be any number. We're going to start with ten. But really it could be any number of pennies, so I am just going to write "*n* pennies" for a minute to stand for any number. Is there a way to play so you are guaranteed to win? I would like to know, in the end, how to play with every number of pennies."

Although there is evidence from the videotape and from written work on the problem that many students understood the expectation that they solve the problem in general, some students struggled to give meaning to the request for this type of solution. For example, after a week of class, Lisa argued that the answer to the Pizza Cuts Problem (Figure 1) is "infinitely many pieces" because "you could just keep drawing lines, drawing lines, drawing lines, until you get as many as you want." Andy, in the first weeks of the term, interpreted a "solution that worked for any value of *n*" to be one that worked for any *specific* value of *n*, and so, for example, he addressed only the case $n = 10$ of his written work on the Penny of Death. However by the third week of class, Andy gave normative meaning to the expectation that he find a general solution as evidenced by his work on a problem of counting all possible squares in an *n* by *n* grid.

What does it mean to *do mathematics*? The instructor described doing mathematics as engaging in making sense of problems, collecting data, looking for patterns, conjecturing, finding counterexamples, building models, generalizing, and making arguments based on mathematical definitions and underlying structure of problems.

In the first interview, each informant was asked to work on the Photo Problem (Figure 2) so that we could record and observe the strategies they would use initially on problems of the type they would do in class. All four informants began by thinking aloud about the number of possible positions for each person, and, after a brief time, each conjectured that the number of photos would be 100 because each person has ten places to stand. Beth confessed that, in cases like this, she tried to make an exhaustive list of all possible arrangements. "I have a tendency to do things the long way. Like today in class …. I just wanted to write every number down, and it's probably what I would do in this case if I had time." Only Lisa eventually began to collect data on smaller version of the problem in order to test her conjecture and build a generalization.

While the informants had few successful strategies for solving the Photo Problem, class videotapes suggest that as early as the Pizza Cuts Problem (week two), almost all of the small groups approached finding a solution by generating data for small cases, organizing that data, and looking for patterns. They did not, however, value using the structure of the problem to explain those patterns, even though the instructor prompted them to consider why the patterns made sense. In the final interview (week thirteen), all four informants immediately used these same strategies in their work on the Circle Pattern Problem (Figure 2).

What does it mean to *justify a solution*? The instructor expected that a mathematical justification be either an exhaustive one or a (typically informal) deductive argument based on mathematical structure or relevant definitions. She requested and negotiated such a justification for every problem discussed in class and justification was a required section of every written report of mathematical work. This expectation did not change throughout the term; what did evolve was the meaning students gave to the expectation.

To convey initial meanings, we consider what the informants wrote in the justification section of a graded assignment on the Penny of Death. Since an acceptable justification was negotiated in class with all four students present (see Figure 3), we speculate that differences among individuals are due in large part to the meanings that they gave to "justification" itself.

Lisa was the only informant to express the essence of the justification and the visual model developed in class, indicating that she was able to make sense of its connection to the problem and of its relevance in justifying the solution. While others observed the visual model and all had copied it in their notes, they did not recognize that it illustrated the essential elements of a justification. We assert that it was not simply that the others did not *understand* the justification (although, for some this was certainly true as well); many did not know that what they were hearing was important, and so they did not attend to it.

Instead of providing a mathematical justification to the Penny of Death, the students in the class did one of three things: they simply restated their solution; they justified why doing the problem was valuable experience; or they appealed to the pattern generated in class. For example, Andy wrote, "The pattern you find when you start with one penny up to ten is very useful in solving this problem. By using this method, this is the only way to solve the problem." The activity of exploring why relationships made sense was meaningless for nearly all of the students at the beginning of the term. While they did appreciate that teachers of mathematics must be able to explain "why," they did not see that this "why" rested on the mathematical structure underlying the problems. In addition, they began the semester with no awareness that making a mathematical argument is part of doing mathematics.

By the fourth week of the course, all four informants had realized that the instructor expected something other than what they had been providing as mathematical justification. Their struggle to give meaning to that expectation is evident in their second interview responses to the question: "How do you know when you are done with a problem?" Lisa's response is typical. "I never know when I'm done … I don't think finding a formula is the end of the problem, necessarily. And that's what I'm learning from this class too." Next the interviewer asked about the justification section for written work, and again the informants tried to describe the instructor's expectation. "Like in-depth reasoning of how you got … no not the process we went through like, well yeah. And she wants to know why that answer is correct… I don't know…" (Beth). Based on the interview responses and on classroom observations, we speculate that the class generally interpreted a mathematical justification either as a way of checking whether a formula worked using relevant numbers, or as explaining a problem in the manner of a teacher.

By the eleventh week of the course, while they were not particularly good at creating justifications, the four informants (and the rest of the class) did give normative meanings to the expectation. In particular, they recognized that a justification must focus on the reasons that the mathematics made sense based on their justifications for their written solutions to the Number of Factors Problem (Figure 1). Again it was Lisa, who made the strongest argument by systematically listing the factors of $2^3 \times 3^3$ in the form of a table with (3 + 1) rows and (3 + 1) columns. In this way, she provided what Carpenter, et. al. (2003) called "an example that is more than an example" because it illuminates the structure of the general argument. Both John and Andy provided an acceptable justification that their solutions worked for the case of a single, specific prime to a power (e.g., $3^3$ has four factors: $3^0$, $3^1$, $3^2$, and $3^3$), but neither were successful in addressing cases involving more than one prime. John did not even make an attempt. Andy recognized that this justification was expected and he provided the following: "The reason you multiply was simple. By using the long method from above for finding all the factors of 144

[dividing 144 by successive natural numbers until all factors were generated], it came out to 15. If you add $(p_1 + 1) + (p_2 + 1)$ [his own notation for the exponents] it would come out to be 8, and you know that isn't right … so we multiplied and it came out to the same number of factors as the long way." Beth justified her solution with an inductive argument. "I experimented with several different prime numbers and the solution made sense with all of them." While Beth's justification was not acceptable to the instructor, it still suggested that she had come to see that justification is based on making sense of mathematical work. Furthermore, at the end of the course, the informants indicated that they *valued* mathematical justifications, both for themselves and for their future students.

## Conclusions

Students in this study began the course without normative meanings for doing mathematics, justifying solutions, and in some cases, finding a "solution for all *n*," and they left the course understanding these constructs. In that regard, this study suggests that participation in a highly reformed classroom culture of doing mathematics can increase mathematical sophistication in prospective elementary teachers. Just as Yackel (2001) reported in her study of an inquiry-based university-level course, we too observed that the social norms of working in groups on problems, sharing ideas, and responding to the mathematical work of others developed quickly and were established by the second day of class. The sociomathematical norms developed more slowly. It took almost three weeks for all students to give normative meanings to the instructor's request for *general solutions* and to her expectation that they *do mathematics*, and it was not until the eleventh week of the course that most students understood that *justifying a solution* meant providing a mathematical argument that explained why their solution was valid.

Simon & Blume (1996) have suggested that, "The hearing of a logical (from the researcher's perspective) argument, which complies with the established classroom norms for mathematical justification, does not necessarily bring the community members to the understanding of the person presenting the argument. Rather the community members tend to be limited in their sense-making with respect to the argument, in their understandings of the concepts involved" (p. 29). Based on our work, we contend that one of the "concepts involved" in justifying is the concept of *justification* itself. Students in this study were able to recognize and attempt to make sense of mathematical justifications only insofar as they could give meaning to their instructor's expectation for it. In fact, the data suggest that giving normative meaning to generalizing, doing mathematics and justifying is prerequisite to success at each of them, and that as students came to make sense of these concepts, they improved in their abilities to do them and they began to see them as valuable.

We do not claim that understanding the concept of generalizing, doing mathematics or justifying is *sufficient* for success. Even though our observations of classroom discourse and written work indicated that the students began to give normative meaning to the instructor's expectation that they justify their solutions, they did not make large gains in their abilities to do so. In fact, even at the end of the semester, student justifications of their mathematical work rarely satisfied the instructor as a representative of the mathematical community. However, we assert that giving normative meanings to constructs such as finding general solutions, doing mathematics, and justifying mathematical work is a *necessary* condition for success at these activities, and we advocate that normative meaning for these constructs must be, and can be, actively fostered.

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Figure 1. Classroom Problems

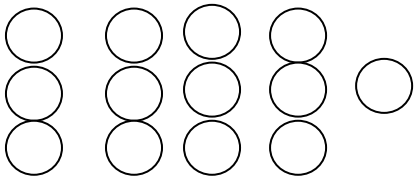
| | **Problem Statement** | **General Solution** | **Sample of an Acceptable Justification** |
|---|---|---|---|
| **Penny of Death Problem** | The game is played by two teams who take turns removing 1 or 2 pennies from an initial set of $n$ pennies. The team that takes the last penny loses. Devise a winning strategy for every number of pennies. | If $n$ is a multiple of 3 plus 1 more, choose to go second and take 1 penny when your opponent takes 2, and take 2 when your opponent takes 1. If $n$ is not a multiple of 3 plus 1 more, choose to go first and take enough pennies to leave your opponent facing a multiple of 3 plus one more. | You will win if you force your opponent to face a multiple of 3 plus 1 pennies because, by taking 1 when she takes 2 and taking 2 when she takes 1, you will (together) remove a total of 3 pennies each round. Since $n = 3m + 1$, after $m$ rounds your opponent will have to take the penny of death. |
| **Pizza Cuts Problem** | You get to make $n$ straight cuts (anywhere you want) across a round pizza. What is the maximum number of pieces of pizza you can make? | If we let $p_n$ be the number of pieces at the $n^{\text{th}}$ cut, then $p_n = p_{n-1} + n$. This can be expressed in closed form as $\frac{1}{2}(2 + n + n^2)$. | To maximize the number of pieces, each new cut should cut as many pieces as possible in two. This will happen when the new cut intersects all previous cuts. The $n^{\text{th}}$ cut will then divide exactly $n$ previous pieces. If we let $p_n$ be the number of pieces at the $n^{\text{th}}$ cut, then $p_n = p_{n-1} + n$. |
| **Number of Factors Problem** | Find a way to compute the number of factors for any natural number (greater than 1) from its prime factorization. | If $n = p^a \times q^b \times \ldots \times r^c$, where $p, q, \ldots r$ are distinct primes, then the number of factors of $n$ will be $(a + 1) \times (b + 1) \times \ldots \times (c + 1)$. | $p^a$ has exactly the following $(a + 1)$ factors when $p$ is prime: $p^0, p^1, p^2, \ldots p^a$. Likewise $q^b$ has $(b + 1)$ factors, … and $r^c$ has $(c + 1)$ factors. All of these factors are distinct because $p, q, \ldots r$ are all different primes. This means that $n$ will have all combinations of these factors for a total of $(a + 1) \times (b + 1) \times \ldots \times (c + 1)$. |

Figure 2. Interview Problems

| | **Problem Statement** | **General Solution** | **Sample of an Acceptable Justification** |
|---|---|---|---|
| **Photo Problem** | Suppose that you are asked to take a group photo of ten people standing side-by-side in one row (like a line-up). How many different pictures could you take where different orderings of people are counted as different pictures? | $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$ | There are ten people to fill the first place in line, and *for every person* who stands first, there are 9 people who could stand second … and *for every two people* standing first and second, there are 8 people who could stand third …etc… so the total number of possible arrangements is 10! |
| **Circle Pattern Problem** | If you have an ($n$, $m$) circle pattern [this means $n$ points on a circle connected at every $m^{th}$ point], what conditions on $n$ and/or $m$ will guarantee you will connect all $n$ points? | $m$ and $n$ must share no common factors, that is the GCD($m$, $n$) = 1. | If $n$ and $m$ have a common factor, say $k$, then as you move around the circle connecting every $m$ points, you will end up back at the starting point after $m \div k$ trips around the circle, hitting $n \div k$ points in. So to hit all points, $k$ must be 1. |

Figure 3: Justification for the Penny of Death Problem as discussed in class:

Class conjecture [written on Board]: With 10, go second and then always take the opposite of your opponent.
Instructor: Why is that?
John: Because it works.
Instructor: Well, why does it work? *Why* does it work?
[Pause]
Daniel: Anything, like 1, 4, 7, 10, 13 … they'd all work like that because you're taking 3 away from it each time and basically you lead [your opponent] down to one.
Instructor: All right. Wait a minute. Let's draw a picture of that. Okay so let's draw a picture in the case of … 13. Daniel says 13 will work like that. So you're taking 3 away every time and leading down to one … so I'm picturing that this is what you're thinking:
[Instructor draws on board as she writes.] Instructor: 3's in a turn, so you're saying 3, 6, 9, 12 and 13 pennies right here [pointing]



Okay, Daniel, now explain from this picture what you're talking about.
Daniel: That no matter what they choose, if they choose 2 first, you choose 1 to make it 3 that you take away from the table [indicated the first group of three]. If they choose 1, you choose 2 …

# An Analysis of Calculus Instructor Grading Inconsistencies Through a Sensible Systems Framework

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**Abstract**

Despite the consensus among mathematics educators that prior knowledge is essential to student success, calculus instructors vary widely in their assessment of prior knowledge errors found on student assignments and exams. This phenomenological study of five calculus instructors at a large research institution investigated the influence that instructor belief systems have on the consistency of grading across instructors. The results showed that the intricacies of instructor sensible systems play a vital role in the assessment of student errors.

**Keywords**: calculus, assessment, prior knowledge, belief systems

## Introduction

Anyone who has had an opportunity to work with students taking their first calculus course has probably encountered a variety of student mistakes; none of which have anything to do with the students' understanding of calculus. Whether students fail to manipulate algebraic expressions correctly, forget the values of trigonometric functions at the special angles, or exhibit difficulty sketching simple quadratic functions; assessing prior knowledge mistakes can be quite cumbersome. Within the context of this study the term prior knowledge refers to any skill or understanding a student must possess before entering a first calculus course. Instructors particularly grapple with grading student work when the student demonstrates an understanding of the calculus problem but is unable to successfully complete it due to their deficiencies in prior knowledge. On one hand, the instructor must consider the ability the student has shown in dealing with the topics of calculus. On the other hand, attention must be given to the students' difficulties using the skills taught in previous courses. This contention between the importance of current course objectives and prerequisite skills is settled differently among instructors. Despite the consensus among mathematics educators that prior knowledge is essential to student success, variances among calculus instructors' beliefs about prior knowledge in a calculus course yield inconsistent grading of student assignments and exams. Using a phenomenological research design, this study investigated the *sensible belief systems* of calculus instructors related to the assessment of prior knowledge errors. More specifically, the goal of this exploration was to determine the influences that provoke differences among calculus instructors' grading of student work.

## Relevant Literature

In many fields of study students have difficulty retaining knowledge from previous course work. Particularly in calculus, the errors that students make have been attributed to prior knowledge and specifically to algebraic misunderstandings in previous research (Edge & Friedberg, 1984; White & Mitchelmore, 1996). One cause of difficulty found by White and Mitchelmore (1996) is students' tendencies to misinterpret the use of variables in calculus

problems. They refer to students that manipulate symbols without an understanding of what they are doing as having an 'abstract-apart' concept of variables whereas students who generalize, symbolize, and abstract variables as having an 'abstract-general' concept. They concluded that "a prerequisite to a successful study of calculus is an abstract-general concept of a variable…" (p. 93). Orton's study (1983a) confirms that problems with algebra (in addition to ratio and proportion) hinder calculus students when dealing with differentiation. At Illinois State University, three groups of Calculus I students were studied to determine the factors of success in a first calculus course (Edge & Freidberg, 1984). Edge and Friedberg used regression models to find that for all three groups' calculus success could be predicted by algebraic skills. Research documenting similar situations in calculus, as well as other disciplines, and in varying degrees supports the continued study of this issue. The lack of specific focus on instructors' grading patterns in relation to prior knowledge errors prompted the study outlined here.

The issue of inconsistent grading directly relates to previous research concerning student study habits and intellectual behavior. In *The Hidden Curriculum*, Snyder (1970) describes the affect that instructional strategies have on the study habits of students. In contrast with what he refers to as the *formal curriculum*, which traditionally emphasizes deep conceptual understanding of the topics covered in each course, the *hidden curriculum* is described as the norms that determine successful degree completion which only students understand as insiders of an institution. As an example, Snyder points specifically to a class whose instructor stressed the importance of being creative and engaged in class discussion. However, when presented with the exam, the students found that in actuality they were expected to simply memorize a large portion of their text and regurgitate that information. Students that prevail in environments for which instructor expectations are unclear or vary are known by Miller & Parlett as cue-seekers (Miller and Parlett 1974). Their study of undergraduate science majors revealed that students who carefully gauge the expectations of instructors, despite contradictions to the formal curriculum, perform much better than those who do not read into the hidden curriculum.

The aforementioned research indicates the importance that instructional strategies have on student behavior. Regardless of teacher intentions, the cues sent to our students are indeed received and acted upon. Specifically, the ways in which teachers score exams and assignments are internalized by students and used to tailor future experiences with mathematics learning. Therefore, it is pertinent to the field of mathematics education that we identify those cues. The exploration of factors that influence grading strategies will not only assist students in understanding what is expected of them, but will also allow instructors the opportunity to adjust if those strategies do not align with the intended curriculum. The study reported here begins much needed dialogue by examining the grading strategies of calculus instructors.

**Theoretical Perspective**

To fully understand the assessment practices of calculus instructors when faced with prior knowledge errors, an examination of each instructor's belief system was required. Belief systems are defined by Phillip (2007) as follows:

> [A belief system is a] metaphor for describing the manner in which one's beliefs are organized in a cluster, generally around a particular idea or object. Beliefs systems are associated with three aspects: (a) Beliefs within a beliefs system may be primary or derivative; (b) beliefs within a beliefs system may be central or peripheral; (c) beliefs are never held in isolation and might be thought of as existing in clusters. (p. 259)

To rationalize previously labeled contradictions between teacher beliefs and practices, Leatham (2006) utilized a sensible system approach by taking into account a holistic view of teacher belief systems clustered around classroom practice. The current study calls upon Leatham's (2006) assertion that, as observers, the perceived beliefs of another are assumed consistent or contradictory based on our own perspectives. "The sensible system framework attempts to minimize these assumptions" (p. 95). He considered the entire system when analyzing teacher behavior that appeared to contradict teacher beliefs. Rather than conclude that the teacher was conflicted when observed instructional behavior did not align with a stated belief, Leatham resolved that there were other beliefs existing within the teacher's belief system that took precedence at the time of the perceived contradictory action. An adaption of the sensible system framework is used here to rationalize the variances among assessments of calculus student errors.

**Methods**

A qualitative research design was used to investigate instructor perspectives on prior knowledge. A phenomenological approach was taken to uncover instructor views of how prior knowledge skills influence student performance in calculus and how prior knowledge errors influence instructor judgment of student understandings. Five calculus instructors at a large Midwestern research institution were individually interviewed. Each participant had taught a 120-student lecture style calculus course within the last five years of being interviewed. The items included in the first component of the interview centered around three main issues: (a) the definition of prior knowledge in a calculus course, (b) the importance of proficiency in prior knowledge skills in a calculus course, and (c) how prior knowledge errors are assessed in a calculus course. The second component of the interviews was task-based, requiring the participants to score a selection of student exam questions. The exams were collected from students at the same institution during the semester immediately prior to the interview dates. These student error examples (SEEs) were chosen to reflect questions commonly seen on the university's exams. The nineteen SEEs presented to the interviewed instructors also included a variety of error types including calculus errors and prior knowledge errors. After scoring each SEE (out of a given point value) the instructors were also asked to identify and classify errors as calculus or prior knowledge as well as comment on how their grading decision was made.

**Analysis**

Analysis of the task-based component of the interviews involved a comparison of the participants' assessments of the nineteen SEEs. Particular attention was given to the instructors' scoring of student work. The differences between the lowest and highest assigned score was identified for each SEE. Interestingly, several of the error examples revealed scores that ranged from below 70% to above 80%. These SEEs were later labeled as having wide score ranges because the scores assigned by the instructors included both what is normally considered to be below average standing (below 70%) as well as above average standing (above 80%) for a single student's work.

To investigate this phenomenon I looked to the work of Leatham (2006) in the study of teacher belief systems. He found that perceived inconsistencies between instructor beliefs and practices could be explained by taking a more holistic look at the teacher's beliefs about pedagogy. In the same vein, I viewed the inconsistent responses from the group of instructors to be a function of the belief systems in play by each individual professor. For each error example

with a wide score range, the instructor comments were analyzed to determine how their individual sensible systems impacted grading decisions.

**Findings**

Seven of the nineteen SEEs were identified as having wide score ranges. To demonstrate how the sensible system framework was applied to each of these cases, SEE 1 will be discussed here. Similar findings were uncovered in each of the remaining six cases of wide score ranges. SEE 1 required the student to find the derivative of a given function using the limit definition of the derivative. The instructors at the ends of the score range were Professor Crumbliss and Professor Edwards. In SEE 1, Professor Crumbliss gave the highest score and Professor Edwards the lowest score. Through a holistic lens of each instructor's perspective on teaching, assessment, and prior knowledge, in particular, the differences in scoring were understood.

SEE 1 was scored out of twenty points. Professor Crumbliss assigned the highest score of 18/20. In his opinion, the algebraic mistake made by the student was of little importance compared to the student's ability to demonstrate conceptual understanding. In his interview he commented that "The most frustrating one is when someone comes up with the equation of a tangent line and it's just perfect and then they go and they simplify it more and there is an algebra mistake there and it's just, I will usually give that full credit because I think that stuff is irrelevant. You know it's more important that they get the concepts." He goes on to state that "There are examples where the calculus is all true and people organize things and make mistakes. I'm not worried, I'll usually circle it and write some type of comment and give them nine out of ten."

<u>Sensible System</u>: Professor Crumbliss felt he could assign an above average score for two main reasons. The first is that he can clearly see that the student understands the process and was able to set everything up correctly. Also, when everything else is correct, the algebra mistakes are seen as irrelevant. Therefore, such errors warrant only minimal point deduction if any at all.

The lowest score given to SEE 1, 10/20, was assigned by Professor Edwards. He described the students' inability to complete each step of the problem correctly as problematic. Specifically, he made the following assertion: "So if a student shows that they have some conceptual understanding I do give them some credit. But calculus is about calculations and you need to get the calculations right. So even if it's a question of a deficiency in prior knowledge it's still something [the student is] responsible for." While examining SEE 1 he commented, "…So they wrote down the derivative correctly and wrote down the correct limit and put in the function and expanded the functions. So the whole problem was in the manipulating…They did that incorrectly and then they got the right limit. So I'd give maybe 10 points [out of 20]."

<u>Sensible System</u>: Professor Edwards explained that "calculus is about doing calculations". He views algebraic manipulations as part of the work of calculus. Partial credit (half in this case) is assigned when students demonstrate good work, which he was able to identify in this case.

This snapshot of the sensible system framework provides an avenue for understanding variances in instructor grading patterns. This examination of the instructors' perspective on the types of errors and conceptual skills demonstrated in student work proved to be especially insightful. The sensible system framework applied to SEE 1 revealed that each instructor was thoughtful in their considerations despite the varied level of scores assigned. Professor Crumbliss was concerned with the ability of the student to demonstrate conceptual knowledge regardless of the existence of algebraic mistakes. Conversely, Professor Edwards' attention was given to the students work as a whole and his grading decisions hinged on how well the student

worked through prior knowledge skills in addition to calculus concepts. As shown here, the existence of a prior knowledge error has a different meaning for each instructor. Each of their perspectives played a vital role in their scoring of student errors.

**Implications**

The importance of prerequisite skills in mathematics courses has been well documented. Particularly at the college level, students need a foundation of prior knowledge to navigate through mathematics requirements and specialized courses in their respective fields of study. However, the increase in college and university enrollment as of late has flooded post secondary classrooms with underprepared students who lack sufficient prerequisite skills. This influx of under-prepared students does not exempt university professors from attending to course objectives designed to build upon the much needed prerequisite courses such as algebra, trigonometry, and geometry. The insights this study provides with respect to instructor belief systems should be carefully considered as mathematics educators develop methods for providing instruction to students lacking necessary prior knowledge skills.

Recent increases in class sizes have also fueled trends towards uniformity across multiple section courses like calculus. Mathematics departments are now looking to provide students with consistency in various aspects of the classroom experience; especially in grading policies. These efforts towards fairness should be tempered with an understanding of the decisions instructors make when assessing their students. The results of this study provide a backdrop for administrators and faculty who manage the coordination of multiple sections classes as they consider the grading practices to be incorporated into redesigned curriculums.

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# A Characterization of Calculus I Final Exams in U.S. Colleges and Universities

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*Abstract: The final exam in a mathematics course is one source of information about the nature and level of student learning that is expected in the course. In this study, a three-dimensional framework was developed to analyze post-secondary calculus I final exams in an effort to determine the skills and understandings that are currently being emphasized in college calculus. Results indicate that Calculus I final exams generally require low levels of cognitive demand, seldom contain problems stated in a real-world context, rarely elicit explanation, and do not require students to demonstrate or apply their understanding of the course's central ideas. Data from a survey that investigated instructors' beliefs about teaching, the role of exams and homework in learning, etc. was completed by the same instructors and was used to investigate instructor beliefs that correlate with exams that are more and less conceptual in their focus. We found that there is a misalignment between the nature of calculus final exams and instructors' perceptions of their exams relative to the extent to which students are asked to explain their thinking and the proportion of exam items that focus on skills and methods for carrying out computations.*

*Keywords*: Calculus; Assessment; Mathematical Reasoning; University level mathematics.

## Introduction

Course exams are among the most revealing artifacts that provide evidence of the mathematical skills and understandings that instructors want their students to know and be able to use. A review of a course exam may provide information about an instructor's expectations for students' level of computational fluency, their depth of understanding specific concepts, and the degree to which students are expected to make connections among the course's central ideas. A close evaluation of an exam's specific problems may also shed light on the nature of the mathematics content that is emphasized during instruction and on homework. Since it is common that exams are the primary means by which instructors gain insight into the nature of students' understanding, it is surprising that so little is known about exams for gateway mathematics courses such as introductory calculus. This study provides one response to this gap in the literature by characterizing the nature of final exams for first-semester calculus courses at a variety of post-secondary U.S. institutions. It also provides some information about how the content of final exams compares with the instructors' perceptions of their exams. The goal of this analysis was to provide a snapshot of the mathematics valued by instructors of introductory calculus and to provide information about the degree to which their exams align with what they profess to value and assess when teaching calculus I.

Our review of the literature related to calculus I assessment revealed that little is known about the content of calculus I exams administered to students in colleges and universities in the United States. However, a review of mathematics research literature focused on student learning of ideas in introductory calculus (e.g., Carlson & Rasmussen, 2008) revealed that calculus I students are generally not developing conceptual understanding of the central ideas of calculus. At the same time, the calculus reform movement has resulted in shifts in calculus curriculum to provide increased focus on student understanding of the course's key ideas. With this increase in conceptual focus in the calculus curriculum, it seems logical that calculus exams would include more questions that assess students' understanding and ability to use the central ideas of the course.

Items on mathematics exams and in textbooks have been characterized in the literature according to their conceptual focus (Bergqvist, 2007; Boesen, Lithner, & Palm, 2006; Gierl, 1997; Lithner, 2000, 2003, 2004; Palm, Boesen, & Lithner, 2006), the degree to which students

are asked to imitate procedures for working specific problem types (Bergqvist, 2007; Lithner, 2000, 2004), and the format of the questions (Senk, Beckman, & Thompson, 2007). These approaches to characterizing exam and textbook items provided insights for the initial draft of a framework intended to characterize calculus I exam items. However, our initial attempts to code calculus I final exams revealed that these characterizations were not sufficient to accurately distinguish one problem type from another as a result of their focus on a single characterization category (e.g., cognitive demand). This led to our undergoing multiple rounds of item coding to establish constructs for characterizing calculus I exam items that provided a more comprehensive characterization.

In addition to devising a framework that allowed us to gain insight into the nature of calculus I final exams across a variety of characterization strands, the present study contributes to the existing body of literature by characterizing a large sample size ($N$=150) of exams from a variety of post-secondary institution categories and combining coded exam data with instructor survey responses to gain insight into the instructor beliefs that govern the selection of exam items.

## Research Questions

The research questions that guided this study were:

1. What is the nature of post-secondary calculus I final exams relative to the levels of cognition they elicit, the representation of both the task statement and the intended solution, and the item format?
2. What instructor beliefs are associated with the selection of items that instructors include on calculus I final exams?
3. What is the relationship between an exam item's representation and format on the cognitive demand of the item?

## Methodology

This study is a part of a larger initiative by the MAA to determine the characteristics of successful programs in college calculus. As part of a larger data corpus, exams and instructor surveys from 253 universities were submitted electronically. 150 of these 253 exams were randomly selected for use in this study. Of the instructors providing data for the larger data corpus, 48% are tenured or tenure-track faculty, 28% are other full-time faculty, 9% are part-time faculty, and 15% are graduate students. Moreover, of the 150 exams randomly selected, 65.4% were administered at national universities[1], 19.6% from regional universities[2], 8.3% from community colleges, 3.8% from national liberal arts colleges[3], and 3.0% from regional colleges[4].

Data analysis in this study consisted of two phases. In the first phase, an exam characterization framework was developed and used to code the 150 randomly selected exams. In the second phase we compared the coded exam data with data obtained from a post-term instructor survey with the intention of determining the extent to which our characterization of the exams corresponds with instructors perceptions of their exams relative to their conceptual

[1] National universities are those that offer a full range of undergraduate majors as well as a host of master's and doctoral degrees.

[2] Regional universities offer a full range of undergraduate programs, some master's programs, and few doctoral programs.

[3] National Liberal Arts Colleges are schools that emphasize undergraduate education and award at least half of their degrees in the liberal arts fields of study.

[4] Regional colleges are institutions focusing on undergraduate education, awarding less than half of their degrees in liberal arts fields of study.

orientation. The exam characterization framework characterizes exam items according to three distinct item attributes: (a) *item orientation*, (b) *item representation*, and (c) *item format*.

## Exam Characterization Framework

*Item orientation.* To classify exam items relative to *item orientation*, we have adapted the six intellectual behaviors in the conceptual knowledge dimension of a modification of Bloom's taxonomy (Anderson & Krathwohl, 2001). In particular, we found that parsing the initial "remember" level of the taxonomy into "remember" and "recall and apply a procedure" was necessary in the context of evaluating the cognitive demand of mathematics tasks. On mathematics exams, students are often required to demonstrate procedural skill that involves a higher level of cognition that simply recalling information, yet enacting the skill may not require students to understand the underlying concepts on which those skills are based.

As in Bloom's taxonomy as well as the modification by Anderson and Krathwohl (2001), the six classifications of the item orientation taxonomy are hierarchical with the lowest level classification requiring students to remember information and the highest level requiring students to make connections (see Table 1). As a result of Lithner's (2004) observations that calculus textbook items can be solved using superficial reasoning strategies, using the item orientation taxonomy, items were coded according to the procedural fluency, understandings, and cognitive processes that are *necessary* to respond to an item. For instance, even though an item may have been designed to assess understandings or specific reasoning abilities, if a student is able to solve the problem by applying a memorized procedure, then the item is classified as "recall and applying a procedure". Additionally, since we had limited insight into students' experiences in their calculus courses, we attended only to the cognitive behavior that a task *required*. That is, we recognized that although a task may seem routine and procedural, the task has the potential to be highly novel for students who, in their courses, were not exposed to or rehearsed the procedure that solves the problem. This resulted in our making a distinction between those procedures that, when correctly applied, give evidence of understanding and those that do not. This distinction is made in Table 1, along with the descriptions of the other levels of the item orientation taxonomy.

*Item representation*. Characterizing items relative to item representation required classifying both the representation of the task as it is stated as well as the representation that the task solicits in the solution. Table 2 describes these classifications relative to the task statement and the solicited solution. It is important to note that a single task statement as well as the solution the task elicits can involve multiple representations. It is also noteworthy that since many tasks can be solved in a variety of ways and with consideration of multiple representations, we coded for the representation requested in the solution by considering only what the task requires for an answer. For instance, a problem that asks students to calculate the slope of a tangent line only requires a student to do symbolic work. Accordingly, we would not code "Graphical" as a representation of the solution since reasoning graphically is not necessary to solve the problem, even though the problem has graphical meaning.

In addition to coding items relative to one of the six representational classifications, if an item requires an interpretation or inference the item is coded with an (*I*). If the item does not require an interpretation or inference the item is coded with an (*N*). In order for an item to be coded as requiring an interpretation or inference, a student must be required to communicate the *meaning* of their work.

*Item Format.* The third and final dimension of the exam characterization framework is item format. The most general distinction of an item's format is whether it is multiple-choice or open-ended. However, there is variation in how open-ended tasks are posed. For instance, the statement of an open-ended task may prompt the student to respond to one question that has one correct answer. Such an item is similar to a multiple-choice item without the choices and is therefore classified as *short answer*. In contrast, a *broad open-ended task* elicits various responses, with each response typically supported by some explanation. The form of the solution in a *broad open-ended item* is not immediately recognizable when reading the task. In addition to coding tasks as short answer or broad open-ended, we also note instances in which a task is presented in the form of a word problem. Also, tasks that require students to explain their reasoning or justify their solution can be supplements of short answer or broad open-ended items. We distinguish between explanations and justifications in that explanations are presented in the form of narrative descriptions, using words, and justifications are presented mathematically (e.g. requiring further symbolic or computational work to demonstrate the validity of a particular previous result). Table 3 contains descriptions of the item format codes.

## Results

*Characteristics of post-secondary calculus exams.* Of the 150 exams coded, 85.21% of the items could be solved by simply retrieving rote knowledge from memory or recalling and applying a procedure, requiring no understanding of an idea or why a procedure is valid. Moreover, students were only required to demonstrate an understanding on 14.83% of exam items.

Coding results from the item orientation taxonomy were used to distinguish exams that were more procedural in nature from those that were more conceptual. An exam was classified as "procedural" if over 70% of the items were coded as either "remember" or "recall and apply procedure"—the lowest two levels of the item orientation taxonomy. Of the 150 exams coded, 90% were classified as "procedural." Additionally, only 2.67% of exams had 40% or more of the items requiring students to demonstrate or apply understanding. The coding results for the item orientation taxonomy are given in Table 4.

In terms of item representation, the predominant proportion of exam items were stated symbolically (73.70%) or required a symbolic solution (89.4%) while items least frequently provided information in the form of a table (1.02%), presented a proposition or statement with the expectation that an example or counterexample be provided (0.59%), or presented a conjecture or proposition with the expectation that a proof be provided (1.29%). The relative percentages of item representations of the exams coded are given in Table 5.

Results also indicate that introductory calculus final exams seldom include tasks that are stated in the context of a real-world situation. Our coding revealed that 38.67% of the coded exams had less than 5% of the items classified as word problems, in either the "short answer" or "broad open-ended" format categories. Further, only 22% of the coded exams had more than 10% of the exam's test items classified as a word problem in the "short answer" or "broad open-ended" format categories. It is also noteworthy that 18% of the exams contained no word problems. The coding results from the item format strand are given in Table 6.

In summary, results from coding the 150 randomly selected exams with the exam characterization framework indicate that Calculus I final exams generally require low levels of cognitive demand, seldom contain problems stated in a real-world context, rarely elicit explanation, and do not require students to demonstrate or apply their understanding of the course's central ideas.

*Contrast between instructors' beliefs and coding results.* To address our second research question, we coordinated the results of the exam codes and instructors' responses to the post-term instructor survey. Particular attention was paid to identifying inconsistencies between our findings about the characteristics of the Calculus I final exams and instructors' perceptions of their exams.

Figure 1 indicates the distribution of survey responses from both instructors with "procedural" and "conceptual" exams to the following survey question: "How frequently did you require students to explain their thinking on exams?" Responses ranged from 1 (not at all) to 6 (very often). We note that an exam item requiring explanation would have been coded as either "understand" or "apply understanding" in the item orientation taxonomy. Results from the item format codes indicate that a total of 3.05% of all items coded ($N = 3{,}735$) required an explanation. Further, only 14.72% of all exam items were coded as "understand" or "apply understanding" in the item orientation taxonomy. However, 68.18% of all instructors who submitted the exams that were coded selected either "4", "5", or "6" on this survey item, indicating that these instructors claim to frequently require that their student explain their thinking on exams. These data reveal that the instructors Calculus I final exams do not align with their perceptions of their exams relative to the extent to which students are required to explain their thinking. Further analysis revealed that the instructors who were classified as "procedural" had the same level of discrepancy between their perceptions of the conceptual focus of their exams, and the actual content of the exam, as those who were classified as "conceptual".

Similarly, there was also discrepancy between our characterization of the final exams and survey responses with respect to the proportion of exam items that emphasized skills and methods for executing computations. Figure 2 indicates the distribution of survey responses from both instructors with "procedural" and "conceptual" exams to the survey question, "On a typical exam, what percentage of the points focused on skills and methods for carrying out computations?" The values on the independent axis correspond to percentages in units of 10 (i.e. 1 represents 0%, 2 represents 10%, etc.). The median responses from both the "procedural" and "conceptual" groups were 50%. Our coding results, however, conclude that 78.7% of exam items require students to recall and apply a procedure. Additionally, 89.4% of all exam items required students to perform symbolic computation.

*Correlating item orientation with representation and format.* In order to determine if particular item representations or formats necessitated higher-order cognitive activity, we calculated the proportions of item representations and item formats within each item orientation category. Table 7 documents the proportion of the most common item representations within each item orientation category[5].

Items that were stated symbolically and required a symbolic solution were most prevalent among items that required students to remember (30.20%), and consumed the vast majority of tasks that required students to recall and apply a procedure (78.98%). Thereafter, we notice that the proportion of items that were stated symbolically and solicited a symbolic solution decreased as tasks demand higher levels of cognitive behavior (10.30% in the "understand" category, 5.50% in the "apply understanding" category, and 0% in the "analyze" category).

## Discussion

[5] By "most common," we refer to an exclusion of item representations that represented less than 2% of the items within a specific item orientation category.

The results of this study revealed that that a high proportion of the 150 coded calculus I final exam items required low levels of cognitive demand, rarely make use of real-world contexts, seldom elicit explanation or justification, and do not provide students with opportunities to demonstrate or apply their understanding. Moreover, we found that there exists inconsistency between our characterization of post-secondary calculus I final exams and instructors' perception of the exams they implement. For the purpose of proposing an inquiry that would benefit from further empirical research, we conclude by proposing a hypothesis that seeks to explain this misalignment.

Based on our tacit assumption that instructors regard assessments as tools that seek to quantify understanding, we conjecture that there is an assumption among instructors that the high proportions of problems that require students to either remember or apply a rehearsed procedure *do* provide insight into the nature of students' understanding. Accordingly, we hypothesize that instructors are attributing the conceptual knowledge that governs their own computational work to students who are able to solve similar procedural problems. As a result of the low proportion of exam items at the "understand" level of the item orientation taxonomy or above, we conclude that a large percentage of problems failed to provide insight into how the student understands the concepts on which their computational or procedural work is based. Hence, it is likely that the majority of calculus I final exams encourage students' to avoid understanding and instead focus on memorizing the problem solving procedures that are associated with specific types of tasks.

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## Appendix

Table 1.
Adaptation of six intellectual behaviors from Anderson & Krathwohl (2001).

| Cognitive Behavior | Description |
|---|---|
| Remember | Students are prompted to retrieve knowledge from long-term memory (e.g., write the definition of the derivative). |
| Recall and apply procedure | Students must recognize what knowledge or procedures to recall when directly prompted to do so in the context of a problem (e.g., find the derivative/limit/integral of *f*). |
| Understand | Students are prompted to make interpretations, provide explanations, make comparisons or make inferences that require an understanding of a mathematics concept. |
| Apply understanding | Students must recognize when to use (or apply) a concept when responding to a question or when working a problem. To recognize the need to apply, execute or implement a concept in the context of working a problem requires an understanding of the concept. |
| Analyze | Students are prompted to break material into constituent parts and determine how parts relate to one another and to an overall structure or purpose. Differentiating, organizing, and attributing are characteristic cognitive processes at this level. |
| Evaluate | Students are prompted to make judgments based on criteria and standards. Checking and critiquing are characteristic cognitive processes at this level. |
| Create | Students are prompted to put elements together to form a coherent or functional whole; reorganize elements into a new pattern or structure. Generating, planning, and producing are characteristic cognitive processes at this level. |

Table 2.
Item representation.

| Item representation | Task statement | Solicited solution |
|---|---|---|
| Applied/modeling | The task presents a physical or contextual situation. | The task requires students to define relationships between quantities. The task may also prompt students to define or use a mathematical model to describe information about a physical or contextual situation. |
| Symbolic | The task conveys information in the form of symbols. | The task requires the manipulation, interpretation, or representation of symbols. |
| Tabular | The task provides information in the form of a table. | The task requires students to organize data in a table. |
| Graphical | The task presents a graph. | The task requires students to generate a graph or illustrate a concept graphically (e.g. draw a tangent line or draw a Riemann sum). |
| Definition/theorem | The task asks the student to state or interpret a definition or theorem, or presents/cites a definition or theorem. | The task requires a statement of a definition or theorem, or an interpretation of a definition or theorem. |
| Proof | The task presents a conjecture or proposition. | The task requires students to demonstrate the truth of a conjecture or proposition by reasoning deductively. |
| Example/ counterexample | The task presents a proposition or statement with the expectation that an example or counterexample is provided. | The task requires students to produce an example or counterexample. |
| Explanation | Not applicable. This code is particular to what is expected in the students' solution. | The task requires students to explain the meaning of a statement. |

*Inferential (*I*) and non-inferential (*N*) are subcodes of item representation.

Table 3.
Item format.

| Item format | Description |
|---|---|
| Multiple Choice | One question is posed and one answer in a list of choices is correct. The student is prompted to select the correct answer among the choices. |
| Short answer | The item asks the student to respond to one question that has one correct answer. The student can anticipate the form of the solution merely by examining the task—this is similar to a multiple-choice item without the choices. |
| Broad open-ended | There are multiple ways of expressing the answer. The form of the solution is also not immediately recognizable upon immediate inspection of the task. |
| Word problem | A word problem is posed in a contextual setting using words, and prompts students to create an algebraic, tabular and/or graphical model to relate specified quantities in the problem, and may also prompt students to make inferences about the quantities in the context using the model. Note that a word problem can be posed as either short answer or broad open-ended or multiple choice. Hence, we code a task as a word problem in addition to identifying it as either short answer or broad open-ended. |

*Explain (*E*) and justify (*J*) are subcodes of Item Format.

Table 4.
Coding results from the item orientation taxonomy.

| Item Orientation | % |
|---|---|
| Remember | 6.51 |
| Recall and Apply Procedure | 78.70 |
| Understand | 4.42 |
| Apply Understanding | 10.30 |
| Analyze | 0.11 |
| Evaluate | 0 |
| Create | 0 |

Table 5.
Coding results for item representation.

| Item Representational Classification (Task) | % | Item Representational Classification (Solution) | % |
|---|---|---|---|
| Applied/Modeling | 13.20 | Applied/Modeling | 6.96 |
| Symbolic | 73.70 | Symbolic | 89.40 |
| Tabular | 1.02 | Tabular | 0.19 |
| Graphical | 10.40 | Graphical | 5.70 |
| Definition/Theorem | 3.51 | Definition/Theorem | 4.36 |
| Proof | 1.29 | Proof | 1.53 |
| Example/Counterexample | 0.59 | Example/Counterexample | 0.59 |
| Explanation | 2.36 | Explanation | 2.36 |

Table 6.
Coding results for item format.

| Item Format | % |
|---|---|
| Multiple Choice | 11.70 |
| Multiple Choice (Explain) | 0.59 |
| Multiple Choice (Justify) | 0.19 |
| Multiple Choice (Word Problem) | 0.40 |
| Short Answer | 76.10 |
| Short Answer (Explain) | 2.38 |
| Short Answer (Justify) | 1.04 |
| Short Answer (Word Problem) | 6.05 |
| Broad Open-Ended | 1.23 |
| Broad Open-Ended (Explain) | 0.08 |
| Broad Open-Ended (Justify) | 0 |
| Broad Open-Ended (Word Problem) | 0.03 |

Table 7.
Proportions of item representations within each category of the item orientation taxonomy.

| Item orientation | Item representation (task) | Item representation (solution) | Percentage |
|---|---|---|---|
| Remember | Symbolic | Symbolic | 30.20% |
| | Definition | Definition | 27.76% |
| | Graphical | Symbolic | 21.63% |
| Recall and apply procedure | Symbolic | Symbolic | 78.98% |
| | Graphical | Symbolic | 5.10% |
| Understand | Graphical | Symbolic | 20.00% |
| | Symbolic | Explanation | 13.33% |
| | Symbolic | Symbolic | 10.30% |
| | Applied/modeling | Explanation | 10.30% |
| Apply understanding | Applied/modeling | Applied/modeling; symbolic | 65.18% |
| | Applied/modeling | Symbolic | 5.76% |
| | Symbolic | Symbolic | 5.50% |
| Analyze | Explanation | Explanation | 25.00% |
| | Definition | Definition/Explanation | 25.00% |
| | Applied/modeling | Applied/modeling; symbolic | 25.00% |
| | Symbolic | Explanation/Graphical | 25.00% |

Figure 1.
Instructors' response to the question: "How frequently did you require students to explain their thinking on exams?"

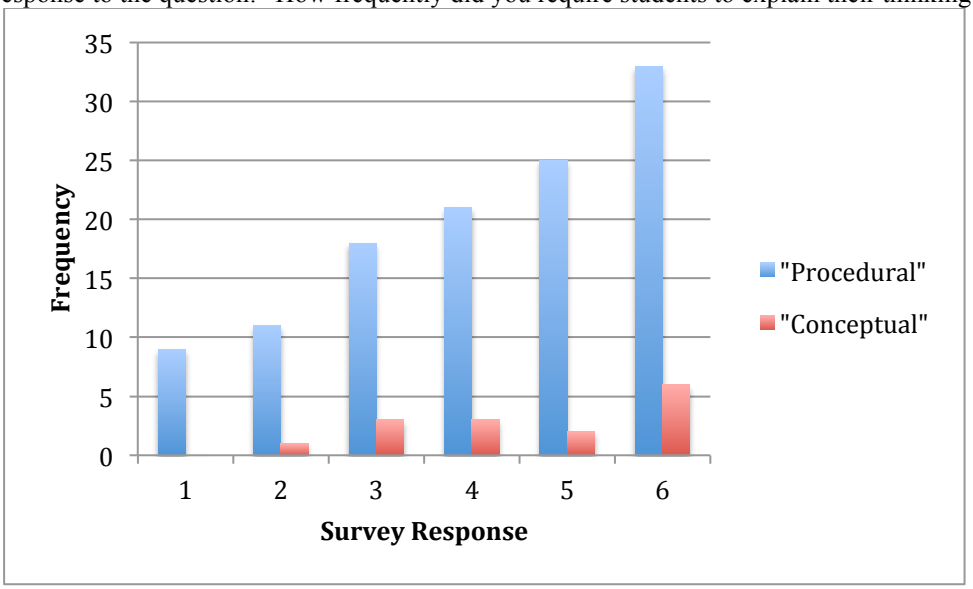
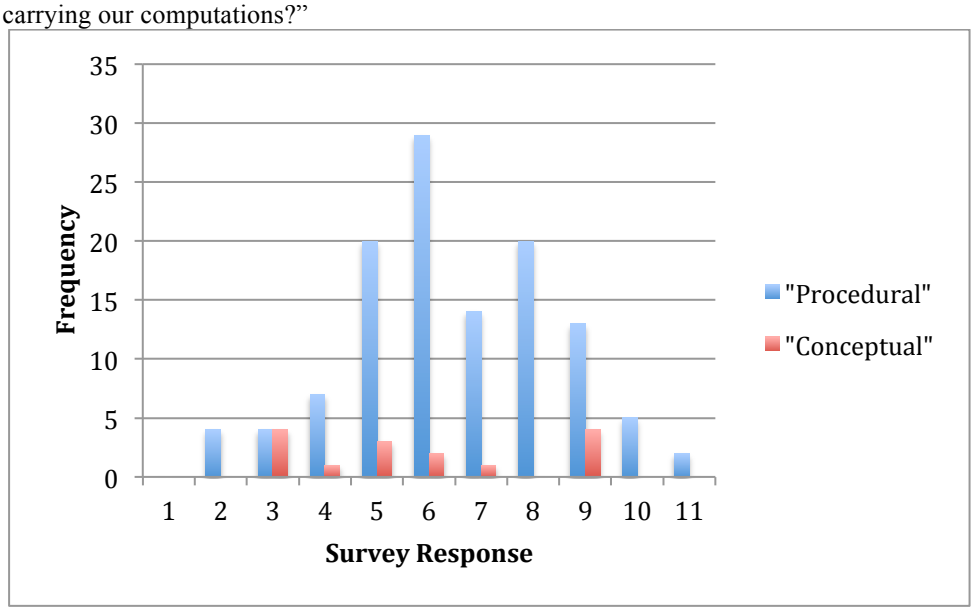



Figure 2.
Instructors' response to the question: "On a typical exam, what percentage of the points focused on skills and methods for carrying our computations?"



# The Search for the Normative Identity in a College Algebra Class

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**Abstract:** This paper presents results from a qualitative study examining talk about classroom norms from two students and the instructor from a single college algebra classroom. Three cases are presented and interpreted using an interpretive scheme developed by Cobb & Hodge (2007). This study considers the following questions: (a) *How do participants talk about the normative identity?* and (b) *How do participants talk about themselves with respect to the normative identity*? Results include participants' observations, interpretations, and evaluations of students' general social and specifically mathematical obligations. Results indicate that participants were aware of classroom norms and were able to describe several students' obligations in rich detail. Further, participants described their roles in negotiating these classroom norms, illuminating issues of authority and agency for all participants. Taken together, the results provide unique insights into classroom activity and culture, which have implications for future work on identity and for teaching at the collegiate level.

**Keywords:** identity, norms, college algebra, authority, agency

Students' beliefs about themselves as learners and as potential mathematicians, their beliefs about the purposes of mathematical activity in the classroom, and their valuations of mathematics writ large influence their motivation to engage in the classroom and beyond. As evidenced by Boaler & Greeno (2000), students' experiences in their mathematics classroom can negatively impact their beliefs about the discipline and about themselves. In instances where students "leave mathematics because they do not want to author their identities as passive receivers of knowledge," it is not only individual students who are affected by disengagement; the discipline of mathematics suffers the loss of potential talent (Boaler & Greeno, 2000, pp. 188-189). Although it is not intentional, entire groups of students may be leaving the discipline because they do not identify with mathematics as it is realized in the classroom. Despite researchers' and teachers' awareness of these issues, student apathy and aversion to the study of mathematics still persist – "there appears to have been little progress towards improving the situation… Many students hold unhelpful and unhealthy views of mathematics, and participation in mathematics classes at the higher levels continues to diminish" (Grootenboer & Zevenbergen, 2008, p. 243). Researchers have begun to understand the profound ways in which students' developing identities as knowers and doers of mathematics impact their understanding of the discipline of mathematics and their place within it (Boaler, 2002; Boaler & Greeno, 2000; Cobb et al., in press; Cobb & Hodge, 2002, 2007; Martin, 2000). As such, researchers have recently employed the construct of *identity* to gain insight into how students' beliefs, attitudes, and motivations are negotiated and renegotiated within the mathematics classroom context.

Although the construct of identity has been given more attention lately, mathematics education researchers are only just beginning to make sense of how it can be used to illuminate the relationship between learning, culture, and participation: "Limited attention has been given to the identities that students are developing… despite a growing body of evidence that indicates that the development of students' mathematical reasoning is intertwined with who they are becoming in the mathematics classroom" (Cobb, 2004, pp. 334-335). Operationalizing these theories of identity is a fairly recent endeavor. Thus, the purpose of this study is to gain insight

into the local, social world of the mathematics classroom in order to enhance current frameworks and interpretive schemes used in this work. In particular, I focus on the identities that students are developing at the collegiate level, as little work has been done with these populations. Recent work on identity focuses on defining and investigating different facets of identity (Boaler, 2002; Boaler & Greeno, 2000; Cobb et al., in press; Cobb & Hodge, 2002, 2007; Martin, 2000). In order to do so, Cobb and Hodge (2007) defined three types of identities – the core, personal, and normative identities, which I detail below. The framework that incorporates these identities is the result of years of classroom observations and thoughtful reflection on identity and establishes the base for my analytic framework.

**Core, Personal, and Normative (mathematical) Identities.** Building on Gee's notion of identity (1999), Cobb & Hodge (2007) stated that the *core identity* for mathematics students is chiefly "concerned with students' more enduring sense of who they are and who they want to become" (p.167). This part of identity might be how most people think about identity – a potentially more stable, longstanding understanding of the self. Generally speaking, aspects of a person's core identity might include their religious or political affiliations, their career, or their gender, for example. Within the context of mathematics, a person's *core mathematical identity* might describe how they engage with the discipline (e.g., as a student, teacher, accountant, researcher, etc.) or, possibly, their long-term beliefs about mathematics (e.g., girls aren't good at math, mathematics is necessary for my future success). Alternatively, a student's *personal mathematical identity* is defined as "an ongoing process of being a particular kind of person in the local social world of the classroom" (Cobb & Hodge, 2007, p. 168). Therefore, it considers the individual aspects of identity that are immediate to the classroom context. Lastly, the *normative* (*mathematical*) *identity* is "the identity that students would have to develop in order to affiliate with mathematical activity as it is realized in the classroom" and in order to "develop this sense of affiliation, a student would have to identify with the obligations that he or she would have to fulfill in order to be an effective and successful mathematics student in that classroom" (Cobb & Hodge, 2007, p. 166). These obligations include general social norms and specifically mathematical norms.

In some instances (e.g., a reform-based classroom), it maybe quite normal for students to question the claims made by their peers. Students in these classrooms might be expected to work in groups, develop nonstandard problem solving methods, or discover mathematical ideas for themselves. In contrast, a more traditional classroom might foster very different types of skills and behaviors – emphasis on efficiency or speed, development of standard methods, etc. Therefore, which characteristics are valued is highly dependent on classroom values and norms. While the core and personal mathematical identities describe individuals and their beliefs and values, the normative mathematical identity is constructed to understand how students are expected to participate in class. As such, the normative identity is a concept that not only differentiates between "traditional" and "reform" classrooms, but also details the nuances of engaging in mathematical activity. According to Cobb and colleagues, the normative identity is co-constructed by the students and the teacher: "the normative identity is a collective or communal construct rather than an individualistic notion" (Cobb, Gresalfi, and Hodge, in press, p. 1). It can be viewed as the archetype student within a *particular* classroom context – the embodiment of classroom social and mathematical norms. Although Cobb and colleagues have conducted a few studies using this interpretive framework with middle-school students (e.g., Cobb, Gresalfi, and Hodge, 2009) and with middle school teachers (Gresalfi and Cobb, 2011), these constructs are still highly theoretical and deserve more attention from the field. Many

issues remain uninvestigated… First of all, is there evidence to support the existence of such an identity? Solomon (2007) posited "there is more than one way of being a successful student in undergraduate mathematics" (p. 15). So, is there really only *one* normative identity in a given class? What would this student look like? Talk like? How would he / she engage with the mathematics? Further, how so students make sense of how they view themselves with respect to the normative identity? Do students believe that they are like this normative student? If not, have they found another way to be successful within that class? Related to this, who constructs this identity? According to Cobb & Hodge, the normative identity is co-negotiated by classroom members. Therefore, students and teacher *should* have the ability to challenge the existing normative identity. Do students believe that they have this kind of agency?

## Research Questions

Toward this end, I pose the following research questions, which are the focus of my study:

(1) How do students talk about the normative identity? How does their teacher talk about the normative identity?

(2) How do these students talk about themselves with respect to the normative identity?

## Methods

### Context and Participants

**Context.** This study took place in a single mathematics classroom at a large, Research I institution. The topic of the course was College Algebra, a three-credit course offered every semester through the Department of Mathematics. After earning credit in this course, students are still required to take a second mathematics course to graduate, regardless of major. Therefore, it is not a terminal course. At the time of the study, students had at least three section options available to them: (a) a large-lecture section (with upwards of 250 students in each section); (b) an on-line section; or (c) a small class setting (with up to 35 students in a single section). Over 1,000 students enrolled in College Algebra – the vast majority of students electing to take it in a large-lecture setting. The student participants from this study, however, chose to enroll in a small class setting as part of a special program designed to provide students with extra support. As of Spring 2011, there were three such programs offered through the mathematics department (all of which have been drastically reduced or have been cut altogether). Knowing that these special support programs were going to be in danger, I felt that it was extremely important to collect information from classroom participants in these unique settings before the opportunity was lost.

**Participants.** The participants in this study included three members – two students and one instructor – from the same section of an enrichment section of a college algebra course in the spring of 2011. Two students and the Graduate Teaching Assistant, who served as the primary instructor and lecturer, volunteered to participate. Demographic and background information of the participants can be found in Figure 1 below (Note: all names are pseudonyms).

### Data Collection and Analysis

For this study, I relied solely on interview data. There are three reasons that I chose to use interview data as my only data source: First, I believed that bringing together multiple perspectives would only provide a richer picture of the classroom norms. Relying only on the perspective of the researcher (in the case of classroom observations) ignores the agency that students have in interpreting and contributing to the normative identity. Moreover, I believe that mathematics education researchers should support students in helping them to find a voice. Second, there is precedence for researchers to rely of interviews: "Studies of identity have relied on interviews. This method allows one to examine the personal narratives of adolescents and

adults and relate them to identity formation" (Yamakawa et al., 2009). Lastly, I was interested in seeing if and how participants were able to describe classroom norms. Past studies have been focused on younger populations and did not rely on student interviews for descriptions of classroom norms. I believed that due to their age and the fact that they have been recently immersed into a new culture (i.e., college), participants might be better positioned to articulate their observations and evaluations of classroom obligations.

In their work, Cobb and colleagues have characterized eight norms that are important to the construction of the normative identity (see Figure 2 below). Cobb & Hodge (2007) developed these eight norms through a constant comparative process (Glaser & Straus, 1967). As previously noted, researchers must pay attention to *general social norms*. These are normative ways of acting and speaking that may not be specific to mathematical activity – one would expect students to explain their reasons in other classes, for example. Additionally, researchers should attempt to make sense of the *mathematical norms* of the classroom. Using these eight norms as a guide, I developed a semi-structured interview protocol and a Likert-scale survey designed to illuminate participants' views on classroom norms and obligations. Before I began analysis, I transcribed each hour-long interview in full. I then looked through each interview, coding sections of interview text using this framework to develop three individual cases. After gaining familiarity with each case, I then conducted a cross-case analysis designed to illuminate patterns across participants' observations.

## Results and Implications for Future Research

Descriptions of the social norms that students are obliged to follow were very similar across participants. Therefore, interpretations about the social aspects of the normative identity were fairly easy to interpret. For example, both students repeatedly described the importance of taking notes and attempting to understand the solution methods presented by Oliver (the instructor), as the following excerpt illustrates:

> Ashley: Umm, I write exactly everything he writes on the board. Umm, I'm really bad at figuring out like the things he says, like verbally. So, but he generally--, if there's things that we need to know that he just said, he will write it on the board. It's just like everything he writes on the board. Yeah, and then if he does something in a problem that I think is important or he skips a step, then I'll like do that. Make sure I do that step that he skipped. So, it will be like cru-cick [[*moves hand as if checking off a list*]], really step-by-step for me to figure out another problem that we have to do.

As such, determining how participants viewed the general classroom obligations was apparent. Specifically mathematical norms, however, were less clear. For example, I believed (at first) that each participant had responded inconsistently to questions about understanding. I had difficulty reconciling how Malik in particular could explain the importance of memorizing at one point in the interview:

> Malik: [The exams ask you to] memorize. (…)'cause it's basically problems that you did in class and you really have to remember what the steps were. So, it's like memorize these steps, but like memorize and understand why we are doing these steps to get to this answer.

and then describe so passionately how students cannot rely on memorization to be successful at the next.

> Malik: You have to get it. You have to really learn. You can't just memorize anymore.

Similar phenomena occurred with the other two participants. Ultimately, after further

investigation, I believe that this is primarily due to the complexity and potential ambiguity of words like *understanding*, *tools, argument*, or *purpose*.

Generally, though, participants' descriptions were alike in the sense that they noticed the similar aspects of classroom practice. Taken together, the overlapping descriptions from participants point to agreement on many classroom norms, which are intended to characterize the normative identity. These characteristics are provided in Figure 3 below. It is important to note that although this table outlines normal ways of behaving in class and describes the obligations that students face, it does not seem to prescribe one normative identity. In other words, it seems that there are various ways to participate in class. For example, it was normal and acceptable for some students to speak up and for others to remain silent when confused. Whether a student spoke up or relied on another student to speak on their behalf did not seem to have direct implications for his or her success in the course, according to participants. Therefore, as this example demonstrates, the construct of *the* normative identity, as the embodiment of classroom norms, might be too narrow. That is, if there are multiple visions of success in the class, a single normative identity might be misleading.

Although Ashley and Malik's observations about classroom norms were quite similar, their talk about themselves with respect to those norms differed. Malik described himself as the model student. He believed that his willingness to ask questions and to vocally participate in class was important to the success of himself and his peers. Ashley described her semester as being "eye-opening." She felt that she was coming to affiliate with mathematical activity in the classroom in ways that were supporting her success. In contrast to Malik, though, she did not feel that it was always necessary or beneficial to speak up in class. Also, she thought that even though memorization of rules and procedures was necessary for success in the course, that this was difficult for her. Although Malik and Ashley seemed to understand their respective obligations and worked hard to fulfill them, it was clear from their talk that Malik and Ashley both had some criticisms of the classroom norms. For example, Malik felt that he did not "understand a word that [the instructor says during lecture]." Ashley felt that examinations were generally too long and they covered too much material. Ashley and Malik had criticisms of the classroom norms, but felt that they lacked the agency (or perhaps the expertise) to shape classroom practice to better suit their needs. As indicated by this last quotation, Ashley felt in fact that she had very little affect on the classroom culture:

> Interviewer: Do you feel like you personally have influence on how this class is run?
>
> Ashley: Uhhh, how it's run? I guess not really (…) I think yeah. I think **he** (the instructor) **has** a pretty set-up, a pretty planned-out of how **he wants** the course to go, how **he wants** the course to look because it looked the same everyday since I've been there. I am pretty sure that if I wasn't there that one day, **he still did the same thing**. (emphasis added)

Lack of agency led students to feel that they had a limited role in negotiating classroom norms.

I believe that this study illuminates the potential for student interviews to be a valuable source of data, especially when students have been placed in a new context thereby potentially heightening their awareness of classroom norms. Additionally, this study contributes to a larger conversation about student agency and the distribution of authority in the classroom. If we expect students to affiliate with classroom activity as it is realized in the local, social classroom microculture, it is important the mathematics education researchers listen to what students have to say and provide them space to not only voice criticism, but also to develop a sense of agency that would support them in positively shaping classroom norms.

Figure 1

*Participant Demographic Information and Background*

| | Malik | Ashley | Oliver |
|---|---|---|---|
| **University Classification** | Freshmen | Sophomore | Instructor |
| **Gender** | Male | Female | Male |
| **Racial/Ethnic Group** | Black | White | White |
| **English as a First Language** | Yes | Yes | No |
| **Major** | Marketing | Elementary Education | N/A |
| **Course Background** | First time taking the course | Third time taking the course | Fifth time teaching the course, Second time teaching the enrichment section |

Figure 2

*Social and Mathematical Norms*

| **Type of Norm** | |
|---|---|
| **General Social Norms** (Cobb et. al, 2001) | Norms for explaining and justifying reason<br>Normative ways of listening to and attempting to understand others' explanations<br>Norms for indicating confusion<br>Norms for indicating and giving reasons for disagreement with an invalid solution |
| **Specifically Mathematical Norms** (Cobb and Hodge, 2007) | Norms for what counts as an acceptable mathematical argument<br>Normative ways of reasoning with tools and written symbols<br>Norms for what counts as mathematical understanding<br>The normative purpose for engaging in mathematical activity |

Figure 3

*Summary of Results: Classroom Norms Characterizing the Normative Identity*

| **General Norms** | |
|---|---|
| Norms for explaining and justifying reason | Students were expected to provide explanations in their written work in order to convince the instructor that they understood the problem and were not guessing or solely providing a calculator solution. |
| Normative ways of listening to and attempting to understand others' explanations | A student's primary obligation was to listen to the explanations of the instructor. Understanding the solution methods demonstrated by the instructor did not typically take place in class, but was gained after looking back at thorough notes taken during lecture. Rarely did students work together or listen to one-another unless one student did not understand the instructor's explanation and were looking a simpler explanation. |
| Norms for indicating confusion | Although it would be acceptable to voice confusion, most students rarely did so. A few students consistently asked questions in class, while others remained quiet. Many did not feel comfortable asking questions in class. Instead, some asked questions outside of class time during office hours, sought help from a tutor, or relied on other students to ask a similar question. |
| Norms for indicating and giving reasons for disagreement with an invalid solution | Students were unlikely to voice disagreements with an invalid solution should they become aware of one, often waiting for the instructor to determine its invalidity and to provide reasons why the solution was not valid. |
| **Specifically Mathematical Norms** | |
| Norms for what counts as an acceptable mathematical argument | An acceptable mathematical argument is one in which each step is presented in the correct order and leads to the correct solution. The instructor has the ultimate authority to determine if the argument and solution are correct. |
| Normative ways of reasoning with tools and written symbols | In this class, the tools used for mathematics included notes, calculators, textbooks, and outside resources like tutoring and office hours. |
| Norms for what counts as mathematical understanding | All participants recognized that there are different levels of understanding required for success in the course. *Instrumental understanding* was necessary for examinations, but *relational understanding* indicated a higher level of student success (Skemp, 1977). Mathematical understanding was gained with practice. Students needed to do homework and practice using the rules and procedures taught in class. |
| The normative purpose for engaging in mathematical activity | There were multiple purposes for engaging in mathematical activity in class. One reason was to prepare students for future financial decisions (e.g., pricing mortgages). Another reason was to build a logical foundation upon which students might draw in the future. |

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# Understanding how precalculus teachers develop mathematic knowledge for teaching the idea of proportionality

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**Abstract:** The purpose of this study was to better understand how a precalculus teacher develops mathematical knowledge for teaching the idea of proportionality. We were also interested in understanding what instructional supports might foster shifts in a teacher's thinking, and then how shifts in her thinking affect her classroom decisions.

The teacher was using a research-based precalculus curriculum designed to help students acquire improved problem solving abilities, and deeper understandings and connections among the courses central ideas. The findings revealed that the research-based professional development and support tools for teaching precalculus can lead to improvements in the teacher's mathematical content knowledge and aspects of her teaching practice. We also observed that the teacher was still making new mathematical connections during her second year of using the materials. We also observed that shifts in a teacher's content knowledge do not always improve the teacher's ability to leverage her student's thinking during teaching.

**Key Words:** precalculus, proportional reasoning, mathematical knowledge for teaching, linearity

Research questions

We are seeking more detailed understanding of how a teacher develops mathematical knowledge for teaching (MKT), and what mechanisms may support or prevent the teacher in making shifts to more effective teaching. To investigate this, we are focusing on a case study of a teacher teaching about proportional relationships. This teacher is using a research-based curriculum, which was designed as part of an NSF grant. We based our investigation on the framework developed by Silverman and Thompson (2008). The framework characterizes the process by which a teacher's MKT may develop. This provides a hypothetical trajectory for teacher learning that allows us to evaluate the development of a teacher's MKT and identify when barriers are preventing further development of her MKT. Developing a more detailed description of how one teacher's MKT about proportionality emerges over time may help to expand our understanding of Silverman and Thompson's general framework.

With that in mind, the questions we are trying to address follow:

1. How does a teacher's mathematical thinking about proportional relationships shift as she teaches with these activities?
    a. What new mathematical understandings if any emerge as she teaches?
    b. What new mathematical connections are made between ideas?
2. How does a teacher's understanding of student thinking shift as she teaches and reflects on her teaching?
    a. What insights into how her students think does she develop?
    b. How does the teacher perceive the understandings she wants her students to develop as supporting their future learning?
    c. What strategies does she develop to support students in developing new ways of understanding?
3. How do these shifts in thinking about the mathematics and understanding of student thinking change the choices the teacher makes in the classroom?

Relationship of this research to current literature

There seems to be general agreement that there is specialized mathematical knowledge required to teach mathematics effectively. Much work has been done to understand MKT in elementary mathematics (Ball, 1991; Ball, Hill, & Bass, 2005; Hill & Ball, 2004; Ma, 1999)). This research has focused on observing what actions a teacher performs in the classroom, and then on what knowledge a teacher needs to be able to do those actions effectively. Ma (1999) describes a deep connected mathematical knowledge which is necessary for effective elementary mathematics teaching which she calls profound understanding of mathematics.

We are interested in the MKT of teachers who teach more advanced mathematics, specifically precalculus. We want to better understand how teachers develop new meaningful ways of thinking about the mathematics connected with student thinking that allow them to develop activites and conversations that support student learning. In The Teaching Gap (Stigler & Hiebert, 2009) states that when teachers make changes, they are often superficial, because their underlying beliefs and thinking do not change. The framework developed by Silverman & Thompson (2008) describes a process by which MKT may develop for teachers of more advanced mathematics. According to this framework, the teacher first develops a significant mathematical understanding through reflection, and realizes its value to her students. Through further reflection and attention to student thinking the teacher develops ideas of how to support her students in developing a new way of thinking. It is through on-going reflection that a teacher is able to develop new ways of thinking and new approaches to teaching.

Theoretical perspective and/or conceptual framework

We are basing our study on the framework developed by Silverman & Thompson (2008) to describe the process of developing MKT. This process includes a number of stages that are not necessarily linear, but iterative and interrelated. The stages are characterized by ways of thinking that emerge as a teacher develops MKT. We can look for these ways of thinking as evidence of developing MKT. This framework also provides a hypothetical trajectory for developing MKT, which may allow us to provide interventions that encourage a teacher to develop MKT.

According to the Silverman and Thompson framework, the stages in developing MKT include[1]:

- *The teacher develops or becomes aware of her own personal Key Developmental Understanding (KDU).* A KDU (Simon, 2006) is an understanding that is a leap in understanding, is connected to many other concepts and provides a powerful foundation for learning. Courtney (2010) in his dissertation notes that teachers are resistant to the type of reflection required for them to develop a KDU. In this study we attempeted to provide activities and tools that encourage this type of reflection. We hypothesize that the teacher is challenged to reflect on her own understandings as she prepares to teach, and works through problems in the curriculum. Ma (1999) notes that Chinese teachers develop their deep mathematical understandings primarily through studying the curriculum materials with teaching students as the primary focus.
- *The teacher develops models of how students may understand through de-centering*[2]. We are interested in how the teacher attends to student thinking and what models she develops of

[1] The description provided here in italics is an abbreviated summary of our interpretation of the Silverman and Thompson framework. We have added notes that describe how the framework relates to our study.

student thinking. The "Teacher Notes" provided as part of the research-based curriculum provide descriptions of student thinking that support the teacher in developing her own model.

- *The teacher develops images of how someone else could develop the KDU they have developed.* We are interested in how the teacher imagines someone else developing the same understandings they have developed. This means developing "personal images that are capable of conveying what one has in mind to someone who does not already understand what one intends to convey."(Thompson, Carlson, & Silverman, 2007) Thompson, et al suggest that this is difficult because teachers have learned to focus on procedures they intend for students to learn, not on ideas they intend for students to develop.
- *The teacher develops a mini-learning theory, including ideas for activities and conversations to develop this way of thinking.* We are interested in how the teacher's ways of thinking affect decisions the teacher makes in the classroom.
- *The teacher sees how this new way of thinking empowers other learning.*

We developed activities and asked questions that attempted to understand how the teacher was developing MKT. The teacher was using a curriculum, which begins with a module that focuses on developing the ideas of quantity, variable, proportionality, rate of change and linearity. The module also emphasizes the network of connections among these ideas. The curriculum presents proportional relationships as relationships between two quantities whose values change together or co-vary in a way such that as the quantities change together they remain in a constant ratio to each other. This emphasis on co-varying quantities focuses attention on the fact that proportionality is describing a relationship between varying quantities, and supports the approach the curriculum uses to support students in solving problems by developing meaningful models of quantities and how they are related.

The understanding of proportional relationships is developed in ways to support future learning and reasoning about average rate of change, exponential functions, and angle measurements in trigonometry.

The curriculum intends to support teachers in developing coherent mathematical thinking with their students through challenging problems set in realistic situations. The curriculum materials provide support for the teacher in the form of workbooks with detailed activities and teacher's notes about how student thinking develops. Teachers also have access to Power Point slides and an on-line forum to exchange ideas with their peers and get support from a curriculum expert. Teachers also attend regular bi-weekly Professional Learning Community (PLC) meetings with other teachers using the materials.

## Research methodology

This case study follows a precalculus teacher, Elizabeth who has taught for 7 years, as she teaches a class of 40 high schools students. Elizabeth has taught with the research-based curriculum for one full year and is now in her second year of implementing the materials. The data we report here focuses on her classroom practices during the first two months of the second year of using the curriculum.

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[2] We used the idea of de-centering to describe a behavior in which one attempts to understand the mathematical thinking and/or perspective of someone else for the purpose of adjusting her behavior in order to influence another in specific ways. (Piaget, 1995; Steffe & Thompson, 2000)

Elizabeth recorded her daily goals for student learning before each class and videotaped all lessons. Immediately following class Elizabeth recorded in a journal, instances she recalled in which student thinking was revealed. The intention was that Elizabeth would notice and become more aware of what her students actually said or did in class without feeling the pressure to make an interpretation or judgment. She was also asked to make comments on her mathematical understandings as they related to the student thinking she observed. This form of data collection was made to promote reflection and was based on Mason's (2002) suggestion that humans have a tendency to interpret before observing and noticing, thus limiting the quality of an individuals observations.

The interviews were performed to investigate the teacher's thinking about the mathematical ideas, her attention to her students' thinking and how their thinking had influenced her classroom decisions. The questions were based on teacher reflections, student work collected by the teacher and on notes from classroom observations and videos. The questions posed during the clinical interviews included prompts as deemed useful for advancing both her MKT and classroom practices. The clinical interviews were conducted after the completion of each unit of connected ideas.

Results of research

During the first year of using the curriculum Elizabeth developed new understandings of teaching proportionality including the value of students understanding the three ways of expressing that two changing quantities are related proportionally (i.e., if $a$ and $b$ are related proportionally then as they change together: i) the ratio of $a$ to $b$ remains constant; ii) $a$ and $b$ are related by a constant multiple; iii) if $a$ is scaled by a constant, $b$ is scaled by the same constant). She was also able to see how the idea of proportionality is related to the ideas of constant rate of change, slope and linearity. We also documented that while her understandings are consistent with the curriculum, she is still developing ways of leveraging student thinking and helping students develop meaning. Finally, we give an example that demonstrates Elizabeth identifying an important mathematical understanding in a clinical interview.

First, we discuss how Elizabeth's mathematical understanding had shifted after one year of using the research-based curriculum, while also noting that some connections between the idea of proportionality and other ideas were missing in her conception of proportionality. Before using the curriculum, when Elizabeth was probed to discuss the idea of constant rate of change, she did not spontaneously see how the idea of proportionality could be used to relate ideas of constant rate of change and slope of a line.

> *I: What connections do you expect students to make between constant rate of change and the slope of the line?*
> *E: Yes, so constant rate of change. First off I would expect them to know that graphically it's a straight line and then second I would expect them to be able to dictate to me that for, uh, over equal amounts of changes of the independent variable the dependent variable is also changing by a same amount*
> *I: What about if you change the input by half the original amount?*
> *E: They should know proportionally that the output should change. You should change it by the half amount as well*
> *I: So, would that be part of your expectations for their understanding?*
> *E: Oh, I'm not sure I've ever really gotten into that not necessarily in a trig pre-calculus class to be honest. ... I can't say that I did in a trig pre-calculus this year.*

We can contrast this with a recent classroom interaction, which was observed in her second year of teaching with the materials. The problem is shown in figure 1. Elizabeth guided the class through a conversation that included first identifying that a linear relationship has a constant rate of change, and that in this problem the constant rate of change is -2.5. The class then discussed that the change in y is proportional to the change in x. The students worked this problem using these understandings. This approach to thinking about rate of change and proportionality is quite a contrast to her earlier statement of what she wanted students to understand. In her teaching here, there is a clear connection between the ideas of slope, constant rate of change and the idea that the change in x and the change in y are proportional. We noticed that she did not use the constant multiple representation of a proportional relationship with her students. In an interview we asked her if she could imagine using the constant multiple representation of proportionality in this problem. Elizabeth noted that she is not comfortable with this representation of proportionality, and then noted the connection to the fact that her students also did not seem comfortable with this representation Elizabeth identified this as an area for her to further develop her own personal understanding.

According to her journal entries and informal conversation, Elizabeth finds it challenging to understand student thinking and then find ways to support students in developing their own understanding. Figure 2 shows student work on a problem, which was given on a quiz. The student's answer contains correct ideas, but the student applies the ideas incorrectly. Elizabeth expressed that the student probably understood the ideas, but just confused himself with the words. Another possibility for this confusion is that the student did not have a well-developed meaning of the idea of quantity and changes in the quantities. When this possibility was suggested to Elizabeth she was able to describe questions and activities that could possibly promote student understanding of these ideas.

Figure 3 contains another quiz problem. As Elizabeth was reviewing the quiz, one student asked how you decide which number goes on top of the ratio. Elizabeth guided them to use unit analysis. Later in an interview the researcher asked what understanding that student was missing that caused them to ask that question. After thinking for a while, Elizabeth said "*We could look at what we are trying to model. We are trying to model the cost with respect to distance... So, we want to look at how does the cost change as the miles change? Therefore their constant rate should be the change in cost with respect to miles. That sounds like a tough conversation.*" Elizabeth acknowledged that this understanding of the meaning of the rate of change would help students in solving this problem, and then went on to say she would further support the students' discussing the corresponding changes in cost and miles on a graph so they understood the meaning of the rate. This is a different approach from using unit analysis, and could be an insight that supports her future teaching decisions.

<u>Applications to/implications for teaching practices or further research</u>

This study supports that the research-based curriculum used by this teacher, and the accompanying professional development tools, may be useful for supporting teachers in advancing their MKT relative to specific content. Our findings support those of Silverman and Thompson in revealing that teachers are more effective in advancing their students' knowledge if they have deep personal understanding of ideas and rich connections among ideas. A major challenge in advancing a teacher's MKT is to support them in developing models of student thinking, and ideas about how to support students in developing meaningful mathematical thinking.

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## Figures

A linear relationship has a slope of −2.5 and passes through the point (−5, 8). Use the given point and the slope to complete the table of values for this relationship.

| $x$ | $y$ |
|---|---|
| −6 | |
| −3 | |
| −0.8 | |
| 0 | |
| 2.3 | |
| 5 | |
| 8 | |

Figure 1

b. If two quantities are related linearly, then they must be proportional.

THE STATEMENT IS ~~FALSE~~ TRUE BECAUSE IN ORDER FOR TWO QUANTITIES TO BE LINEAR, THEY MUST PROVE TO HAVE A CONSTANT RATE OF CHANGE OF X AND Y OR $\frac{\Delta y}{\Delta x}$. THIS $\frac{\Delta y}{\Delta x}$ MUST BE PROPORTIONAL THROUGHOUT.

Figure 2

2. On a week-long tornado hunting trip, Erica rents an SUV in Minneapolis for $440 a week. When filling up her second tank of gas, Erica noticed that she had spent $84 on gas to travel 350 miles. (Assume that the cost of gas per mile is constant throughout her trip.) $\frac{\$84}{350 \text{ miles}}$

a. (4 points) Thinking about the quantities that are changing and not changing, define a formula to determine the total cost of the trip in terms of the number of miles that Erica drives. Make sure to define the variables.

(Constant rate of change $\frac{\text{in gas cost}}{\text{change in miles}} = \frac{\$84}{350\text{mi}} = \frac{\$.24}{1\text{mi}}$)

Let C = total cost of trip in dollars for the rental, n = # of miles driven throughout trip

$C = .24n + 440$

Figure 3

# Expanding Toulmin's Model: The Development of Four Expanded Argumentation Schemes from Analysis in Linear Algebra

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**Abstract**
In this presentation, I define four types of argumentation schemes that are expanded versions of Toulmin's (1969) model of argumentation. These expanded schemes—Embedded, Proof by Cases, Linked, and Sequential—developed out of necessity when the original 6-part Toulmin scheme proved inadequate while analyzing argumentation of an inquiry-oriented linear algebra classroom community. Aspects of these four expanded schemes were adapted from and are compatible with those presented by Aberdein (2006, 2009). Within this presentation, I investigate how Toulmin's is used within mathematics education research, as well as other fields of research, propose how the expanded schemes provide needed detail when analyzing complex argumentation, and provide examples of each from the whole-class discussion of the introductory linear algebra course.

**Key words**
Toulmin's Model, Linear algebra, The Invertible Matrix Theorem, Argumentation

This presentation highlights portions of my dissertation research, which had two main aspects: (a) research into the learning and teaching of linear algebra, and (b) research into analyzing the development of mathematical meaning for both students and the classroom over time (Wawro, 2011). In this study, I considered the development of mathematical meaning related to the Invertible Matrix Theorem (see Figure 1) for both a classroom community and an individual student over time. In this particular linear algebra classroom, the IMT was a core theorem in that it connected many concepts fundamental to linear algebra through the notion of equivalency. As the semester progressed, the IMT took form and developed meaning as students came to reason about the ways in which key ideas involved were connected. As such, the two research questions that guided my dissertation work were:

1. How did the collective classroom community reason about the Invertible Matrix Theorem over time?
2. How did an individual student, Abraham, reason about the Invertible Matrix Theorem over time?

**The Invertible Matrix Theorem**

Let $A$ be an $n \times n$ matrix. The following are equivalent:

a. The columns of $A$ span $\mathbf{R}^n$.
b. The matrix $A$ has $n$ pivots.
c. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$.
d. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a way to write $\mathbf{b}$ as a linear combination of the columns of $A$.
e. $A$ is row equivalent to the $n \times n$ identity matrix.
f. The columns of $A$ form a linearly independent set.
g. The only solution to $A\mathbf{x}=\mathbf{0}$ is trivial solution.
h. $A$ is invertible.
i. There exists an $n$ x $n$ matrix $C$ such that $CA = I$.
j. There exists an $n$ x $n$ matrix $D$ such that $AD = I$.
k. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
l. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.
m. Col $A = \mathbf{R}^n$.
n. Nul $A = \{\mathbf{0}\}$.
o. The column vectors of $A$ form a basis for $\mathbf{R}^n$.
p. Det $A \neq 0$.
q. The number 0 is not an eigenvalue of $A$.

*Figure 1.* The Invertible Matrix Theorem

To address both questions, I utilized Toulmin's Model of argumentation to analyze the structure of explanations related to the IMT both in isolation and as they shifted over time. Microgenetic analysis (Saxe, 2002) of these various arguments at both the collective and the individual level revealed four different complex structures of argumentation that were utilized when reasoning about the IMT, each of which is an expansion of Toulmin's classic 6-part scheme: (a) Embedded structure, (b) Linked structure, (c) Proof by Cases structure, and (d) Sequential structure.

The introduction and use of the expanded Toulmin scheme is a contribution to the field in that it advances the way Toulmin's Model of Argumentation is used in mathematics education

research. While Toulmin's Model has been used in this field since Krummheuer (1995), the majority of research has only utilized the "core" of the argument—data, claim, and warrant (see Figure 2). The work of Inglis, Mejia-Ramos, and Simpson (2007) argued for the use of the full 6-part Toulmin scheme, placing special emphasis on modal qualifiers, and Weber, Maher, Powell, and Lee (2008) made use of the 6-part scheme to analyze classroom-level debate. The need for the expanded Toulmin scheme in the present study may have been a function of the nature of the mathematical content at hand—that of linear algebra. An introductory linear algebra course often serves as a transitional point for students as they progress from more computationally based courses to more abstract courses that feature reasoning with formal definitions and proof construction. Thus, the complexity of proof-oriented argumentation involving formal, abstract concepts may have played a large role in why the expanded structures were needed. How may the expanded Toulmin scheme be helpful or necessary in analyzing other linear algebra data sets, or even other content domains? As such, the expanded structures may prove to be a powerful tool in developing theory and analyzing student development with regards to the discipline-specific practice of proving.

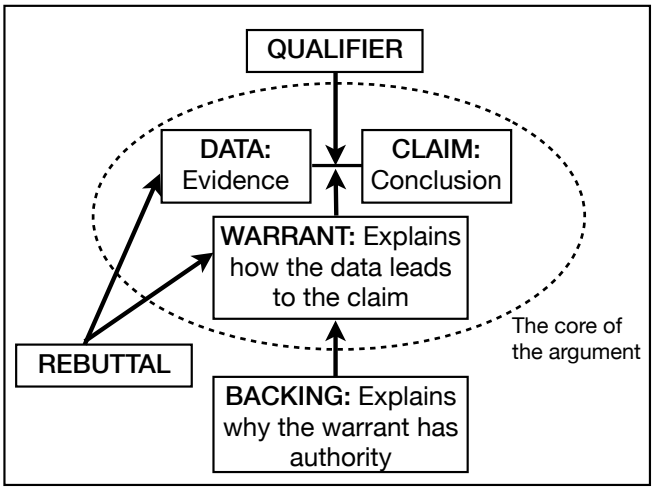


*Figure 2.* Toulmin's Model of Argumentation

## Theoretical Framing and Methods

The theoretical perspective on learning that undergirds my work is the emergent perspective (Cobb & Yackel, 1996), which coordinates psychological constructivism (von Glasersfeld, 1995) and interactionism (Forman, 2003; Vygotsky, 1987). I take the perspective that the emergence and development of mathematical ideas occurs not only for each individual student but also for the classroom as a collective whole. Many researchers acknowledge the role of the collective on the mathematical development of a learner and vice versa (Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Rasmussen & Stephan, 2008; Saxe, 2002). Through this viewpoint, the interrelatedness of the individual and the collective come to the fore, highlighting how the activity of one necessarily affects that of the other.

Data for this study came from the fourth iteration of a semester-long classroom teaching experiment (Cobb, 2000) in an inquiry-oriented introductory linear algebra course. Data sources were video and transcript of whole class and small group discussion, as well as video and transcript of Abraham's interview and written work. The overarching analytical structure of my

methodology was influenced by a framework of genetic analysis through the notion of cultural change, using two interrelated strands of microgenesis and ontogenesis (Saxe, 2002). In this paper I report on my use of Toulmin's Model of argumentation to analyze the structure of explanations related to the IMT both in isolation and as they shift over time.

Toulmin's 6-part scheme for substantial argumentation consists of *claim* (C), *data* (D), *warrant* (W), *backing* (B), *qualifier* (Q), and *rebuttal* (R), each with its explicit role in any given argument (see Figure 2). The operational definition for *argument* in this present study on student reasoning in linear algebra is "an act of communication intended to lend support to a claim" (Aberdein, 2009, p. 1). To address the research question of how the classroom community reasoned about the IMT throughout the semester, I coded arguments that explicitly involved developing meaning for the concepts in the IMT or for relationships between the ideas in the IMT, and each of these arguments occurred during whole class discussion. I drew data from ten class sessions throughout the semester, in which conversation was directly related to the development of the IMT. To address the second research question, data from two individual semi-structured interviews I conducted with Abraham, and I also analyzed Abraham's written work.

## Results

Within the whole class discussion data set of the ten class sessions, I coded 118 different arguments using Toulmin's model of argumentation. Of these coded arguments, 81 of them were of a form that consisted of some subset of the six parts of the layout. Additionally, there were 15 arguments in which the instructor played a distinct role in the development of the arguments. Rather than being a contributor to the argument directly (for example, by providing the warrant to someone's claim), she called for data, warrants, or backing to be provided by either the speaker or another member of the class. This speaks to the teacher's unique role to move the mathematical agenda forward as well as to push for developing the social norms of explaining one's thinking and justifying one's claims (Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, & Wawro, 2009).

Within the remaining 22 arguments, the 6-part layout, with each part occurring at most once, seemed insufficient to capture the complexity of the arguments that transpired during whole class discussion. Some layouts were a string of the six components (such as $C\text{-}D_1\text{-}Q\text{-}D_2\text{-}W_2\text{-}Q\text{-}D_3\text{-}W_3$) and occurred, for instance, when multiple members of the classroom were working together to justify a relatively new claim (such as why if the determinant of a matrix is zero, then the column vectors of that matrix have to be linearly dependent). Other arguments were structurally complex (such as a student proving a claim by presenting justifications for all possible cases) in ways that necessitated an expansion of some aspect of the original Toulmin's model of argumentation. The four varieties of this that I encountered in my analysis were:

1. *Embedded structure*: When data or warrants for a specific claim were so complex, they had minor embedded arguments within them;
2. *Proof by Cases structure*: When claims were justified using cases within the data and/or warrants;
3. *Linked structure*: When data or warrants for a specific claim had more than one aspect that were linked by words such as "and" or "also"; and
4. *Sequential structure*: When data for a specific claim contained an embedded string of if-then statements, where a claim became data for the next claim.

Instances of the necessity of the expanded structures also occurred when analyzing how Abraham reasoned about the IMT throughout the semester. Two of the twelve arguments he presented in whole class discussion were best captured with the expanded structure, as well as three of his 22 arguments from small group work. Finally, 21 of the 58 arguments from the two interviews were best captured with one of the expanded structures. One possible explanation for this distinction could be the nature of an individual interview: one student is given undivided attention for over ninety minutes and is asked to share his ways of reasoning. This is different than in, say, small group discussion, during which Abraham spoke so interactively with his group members that it was rare to have him speak without interruption for a long enough period of time (i.e., to be able to contribute an entire argument to only him) that the resulting argument necessitated the expanded Toulmin's model.

These four expanded structures are compatible with the work of Aberdein, a researcher in the fields of logic and humanities who has done a variety of research regarding argumentation in mathematics. In particular, Aberdein (2006, 2009) expanded Toulmin's basic framework to include more complex proof structures, such as induction or proof by contradiction. Often with my analysis of both whole-class discussion and individual argumentation, the complexity of the provided justifications did not seem adequately captured with only the "data-claim-warrant-backing" structure. As such, I adapted Aberdein's notion of the expanded layout in order to characterize these more complex structures.

In the remainder of this section I discuss and present an example of two of the expanded structures: Embedded and Linked. I draw my examples from whole class discussion of the IMT. During the presentation, I will discuss each of the four structures and present examples from not only the collective but also the individual unit of analysis.

**Embedded Structure**

I define an *embedded structure* as a Toulmin scheme within which one or more of the data, warrant, or backing is itself composed of a Toulmin scheme, minimally a C-D pair. A simple example of an embedded structure is provided in Figure 3. This argument occurred on Day 24 of the semester, during which the class was investigating determinants and their connections to other ideas in the Invertible Matrix Theorem. The instructor was making explicit how the formula for a 2x2 matrix $A$, det $A = ad - bc$, connected to the row-reduced echelon form of $A$. In previous class sessions, the class members had discussed why row-reducing an $n$ x $n$ matrix $A$ augmented with the $n$ x $n$ identity matrix not only was a valid method to determine if $A$ was invertible, but it also allowed you to compute $A^{-1}$. In other words, the class discussed why $\left[A \mid I\right] \sim \left[I \mid A^{-1}\right]$ "worked." On Day 24, the teacher revisited this computational method with the generalized matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The class had reached the point $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$ in their work when the argument occurred. The teacher made the claim that if $ab - bd = 0$, then the left-hand side of the above augmented matrix cannot row-reduce to the identity matrix, and the class assisted her in creating a justification for that claim (see Figure 3).

The original 6-part Toulmin scheme is not sufficient to capture the complexity of this argument. In a Toulmin scheme, a warrant serves to explain why the data is relevant to the claim. For the argument shown in Figure 3, the warrant explains why having a zero in the bottom corner

(the data) relates to not row-reducing to the identity matrix (the claim). The data-claim pair, seen in Figure 3 as being part of the data, does not serve this purpose of connecting the data to the claim; rather, it serves to explain the data in more detail.

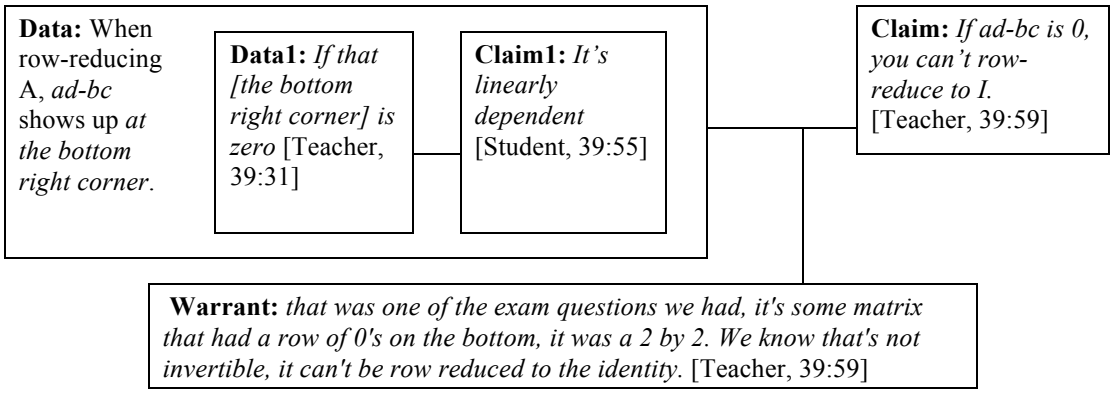


*Figure 3.* Example of an Embedded Structure: Explanation of why if the determinant of a 2x2 matrix is zero, the matrix cannot row-reduce to the 2x2 identity matrix.

Purely embedded structures are the least sophisticated of the four complex structures, yet they are foundational to the other three. In each of the remaining three complex structures—Proof by Cases, Linked, and Sequential—an embedded structure takes a more specific form within either the data, warrant, or backing of a given Toulmin scheme.

**Linked Structure**

I define a *linked structure* as a Toulmin scheme within which the data and/or warrant for the claim are composed of more than one embedded sub-argument that are linked by words such as "and" or "also." This differs from the Proof by Cases structure in that the sub-arguments are not related in the same manner. Furthermore, this structure goes beyond, for instance, a Toulmin scheme with multiple data. As Aberdein points out, Toulmin himself allowed for multiple data but that the linked structure expands upon Toulmin by "permitting multiple propositions within a node to be distinguished as separate nodes...However, this is necessary unless the propositions are individually attached to other nodes" (Aberdein, 2006, p. 7). In other words, the difference in the Linked structure is that the multiple data are actually sub-arguments themselves.

An example of a Linked structure occurred on Day 20 of the semester. On Days 19 and 20, the class investigated notions related to one-to-one and onto transformations: examples of each, non-examples of each, and other concepts to which they were similar. The students were parsing out the relationship between one-to-one and onto (which are properties of linear transformations) and linear independence and span (which are properties of sets of vectors). On Day 20, students began to explore the connections between onto and span, as well as between one-to-one and linear independence. In Figure 4. Abraham explains how the claim of "being linearly independent is the same as being onto" if the matrix is square made sense to him.

Abraham, after he made his claim, began by qualifying his claim by stating he "just remembers" the data he was about to share with the class. He then stated two sub-arguments, which are from "the $n$ x $n$ theorem": "If a matrix is square and linearly independent" (Data1)

then “it also spans” (Claim1) and “if it spans” (Data2) then “it’s also linearly independent” (Claim2). One may notice the metonymic nature (Lakoff & Johnson, 1980) of these statements: a matrix does not, in the strict mathematical sense, have the properties of linear independence or span, but rather the column vectors of that matrix do. The wording of the claims and data reflect this in order to be as true as possible to students’ original utterances. Furthermore, these four statements together comprise the data for the original claim. They are separated into sub-arguments because, mathematically, they are quite different and the validity of those data-claim pairs had to be established previously in the semester.

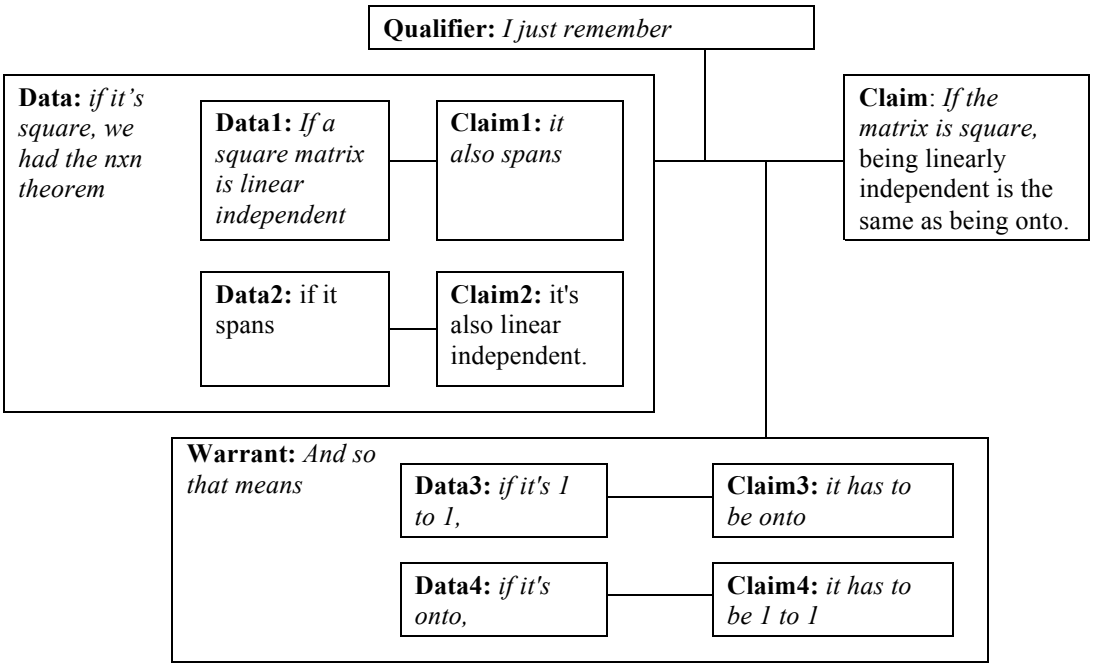


*Figure 4.* Example of a Linked structure: Abraham explains a connection between linear independence and onto.

Abraham’s warrant in this argument (see Figure 4), which began with “and so that means,” has a structure similar to that of his data. What was left unsaid in his explanation is why his warrant connected the data to the claim. His claim discussed the possible equivalency of linear independence and onto, he stated (without support) the equivalency of linear independence and span in his data, and he stated (again, without support) the equivalency of one-to-one and onto in his warrant. Analyzing the relevant classroom mathematical practices from this community that allowed Abraham to make these statements without support is beyond the scope of this proposal.

## Conclusion

I present four types of argumentation schemes that are an expanded version of Toulmin’s model: Embedded, Proof by Cases, Linked, and Sequential. These argumentation structures were developed out of examining the transcript and video from one particular linear algebra class as its members reasoned about the Invertible Matrix Theorem and the mathematical concepts

involved in that theorem. These four expanded structures were adapted from and are compatible with the expanded Toulmin schemes presented by Aberdein (2006, 2009). His work in informal logic and argumentation in mathematics presented analysis of logical structure and proof; however, it does not seem that his work analyzed the argumentation practices that occurred in actual discourse. While expanded Toulmin structures were quite useful in mapping out the proofs' structures, the source of the proof was left unstated. Was the proof given in a textbook and mapped out by Aberdein? Did he himself develop the proof and, if so, in what form? Was it written or communicated verbally to others, and then analyzed via the expanded structures? Thus, Aberdein's use of Toulmin's model is distinct from its use in the work presented here. The research in linear algebra presented here investigated the ways in which the members of a classroom reasoned about the Invertible Matrix Theorem. The analysis in this section focused on whole class discussion and examined the structure of arguments given by members of a classroom as they justified claims in situ. Thus, this study contributes by investigating the "argumentation of natural language" in an inquiry-oriented mathematics classroom and found it beneficial to adapt Aberdein's notion of the expanded Toulmin layout to do so.

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# A HYPOTHETICAL LEARNING TRAJECTORY FOR CONCEPTUALIZING MATRICES AS LINEAR TRANSFORMATIONS

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**Abstract:** In this presentation we articulate a hypothetical learning trajectory (HLT) designed to support students' development and elaboration of a transformation view of matrix multiplication. The major learning goals of this HLT are (a) interpreting a matrix as a mathematical object that transforms input vectors to output vectors, (b) interpreting matrix multiplication as the composition of linear transformations, (c) developing the imagery of an inverse as "undoing" the original transformation**,** and (d) reasoning about matrices as objects that geometrically transform a space. Within this HLT, we extend students' conceptualization of the "matrix acting on a vector" view to a more global view of a matrix transforming an entire space, as opposed to the localized view wherein matrices are interpreted as transforming one vector at a time.

**Keywords:** Linear algebra, hypothetical learning trajectory, linear transformation

Student difficulties in learning fundamental concepts in linear algebra are well documented (e.g., Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000; Harel, 1989; Hillel, 2000; Sierpinska, 2000). Modeling and quantitative reasoning form the basis for conceptual understanding of algebraic ideas (Kaput, 1998; Thompson, 1994). Symbolization of algebraic ideas relies heavily on the use of variables and functions (Arcavi, 1994), and research shows that students at the undergraduate level continue to struggle in their interpretations of variables and functions (Oehrtman, Carlson, & Thompson, 2008; Jacobs & Trigueros, 2008). We posit that this difficulty is amplified in the realm of linear algebra, where students must come to reason about and symbolize *systems* of quantitative relationships.

A result of our work that we present in this piece is an instructional sequence designed to support students in developing a quantitatively based view of matrices as transformations. Specifically, we articulate a hypothetical learning trajectory (HLT) (Gravemeijer, Bowers, & Stephan, 2003; Simon, 1995) designed to support students' development and elaboration of a transformation view of matrix multiplication. The major learning goals of this HLT are (a) interpreting a matrix as a mathematical object that transforms input vectors to output vectors, (b) interpreting matrix multiplication as the composition of linear transformations, (c) developing the imagery of an inverse as "undoing" the original transformation**,** and (d) reasoning about matrices as objects that geometrically transform a space. These learning goals include a student transition from a localized view wherein matrices are interpreted as transforming one vector at a time to a more global view of a matrix transforming an entire space.

## Theoretical Framework and Literature Review

This work draws on three instructional design heuristics of Realistic Mathematics Education (RME) as summarized by Cobb (2011). First, an instructional sequence should be based on experientially real starting points. In other words, tasks that comprise an instructional sequence should be set in a context that is sufficiently meaningful to students that they have a set of experiences through which to meaningfully engage in, interpret, and make some initial mathematical progress. Second, the task sequence should be designed to support students in making progress toward a set of mathematical learning goals associated with the instructional sequence. Third, classroom activity should be structured so as to support students in developing models-of their mathematical activity that can then be used as models-for subsequent mathematical activity. In other words, the process of students' reasoning on a task becomes reified so that the outcome of that process of reasoning can serve as a meaningful basis and starting point for students' reasoning on subsequent tasks.

In order to operationalize these RME heuristics into content-specific deliverables that are more explicitly related to instruction, a number of researchers have used the construct of an HLT (e.g., Gravemeijer, Bowers, & Stephan, 2003; Larson, Zandieh, & Rasmussen, 2008; Simon, 1995; Simon & Tzur, 2004). In a classroom setting, individual student thinking shapes class discussion and development of mathematical meaning at the collective level, while whole class discussion influences the thinking of individual students (Cobb & Yackel, 1996). Focusing on the classroom as the unit of analysis, we utilize the notion of an HLT as a construct appropriate to guide the mathematical development at the collective level. As such, we draw on the definition of HLT as a storyline about teaching and learning that occurs over an extended period of time (Larson et al., 2008). The storyline includes four interrelated aspects:

1. Learning goals about student reasoning;
2. The evolution of students' mathematical learning experience;

3. The role of the teacher; and
4. A sequence of instructional tasks in which the students will engage.

This framing highlights the multi-dimensional structure of classroom instruction. As the first and second aspects highlight, a teacher must consider the learning goals she has for her classroom, as well as envision the evolution of students' mathematical development as these goals are actualized. The third and fourth aspects of an HLT—the role of a teacher and the sequence of instructional tasks in which the students engage—speak to *how* these could be carried out within a given classroom.

**Toward Conceptualizing Matrices as Linear Transformations**

Research on the learning of linear algebra identifies three common student interpretations of a matrix times a vector: matrix acting on a vector view (MAOV), vector acting on a matrix view (VAOM), and systems views (Larson, 2011). The MAOV view is based on the idea that the matrix acts on or transforms the input vector, thus turning it into the output vector. The VAOM view is based on the idea that the vector acts on the matrix by weighting the column vectors of the matrix, whose sum results in the output vector. A systems view of matrix multiplication is typified by an effort to reinterpret matrix multiplication by thinking of it as corresponding to a system of equations. The HLT we detail offers a means by which instructors can support students in developing and extending the MAOV view of a matrix times a vector to a more global view of how a matrix transforms an entire space and how transformations can be composed.

A transformation is a broad mathematical concept that can be represented in a number of ways. For example, a matrix is one specific way in which certain types of transformations (e.g., linear transformations) can be represented. A transformation (function) $T$ from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$. A linear transformation $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a map that satisfies the following properties: (a) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for every $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$, and (b) $T(a\mathbf{v}) = aT(\mathbf{v})$ for every scalar $a$ and every $\mathbf{v}$ in $\mathbf{R}^n$. It can be shown that every transformation given in terms of matrix multiplication is a linear transformation when defining $T(\mathbf{v})=A\mathbf{v}$ for a given $n$ x $m$ matrix $A$. For instance, one may consider the transformation $T$ from $\mathbf{R}^2$ to $\mathbf{R}^2$ that rotates the plane ninety degrees counterclockwise; this transformation can be defined by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It is this conceptualization, which we refer to as conceptualizing matrices as linear transformations, that is the subject of this presentation.

## Methods

This research-based HLT grows out of a larger design research project that explores ways of building on students' current ways reasoning to help them develop more formal and conventional ways of reasoning, particularly in linear algebra. The instructional sequence described in this paper was developed and iteratively refined over the course of four semester-long classroom teaching experiments (Cobb, 2000) that took place in inquiry-oriented introductory linear algebra classes at public universities in the southwestern United States. We use the term inquiry-oriented in a dual sense, where the term inquiry refers to the activity of the students as well as the teacher (Rasmussen & Kwon, 2007). Students engage in discussions of mathematical ideas, questions, and problems with which they are unfamiliar and do not yet have ways of approaching – so that evaluating arguments and considering alternative explanations are central aspects of student activity. Teacher activity includes facilitating these discussions, an

activity which demands that the instructor constantly inquires into students' thinking. Students in these courses were generally sophomores or juniors in college, majoring in math, engineering, or computer science, and were required to have successfully completed two semesters of calculus prior to enrollment in the course.

During each classroom teaching experiment, we videotaped every class period using 3-4 video cameras that focused on both whole class discussion and small group work. We also collected student written work from each class day. As a research team, we met approximately three times a week in order to debrief after class, discuss impressions of student work and mathematical development, and plan the following class. We also used these meetings retrospectively to inform decisions regarding the following iteration of the classroom teaching experiment, as what we analyzed one semester became refined and informed the next iteration of the curriculum. One of our goals was to produce an empirically grounded instructional theory, and doing so involves a number of stages. One stage is an iterative cycle of the creation, implementation, and refinement of HLTs. Over the four years, we have refined not only our instructional tasks, but we also have deepened what we know about student thinking in linear algebra, refined the learning goals for our course, and increased our awareness of the role of the teacher. Examples presented in this paper were taken from the fourth and latest classroom teaching experiment. The HLT detailed here will, in subsequent work, be the basis for comparison to the actual learning trajectory in multiple classrooms.

## Results

The hypothetical learning trajectory developed in this report encompasses four learning goals: (a) Interpreting a matrix as a mathematical object that transform input vectors to output vectors; (b) Interpreting matrix multiplication as the composition of linear transformations; (c) Developing the idea of an inverse as "undoing" the original transformation; and (d) Coming to view matrices as objects that geometrically transform a space. We do not claim that these learning goals become actualized in a sequential manner. Rather, these four learning goals interweave and aid students in developing a robust conceptual understanding of matrices as linear transformations. For instance, one may see learning goal (a) as a local view of linear transformation, whereas learning goal (d) may be interpreted as a more global view. The global view is not meant to replace the local view; rather, it elaborates it. We want students to be able to draw on and coordinate both views, moving flexibly between them as need be.

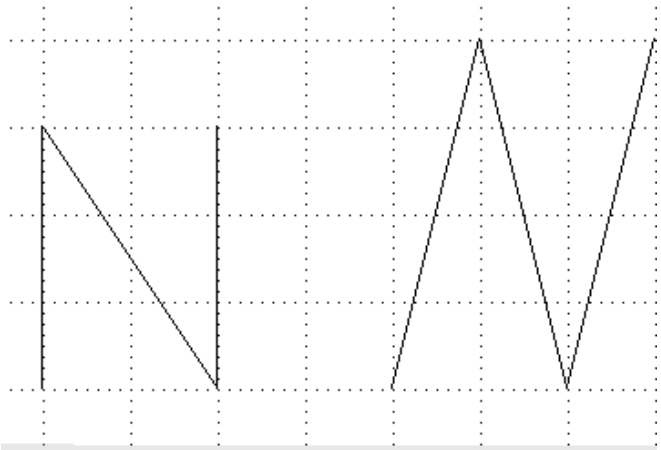
Prior to the instructional sequence driven by this HLT, the class had engaged in an RME inspired instructional sequence focused on helping students develop a conceptual understanding of linear combinations, span, and linear independence (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2011). The class also spent time developing solution techniques for linear systems to help answer questions regarding span and linear independence of sets of vectors. This led to the definition and exploration of equivalent systems, elementary row operations, matrices as an array of column vectors, augmented matrices, Gaussian elimination, row-reduced echelon form, pivots, and existence and uniqueness of solutions. This broad set of ideas was unified by developing and proving conjectures regarding how these ideas fit together for both square and non-square matrices.

Our definition of an HLT has four components, and we organize our results around the first aspect: learning goals about student reasoning. Within this structure we discuss how the role of the teacher (the third aspect) and the sequence of instructional tasks (the fourth aspect) work toward these learning goals. For the purpose of this proposal, we only elaborate on the first

learning goal and the associated aspects of the HLT. A discussion of the other three learning goals will be included in the presentation and full paper.

**Introduction to the Concept of Matrices as Transformations**

The first learning goal of this HLT is conceptualizing matrices as mathematical objects that transform input vectors to output vectors. That is, in contrast to interpreting $A\mathbf{x}=\mathbf{b}$ in terms of a vector equation or a system of equations, the goal here is to encourage conceiving of $A\mathbf{x}=\mathbf{b}$ as a matrix $A$ acting on the vector $\mathbf{x}$ to produce the vector $\mathbf{b}$. This goal involves a major interpretive shift for students, but their prior experiences working with functions serve as a good starting point for this new conceptualization of matrices. The role of the teacher in supporting this shift may include formally defining the term "transformation," discussing how $A\mathbf{x}=\mathbf{b}$ can be interpreted as an example of a transformation by defining $T(\mathbf{x}) = A\mathbf{x}$, and defining the terms "domain" and "codomain." We have found that it is important for the teacher to explicitly connect the notion of transformation to students' experiences with the notion of function by encouraging the students to draw parallels between the two contexts for the ideas of domain, range/codomain, and input/output elements. Through these introductory whole-class discussions, we conjecture that students begin to lay a foundation for thinking of input-output pairs of vectors that are related through a matrix transformation. Rather than provide further specifics of this introductory aspect, we shift out focus to the main task of this HLT, students' mathematical development through interaction with this task, and the role of the teacher.



Suppose the "N" on the left is written in regular 12-point font. Find a matrix $A$ that will transform N into the letter on the right, which is written in italics in 16-point font.

$A =$

*Figure 1.* The Italicizing N Task

**The Italicizing N Task**

The Italicizing N task (see Figure 1) is the first task in our instructional sequence, through which students embark on their initial exploration of matrices as linear transformations. The students' goal within the task is to determine a matrix $A$ that represents the requested

transformation of the "N" described within the task. Preliminary steps for students include determining that $\mathbf{R}^2$ is the domain and codomain, and determining that $A$ will be a 2 x 2 matrix; for students, neither of these is immediately intuitive or obvious. Furthermore, students must grapple with how to interpret and symbolize the representations of the "N." Examination of past student work has revealed two main strategies: using vectors in $\mathbf{R}^2$ or using points in the *x-y* plane. For example, Figure 2 contains student work that seems to notate the lines in the "N" as vectors in $\mathbf{R}^2$ with an origin that "floats" within the "N." On the other hand, Figure 3 shows student work that treats the corners of the "N" as points on the *x-y* plane with an origin anchored at the lower left vertex of each "N." The teacher plays a crucial role in setting up this task by supporting students in developing a shared interpretation of the setting and goals of the task, as well as supporting students in interpreting matrix multiplication as a transformation as they work on the task. In addition, it is the role of the teacher to decide what groups should share their work in whole class discussion, and in what order those groups should share so that ideas are purposefully sequenced. This allows the teacher to push students to make connections among various approaches and bring out key mathematical ideas. For instance, the teacher may choose to highlight the distinctions in notation shown in Figures 2 and 3, asking students to justify their choices and discuss if they both valid. The teacher, as a member of the mathematical community, is in a position to raise questions, such as why anchoring the origin would be advantageous, that the students are not necessarily in the position to make on their own. It is this interaction between the role of the teacher and students' mathematical progress on an instructional task that helps to actualize the teacher's learning goals for the classroom.

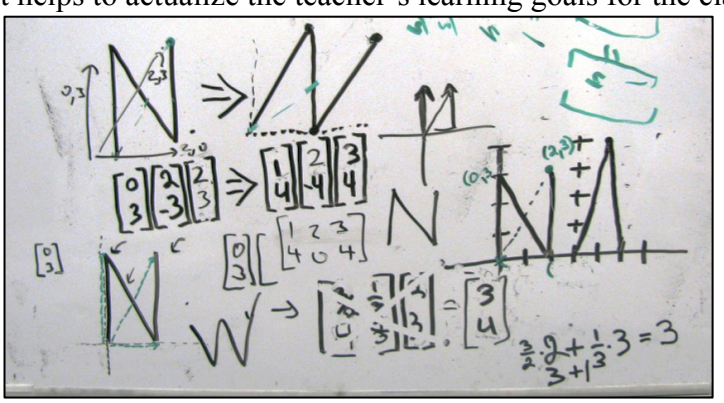


*Figure 2.* Student work that notates the lines in the "N" as vectors in $\mathbf{R}^2$

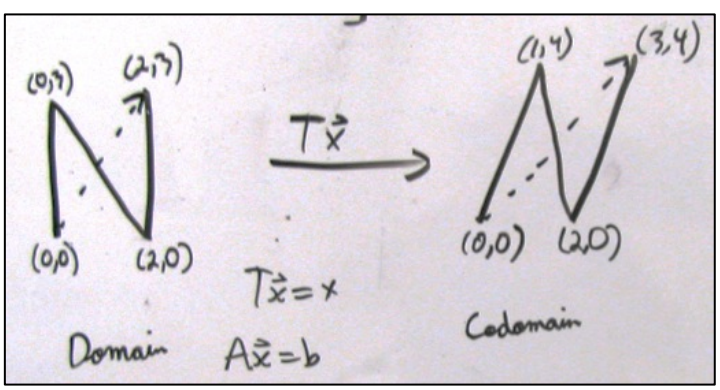



*Figure 3.* Student work that notates corners of the "N" as points on the *x-y* plane

Once the variety of approaches have been discussed and compared within whole class discussion, the students investigate how to determine what the component values within $A$ would be. A prominent approach in the past has been for students to set up two matrix equations, $A\mathbf{x}_1=\mathbf{b}_1$ and $A\mathbf{x}_2=\mathbf{b}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ come from the regular "N" and $\mathbf{b}_1$ and $\mathbf{b}_2$ from the italicized "N" (see Figure 4), such that the $\mathbf{x}_i$ are input vectors and the $\mathbf{b}_i$ are output vectors. The teacher then has the opportunity to ask students about their various choices for input and output vector pairs. This should lead to the notion that $A$ is a unique matrix representation of the transformation (according to the standard basis, which has remained implicit at this point), as well as push students to conjecture about the type and quantity of pairs that are necessary and sufficient to define $A$. The teacher may choose to allude to the notion of basis and how an informed and wise choice of basis may simplify the matrix representation for a transformation $T$. The teacher may also choose to draw attention to the type of input-output vector pairs (in this case, two linearly independent input vectors) that are necessary in order to define the matrix $A$ uniquely. Also, if no students come up with this on their own during class, the teacher has the opportunity to suggest solving $A[\mathbf{x}_1\ \mathbf{x}_2] = [\mathbf{b}_1\ \mathbf{b}_2]$ as an approach for determining $A$. This normally is the first time that the multiplication of two matrices (rather than only a matrix times a vector) appears in the course. Thus, an explicit point of conversation becomes matrix multiplication and how to interpret it within this context. This gives rise to defining matrix multiplication $AB = C$ as $A$ acting on the columns of $B$ so that $C = [A\mathbf{b}_1 \ldots A\mathbf{b}_n]$.

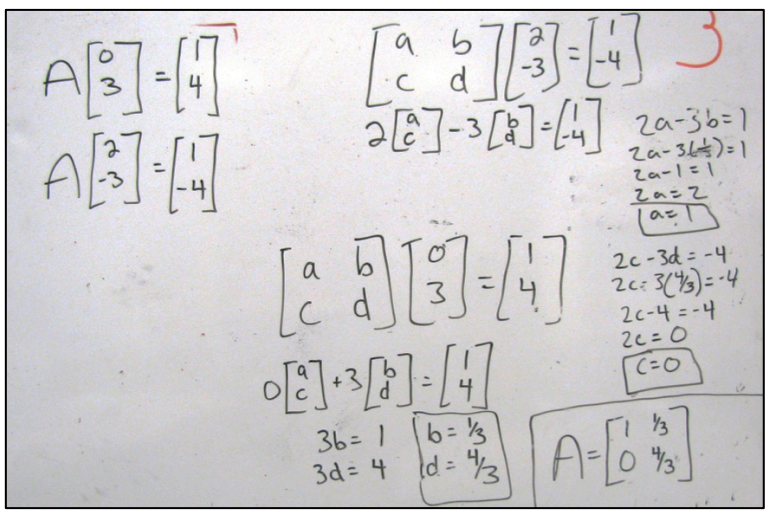



*Figure 4.* Students set up matrix equations to solve for the values of matrix $A$

**Follow-up Activities and Discussion**

The Italicizing N Task is followed up by activities that ask students to investigate other transformations of the plane, such as stretching, rotating, etc. The emphasis here begins to shift away from only considering particular input/output pairs to how the transformation defined by $A$ affects the entire plane, without needing to go through the motions of plotting particular pairs. While still working in $\mathbf{R}^2$, the teacher suggests other transformations (such as stretching photos and skewing images in Quadrant 3) to develop a connection to geometric interpretations of the standard 2 x 2 transformation matrices. This leads into and is not disjoint from the fourth learning goal of coming to view matrices as objects that geometrically transform a space.

## Conclusion

In this work we elaborated the first learning goal of our HLT, highlighting the ways in which students' activity moves them toward this goal, and the role of the teacher in supporting student learning. The teacher's introductory work of developing with the students a way to interpret a matrix as a transformation lays the groundwork for students' development of an MAOV view. Student activity of working to construct a matrix that performs a desired geometric transformation provides them with opportunities to explore what is needed to define such a transformation and how such a transformation might be symbolized with a matrix. This activity lays the foundation for coordinating the conceptualization of the local "one-vector-at-a-time" MAOV view with a more global view of envisioning how the space is changed through the "italicizing." In this way, the Italicizing N task also serves the other three learning goals of the HLT, in that it sets up the class for further investigation into matrix multiplication and how to conceptualize it in a manner consistent with function composition, inverse, and global transformation of a space. Portions of the task sequence that accompanies the other three learning goals are provided in Figures 5 and 6.

Last semester, two linear algebra students—Pat and Jamie—described their approach to the Italicizing N Task in the following way:

> "In order to find the matrix $A$, we are going to find a matrix that makes the "N" taller (from 12-point to 16-point), find a matrix that italicizes the taller "N," and the combination of those will be the desired matrix $A$."

1. Does their approach seem sensible to you? Why or who not?
2. Do you think their approach allowed them to find a matrix $A$? If so, do you think it was the same matrix $A$ we found this semester?
3. Try Pat and Jamie's approach. You should either: a) come up with a matrix $A$ by using their approach, or b) be able to explicitly explain why this approach does not work.

*Figure 5.* The Pat and Jamie Task

Regarding the Italicizing N Task, complete the following:

> Find a matrix $B$ the will transform the letter on the right back into the letter on the left.

1. Find $B$ using either your method or one of your classmate's method for finding $A$
2. Find $B$ using Pat and Jamie's method for finding $A$.

*Figure 6.* The inverse follow-up to the Italicizing N and the Pat & Jamie Tasks

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# Two-Variable Functions: Novice and Expert Shape Thinking

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*Abstract: This study describes two first-semester calculus students' understandings of functions of two variables in a teaching experiment that focused on thinking about function as the simultaneous variation of quantities. The students' actions, responses, and construction of graphs revealed that one student thought about graphs as a malleable wire and another student considered graphs as the result of tracking the values of covarying quantities, which I characterize as novice and expert shape thinking. In this talk, I outline importance of understanding students' ways of thinking about functions of more than one variable, the methodology used to collect and analyze data. I conclude by discussing the implications of thinking about graphical representation of functions using novice and expert shape thinking.*

*Keywords: Two-Variable Functions, Covariation, Quantitative Reasoning, Student Thinking.*

## Background

Numerous research studies (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994) have suggested that the concept of function is one of the most important in learning mathematics. Several studies have suggested that a coherent conception of function involves reasoning about a function relationship as one in which the output variable varies continuously with the input variable (Breidenbach, et al., 1992; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk & Nemirovsky, 1994). A perspective that complements the input-output perspective is of function being rooted in covarying quantities, which involves a sustained image of two or more quantities varying in tandem (Saldanha & Thompson, 1998). Both approaches to understanding the function idea attempt to help students represent and interpret change in a function situation, and have been shown to be foundational for understanding concepts in more advanced mathematics (Carlson, Smith, & Persson, 2003).

*Difficulties Understanding Function*

Not only is the function concept important, but also it is also difficult for students to understand. Carlson (1998) found that students receiving course grades of A in calculus possessed a process view of functions, but this process view did not support understanding the covariant, or interdependent variance of aspects of the situation. Carlson found that 43 percent of the A students found $f(x+a)$ by adding *a* on the end of the expression for *f*, without referring to $x+a$ as an input for the function. In addition, just 7 percent of A-students in a college algebra course could produce a function in which the output and input values were all equal. However, there is little research about how students' conceptions of single-variable functions influence how they think about representing functions of two or more variables.

*Motivation for Research Questions*

Most secondary, introductory algebra, pre-calculus, and first and second semester calculus courses do not require students to think about functions of more than one variable. Yet vector calculus, calculus on manifolds, linear algebra, and differential equations all rest upon the idea of functions of two (or more) variables. Several studies have investigated students' responses when presented tasks focused on two-variable functions, but did not create models of those students'

ways of thinking (Martinez-Planell & Trigueros, 2009; Trigueros & Martinez-Planell, 2007). Because past studies have not focused on modeling student thinking about two-variable functions, we cannot say how or if students use their understandings of single-variable functions to think about functions of more than one variable. Existing research has not yet addressed the extent to which students' documented difficulties with algebraic and graphical representations of function in one variable persist in a multivariable setting.

## Research Questions

As a result of the lack of knowledge about student thinking related to two-variable functions, I posed these *research questions* for investigation:

- What ways of thinking do students reveal in a teaching experiment that is focused on thinking about functions of more than one variable as covariation of quantities?
- If students' ways of thinking change, what mechanisms (e.g. instructional supports, visualization tools) facilitated these changes?
- What implications do specific ways of thinking about functions have for students as they attempt to generalize their thinking to functions of more than one variable.

## Theoretical Framework

This investigation drew from Glasersfeld's (1995) radical constructivist theory of knowing, which drew heavily from the work of Piaget (Piaget, 1971a, 1971b). Glasersfeld argued that each learner's mathematical reality is personal, and therefore unknowable by others. We as researchers cannot ever be completely confident in our understanding of a student's mathematical knowledge because it is personal. We can, however, describe ways of thinking that, were a student to have them, would account for his/her actions and statements. My use of this theoretical perspective does not ignore the many other ways by which I could characterize learning and thinking within a teaching experiment, but I believe this perspective is critical to studying changes in an individual's understanding of a concept such as two-variable functions. Because I assume that students' ways of thinking are the result of their individual interpretation of others' utterances and instructional supports, I am at best modeling the ways of thinking that drive those interpretations, but my only access to creating a model of student thinking is their actions and utterances within teaching episodes. Thus, because I am using students' language and actions to infer about their ways of thinking, I cannot claim it *is* their thinking, but can use a systematic methodology to establish the viability of those inferences to create a model of their thinking.

## Methodology

The framework described above requires that the methodology for this study establish a viable model of student thinking. Assuming that each learner's mathematical reality is independent and unknowable to others introduces a host of difficulties, the foremost being how to explain how a student is thinking if all we have are their actions and verbal cues. I used conceptual analysis of representing a function of two-variables to design the teaching experiment (Glasersfeld, 1995; Thompson, 2008), and to provide insight into possible modes of student thinking (Steffe & Thompson, 2000). Before and after each teaching episode I completed conceptual analyses of the teaching experiment participants' ways of thinking. My conceptual analyses were focused creating models of student thinking based on their actions and verbal cues from that teaching episode. The design of the next session's content took into account how to test

and refine my developing model for each student's thinking about two-variable functions. By iterating this process over the course of the entire teaching experiment, my models were continuously tested and refined as a result of events within individual teaching episodes.

Once the data was collected from the entire teaching experiment, I used an open and axial coding scheme to develop constructs that allowed me to describe and explain ways of thinking that made students' verbal and physical actions during the teaching episodes coherent. This coding scheme was continuously refined as I coded video from each teaching episode until the explanatory and descriptive constructs comprised a model that could viably explain how a student might have been thinking in that episode. Once I developed this stable model of student thinking, I evaluated the model's predictive ability using the remaining teaching episodes. By predictive, I mean how well the constructs and their interrelationships within the model of student thinking could suggest how a student would respond to a particular question about two-variable functions. As a result of this iterative process of analysis, the models of student thinking gained descriptive, explanatory, and predictive power.

## Results

The most critical constructs to emerge from the iterative process of developing models of student thinking were novice and expert shape thinking. These constructs were used to explain how a student imagined the construction of a function's graph and interpreted the given graph of a function in applied problem situations. In the subsequent excerpts I present exchanges between the two students that I believe characterize novice and expert shape thinking. In Excerpt 1, I describe Brian and Neil's way of constructing the graph of a two-variable function, and in Excerpt 2, Brian and Neil discuss how to interpret a given graph of a two-variable function in space.

Neil and Brian were asked to think about a function $h(x) = af(x)$ where $f(x) = x^3 - 2x$ (from a previous activity), where *a* was a parameter value. I suggested that they could think about *a* as a distance perpendicular to the whiteboard. For example, if *a* was -20, the function $h(x) = -20f(x)$ was set 20 units into the whiteboard. Students were to imagine that starting with a large negative value of *a*, the graph of the function was "pulled" out of the whiteboard, and while being pulled, the graph left "a tracing out" which generated a surface in space. Students were then given another function $f(x,y) = xy$ and asked to reflect on how they could use the idea of *a* to help them graph $f(x,y)$ in space. In this excerpt, I had just introduced thinking about the surface traced out by the function $h(x)$ along the *a* axis.

*Excerpt 1.*

Neil and Brian generate a function of two-variables

B: Well, this actually kind of seems like the earlier activity we did, but we are tracing out a function instead of a point, hmm. Umm, but I figured out that before I could say what it traced out, I had to know what the function looked like for a few values of *a*, then I could imagine that it would make a surface in space.

I: Brian, what do you think about what Brian said?

N: I agree with him. But I don't think this is really like the Homer activity at all, the graphs don't even look similar to each other. I think Brian and I, umm, well, we don't think about things in the same way, maybe. I can see how if we did this "tracing out" thing with the fairy dust, we would kind of get a wave, because it would be choppy looking.

I: Okay, so you have both agreed on what the tracing out would look like, but maybe not its relation to other tasks we have done. I'd like you to think about this function, [writes $f(x,y) = xy$], and think about how you might use the way we thought about *a*, to create a graph of this function.

B: Well, its actually not too hard to think of $f(x,y)$ in the same way, because you could either think of *y* as *a*. Then the function is just like the one we were dealing with before, except that $f(x)$ is *x* instead of $x^3 - 2x$. Umm, what do you think Neil?

N: I think I am following what you are saying, but I still think it is really hard to imagine what the graph of $f(x,y)$ looks like. We have sort of worked with combining shapes of graphs, so maybe $f(x,y)$ is a combination of $f(x) = x$ and $f(y) = y$, but I am not sure how to graph the second one. But if I knew what each of the graphs looked like, I might be able to combine what they look like we did, umm, before in an activity.

Brian connected this activity to the Homer task (from the first teaching episode) by the fact the graph was the result of "tracing out a function", where he had been thinking about a one-variable function's graph as the tracing out of a point representing (x, *f*(x)). Brian's response suggests he saw a relation between a graph of a function as a surface in space and a graph of a function in the plane by imagining the process by which the graph was constructed. When Brian said he could think of $f(x,y)$ in the same way as $h(x) = af(x)$, he revealed that he was thinking about *a* as a variable. Thus, I believe he thought about $h(x) = af(x)$ as $h(x,a) = af(x)$, which allowed him to easily extend this way of graphing a function to $f(x,y) = xy$. Neil did not see a connection between this activity and generating the graph of a one-variable function because the surface generated by the sweeping out of $h(x) = af(x)$ did not "look similar" to the graph in the earlier activity. Neil's basis for comparison of the functions was the shape of the generated surface, not the process by which the surface was generated. As a result, Neil's response that $f(x,y)$ is a combination of the shape of $f(x)$ and $f(y)$ was much like his responses in previous tasks, where he thought about graphs as wires that could be manipulated without reference to the axes on which the quantities' values were represented.

*Excerpt 2.*

Neil and Brian interpret the graph of a two-variable function.

I: Okay, so here is a graph of a two-variable function, what function do you think it represents?

B: Well, like we have been doing all along, I think about the function as being generated by a tracing out of a one-variable function. I would use the x-y perspective to think about this, because it looks like it was generated by sweeping out a circle by looking at level curves of *z*.

N: I'm not sure what you mean by continuous, but this function looks like it has some straight edges, but also could look like a circle like you said. I'm not sure what function is represented by a circle, and I don't know if I could find it in my graphing calculator, so I can just see it as a combination of a circle from overhead and some parabolas from the side. So maybe a sum of a circle function and a

parabola function?

B: I don't think it can just be a simple combination, you get the same graph of the function whether you sweep out from the x-y, z-x or z-y perspectives.

N: How can you get the same function? If you are combining different trends in the graph and different shapes, it won't always produce the same surface depending on what perspective you are considering.

Brian focused on the process by which the graph was constructed, which he imagined was a "sweeping out" of the relationship between two quantities' values simultaneously represented on the coordinate axes. In each activity, he paid attention to how the graph represented simultaneous values of attributes in the situation. His attention to the axes suggests that he was thinking about functions' graphs as a direct result of tracking the values of quantities on the axes. Because he was thinking about the graphs as formed by the sweeping out of a point representing two quantities' values, he was able to think about graphs of two-variable functions in the same way as functions of one variable. I refer to these ways of thinking as *expert shape thinking*.

Neil attended to the shape of the constructed graph, and tended to think about "bending" the graph to fit the physical situation it was intended to model. Thus, he did not see the similarities in the activities that Brian described because he was paying attention to how the graph "looked". By thinking about the graph as a malleable wire that could be shaped to fit the situation, Neil revealed that he was not attending to the graph as representing two quantities' values simultaneously. It is not surprising, then, that he saw the graph of a single variable function and the graph of a two-variable function as unrelated. My analogy for Neil's thinking is that he saw the graph of a single-variable function (e.g. polynomial as sum of monomial) as a rope, and using the same way of thinking, he saw the graph of a two-variable function as a wave. In both cases, he was focused on the graph as an object in the quadrants without reference to the quantities' values represented on the coordinate axes. However because the appearance of a rope and a wave are very different, I believe Neil had difficulty he had drawing connections between single and two-variable functions. I refer to these ways of thinking as *novice shape thinking*.

## Further Research and Implications

The results presented here are part of a larger data corpus consisting of three teaching experiments focused on student thinking about two-variable functions, their graphs, and two-variable rate of change. My analyses of these subsequent teaching experiments have indicated that novice and expert shape thinking are constructs with potential to have explanatory and predictive power in how students think not only about two-variable functions, but graphs and functions in general. However, these teaching experiments have also indicated that the construct of expert shape thinking needs further development. In studies in the near future, I plan to have mathematicians, math educators, and graduate students participate in structured interviews in order to gain insight into how they think about one, two and multivariable functions. In doing so, I plan to gain more insight into what ways of thinking support someone in becoming an expert shape thinker. I believe that making these distinctions not only has implications for further research, but also can contribute to classroom practice by helping the teaching community understand what ways of thinking students must have to think about functions and their graphs in sophisticated ways.

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# How and why mathematicians read proofs: A qualitative exploratory study and a quantitative confirmatory study

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*Abstract: In this paper, we report the results of a qualitative study in which we interviewed nine mathematicians about how and why they read the published proofs of their colleagues. We then tested the hypotheses generated from these analyses with a quantitative study with 112 mathematicians. Key results from these studies are (a) mathematicians do not always read a proof to obtain conviction, but usually do so to find techniques that might be useful in their own research, (b) mathematicians use authoritative evidence to gain conviction in the proofs that they read, (c) some mathematicians do not always check every line in a proof, even when they referee, and (d) the consideration of examples was crucial in gaining understanding and conviction.*

Key words: Mathematical practice; Proof; Proof comprehension; Proof reading

## Introduction

In this paper, we examine the goals that mathematicians have when reading the published mathematical proofs of a colleague and the processes that mathematicians use to achieve these goals. A goal of many research programs is to lead students to think and behave more like mathematicians with respect to proof (e.g., Harel & Sowder, 2007), with some researchers conducting teaching experiments designed to have students develop the same standards of conviction as mathematicians (e.g. Harel, 2001; Stylianides & Stylianides, 2008) while others have created learning environments designed to have students engage in proof-related activity that is similar to mathematicians' practice (e.g., Lampert, 1990).

Stylianou (2002) argued that if the goal of mathematical instruction is to have students behave like mathematicians, participate in authentic mathematical practice, and adopt the same values and beliefs as mathematicians, then it is necessary to have an accurate understanding of how mathematicians behave and what they believe. Recently, the RAND Mathematics Study Panel (2003) concluded that concluded that more research on mathematicians' practice pertaining to justification and proof is needed to form a sufficient basis to design instruction and Hanna and Barbreau (2008) argue that recent surprising findings about mathematical practice have important implications for the teaching of mathematics. With regards to reading proofs, Konoir (1993) contended that "getting to know the complex processes and mechanisms of reading text has essential significance for the didactics of mathematics" (p. 251).

While there is a large body of educational literature on mathematical proof, most of this research focuses on students' construction of proof or their beliefs about proof with several researchers noting the relative paucity on the reading of proof (e.g., Weber, 2008, 2010; Hazzan & Zazkis, 2003; Mamona-Downs & Downs, 2005; Selden & Selden, 2003). Further, the research on reading proof has primarily used proof reading as a lens to see what types of arguments students find convincing (e.g., Martin & Harel, 1999). Research on how students and mathematicians read, evaluate, and learn from deductive arguments is only beginning to emerge (e.g., Selden & Selden, 2003; Wilkenson & Wilensky, 2011). In this

paper, we examine the purposes for which mathematicians read the proofs of their colleagues and the processes they use to fulfill these processes.

**Theoretical perspective**

Our investigation is informed by two frameworks. The first is the adapted warrant typology of Inglis, Mejia-Ramos, and Simpson (2007). When an assertion claims every element of a set satisfies a given property, one may check that this assertion is true by verifying that a proper subset of the given set satisfies this property. We refer to this type of argument as *empirical*. One may increase one's confidence that a claim is correct because an authoritative source endorsed that claim. We refer to this type of argument as *authoritative*. Finally, one may produce or observe a deductive argument that derives a particular assertion. We refer to this type of justification as *deductive*.

The second framework is an extension of Boero's (1999) stages of proving. Boero argues that proof production by mathematicians usually has five stages: (i) generating a conjecture, (ii) formally stating the conjecture, (iii) exploring the conjecture, (iv) developing an informal justification, (v) logically chaining the informal justification into a proof. Boero (1999) noted that although empirical warrants (and other informal reasoning) do not appear in the products of proof production ((ii) and (v)), they play a significant role in the generation of these products ((i), (iii), and (iv)).

We contend that the mathematical work of generating a proof does not end in stage (v). After producing a proof, (vi) the mathematician submits this work for review by other mathematicians, (vii) other mathematicians evaluate the argument, and if the review is positive, (viii) other mathematicians read and learn from the argument in a published source. Our paper primarily concerns stage (vii) and especially stage (viii)—that is, how do mathematicians evaluate and comprehend the published proofs of their colleagues—and what warrant types mathematicians use to make these evaluations.

**Procedure**

*Qualitative study*. Nine mathematics professors from a large state university met with the author for an audio-recorded semi-structured interview that was one- to two-hours long. These mathematicians were all full professors and prominent in their fields of research.The goal of the interview was to investigate the reasons why and the ways in which the mathematicians read the published proofs of their colleagues. The analysis in this paper will focus primarily on the participants' responses to the following questions: (1) In your own mathematical work, I assume you sometimes read the published proofs of others. What do you hope to gain by reading these proofs? (2) What do you think it means to understand a proof? (3) What are some of the things that you do to understand proofs better? (4) Does considering specific examples ever increase your confidence that a proof is correct? All interviews were transcribed and analyzed using an open coding scheme in the style of Strauss and Glaser (1996).

*Quantitative study.* The primary results are based on the qualitative study described above. The purpose of the quantitative study is to increase our confidence on the validity of these results.

The preceding analysis yielded 14 grounded hypotheses about mathematicians' practice in reading and evaluating the proofs of their colleagues. However, because our sample size was relatively small (only nine mathematicians) and because our hypotheses were formed from excerpts from a subset of the participants, the generalizability of our results was in question.

To increase our confidence about our grounded hypotheses, we adapted Aiso Heinze's (2010) use of using large-scale surveys with mathematicians. To design the survey, for each grounded hypothesis, we formed a question of the form "When I read a proof in a respected journal, it is not uncommon that [hypothesized behavior]" and, if appropriate, "When I am refereeing a manuscript, it is not uncommon that [hypothesized behavior]". For instance, one such question was, "it is not uncommon for me to see how the steps in the proof apply to a specific example". This increases my confidence that the proof is correct. Participants were asked to respond on a five-point Likert scale whether they agreed or disagreed with each of the statements. To ensure that participants were not agreeing to *any* statement (thereby threatening the validity of our results), we also included six foils of statements that we believed were not indicative of how mathematicians behave. (e.g., "When I read a proof in a respected journal, it is not uncommon that a primary reason for doing so is to explore the writing styles of mathematicians from other countries"). The large majority of participants who read these foils disagreed with them.

We recruited 100 research-active mathematicians at prestigious universities to participate in an internet study via e-mail solicitation. We used the research methods described in Inglis and Mejia-Ramos (2009) to ensure the validity of the data.

**Results**

*Purposes of reading proofs*. All interviewed participants indicated that they read published proofs for reasons other than conviction. Some went further and suggested that they often did not read published proofs for the purposes of conviction at all. All nine interviewees claimed a central reason they read proofs was to assist them in their own research. For instance, one participant explained, "you discover these are things that you can use…that's an important part of reading proofs, that you steal good ideas out of good proofs".

The quantitative study confirmed these findings. The majority of participants agreed that they did not always check published proofs for correctness and that an important reason for reading such proofs was to generate ideas for their own research.

*Proof as cultural artifact*. In synthesizing our interview data, we argue that mathematicians view proofs as a cultural artifact that underwent a cultural history. Some (but not all) of the participants indicated that they were prepared to accept a proof as valid if it appeared in a journal since it was checked by other reviewers. For instance, one interviewee said, "Now notice what I did *not* say. I do not try and determine if a proof is correct. If it's in a journal, I assume it is".

Again, the quantitative study confirmed these findings. The majority agreed that they were highly confident that a proof was valid if it appeared in a reputable journal or was written by an authoritative source.

*Beyond line-by-line understanding.* Some interviewed participants made a distinction between studying the proof at a line-by-line level (i.e., understanding how each statement in a proof was deduced from previous statements) and understanding the proof in terms of general methods. In the words of one participant, "There are different levels of understanding. One level of understanding is knowing the logic, knowing why the proof is true. A different level of understanding is seeing the big idea in the proof". All participants valued understanding the proof beyond a line-by-line reading.

Some participants went further, stating a primary goal of reading the proof was the global understanding and they sometimes did not understand the proof at a line-by-line level. One participant stated, to understand a proof "[To understand a proof] means to understand

how each step followed from the previous one. I don't always do this, *even when I referee*. I simply don't always have time to look over all the details…. When I read the theorem… I find the big idea of the proof and see if it will work. If the big idea works, probably the rest of the details of the proof are going to work too". We were quite surprised that an eminent researcher would not check every line of a proof when refereeing!

The quantitative study confirmed most of these findings. The majority indicated that they sought to understand a proof in terms of its main methods and that if the main method was valid, they would be highly confident the proof was true without checking each of its details. 35% of participants said they did not verify every line in a proof while refereeing.

*The role of examples*. All participants commented on the importance of considering examples when reading a proof, not only for understanding the proof better but also for assessing its correctness. The quantitative study revealed that the majority of participants agreed that considering how the steps of a proof applied to a specific example both increased their understanding of the proof and their confidence that it is correct. As one participant said, examples were an essential component to making sure the ideas of the proof made sense: "I never just read a proof at an abstract level. I always use examples to make sure the theorem makes sense and the proof works... When I'm looking through a proof, I can go off-track or believe some things that are not true". Further, 40% of participants agreed that they are sometimes sufficiently convinced that a problematic assertion is valid to continue with their proof reading, even when they are refereeing a proof.

**Discussion**

In synthesizing our results, we argue a proof can be understood as a cultural artifact, at a line-by-line level, or at a global level. Further, we argue that, at least for some mathematicians, each type of understanding does not rely solely on deductive evidence. Participants were highly confident that a proof was true if it was published in a reputable journal or came from an authoritative source—meaning that they used authoritative evidence to obtain conviction. They trusted either the authority of the author of the proof or the authority of the reviewers and editor who accepted the proof as valid. For both line-by-line checking and understanding the proof at a global level, empirical evidence played a pivotal role, both in checking for correctness and gaining understanding. One surprising finding was how some mathematicians did not put in the effort (perhaps due to lack of time) to completely understanding the proof at a line-by-line level. That some mathematicians apparently do not always check every line of a proof that they referee may explain why most published theorems are true, even if many of these proofs contain logical errors (Davis, 1972; Hanna, 1990).

In terms of their influence on mathematics education, we believe these results inform the goals of instruction and interpreting the results of empirical studies. In terms of the former, some researchers believe an important goal of instruction is to have students not use authoritative evidence to gain conviction (e.g,. Ball & Bass, 2000; Harel & Sowder, 2007). However, mathematicians often accept arguments as correct because they are published in a reputable outlet or come from an authoritative source. Similarly, we expect that students will naturally trust what they read in their textbooks or hear from their teachers. We contend that students are behaving rationally and consistently with mathematicians when they do so. Further, we argue that it would be both impractical and unproductive to curb this behavior. The problem with relying solely on authoritative sources is *not* that students will believe things that are not true but that they will forfeit the opportunity to gain understanding by not seeking deductive justifications.

With regard to interpreting research studies, Fischbein (1981) published a famous study in which many high school students were shown a number theory proof and agreed that it was valid, yet still wanted to check that the theorem was true with specific examples. Some researchers argued this revealed a deficiency in students' understanding of proof since such a check should have been regarded as superfluous (e.g., Fischbein, 1981; Harel & Sowder, 1998). However, if we accept that proofs are not understood at a deductive level alone and empirical evidence can boost one's confidence in a proof, what the students did seems entirely rational and, we argue, is not inconsistent with mathematical practice. We argue that the students were not in a position to gain complete conviction from the proof since they likely were not yet mathematically sophisticated to judge the proof as correct with absolute certainty. Therefore, increasing one's confidence in the theorem by checking it with examples was a sensible thing to do.

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# ASSESSMENT OF COLLEGE STUDENTS' UNDERSTANDING OF THE EQUALS RELATION

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*Incorrect usages of the equal sign by undergraduate students indicate a tendency for students to comprehend the equal sign as an operator symbol or to ignore the equal sign altogether. This article focuses on college students' understanding of the equals sign and the equals relation, and how that understanding is influenced by the context in which the equals sign is presented. This study indicates that college students often fail to correctly interpret the equals relation and suggests two explanations for these misinterpretations: 1) students fail to recognize the extent of the sameness suggested by an equation, 2) when students focus on solving, evaluating, or coming up with "the answer" they fail to recognize the contribution of the equals sign or other indications of an equals relation in a given context.*

*Key words:* equal sign, student understanding of equations, remedial undergraduate education

## Introduction

Many secondary and post-secondary students are not able to apply basic skills and procedures in problem solving situations because they lack understanding of the structures that define and explain these skills and processes (Brown et al., 1988). Students in the United States generally believe that learning mathematics is an exercise in memorizing rules and procedures and using those rules and procedures to derive correct answers to numerical problems. This misconception about what constitutes mathematics is prohibitive to the study of algebra and other subjects dependent upon algebraic understanding. When students believe that a mathematical expression represents a string of operations that are to be performed, they encounter a conflict with implicit objectives of algebra that require a view of the expression as an object that can be manipulated. For students to learn and understand algebra they must have the ability to see a mathematical expression as a structure (Kieran, 1992; Sfard, 1991).

Research has shown that procedural emphasis on arithmetic computations dominates the elementary math curriculum in the United States (Valverde & Schmidt, 1997). Little or no attention to structural understanding can lead to misconceptions about the fundamental structure of arithmetic and impede a students' ability to understand algebraic concepts (Baroudi, 2006; McNeil & Alibali, 2005A). Because of the arithmetic dominated curriculum of most elementary schools, the notion of "equals" and the meaning of the equal sign are misunderstood by most elementary students. Most elementary students do not comprehend that the equal sign is an indication that an equality relation exists between two structures. Instead, they perceive the equal sign as an indication that a particular procedure is to be performed (Knuth, Stephens, McNeil, & Alibali, 2006; Molina & Ambrose, 2006; McNeil & Alibali, 2005B; Falkner, Levi, & Carpenter, 1999; Behr, Erlwanger, & Nichols, 1980).

The problems that misconception of the equals relation pose for learning mathematical notions cannot be overstated. In algebra, students are introduced to properties and structures of arithmetic operations primarily through use of equations. Equations are used in algebra to indicate which mathematical objects, written in different forms, are the same. If students

misinterpret the intent and meaning of the equals sign, then an equation has no value as a means of helping students learn the relation proposed by the equation.

As students mature and their exposure to the equal sign in multiple contexts increases, most obtain the ability to interpret the equal sign as an indicator of an equivalence relation. This is evidenced by the fact that most high school students are able to accept equality statements containing multiple operations on each side (Herscovics & Kieran, 1980; McNeil & Alibali, 2005B). It is not clear, however, whether this ability to interpret the equals sign in terms of an equivalence relation develops into an understanding of equivalent equations in algebra or calculus (Kieran, 1981).

Incorrect usage of the equal sign by post-secondary students as they solve equations or calculate derivatives indicates a tendency to regress back to a comprehension of the equal sign as an operator symbol. It is possible that such misuses of the equal sign are simply careless mistakes made by students because they lack knowledge of an appropriate notation. Is it appropriate, however, to assume that college students correctly interpret the equals sign in all contexts? Should we assume that because a college student has completed many years of math, including an algebra curriculum, that their understanding of the equals relationship is sufficient to allow them to succeed in their continued pursuit of mathematical learning?

The studies that have been done thus far have focused on student understanding of the equals relation in very specific contexts. The instruments used in these studies presented participants with various equations and then asked the participants to interpret the meaning of the equal sign in those equations. The responses were then classified as appropriate if the participants gave a relational interpretation and inappropriate if the interpretation was procedural. There were no studies found that measured understanding of the equals relation beyond interpretation of the equal sign. In this paper, the author reports the results from a study that examined college students' understanding of the equals sign and the equals relation, and how that understanding is influenced by the context in which the equals sign is presented.

## Student Understanding of Equals Relation

Rittle-Johnson and Alibali (1999) conducted a study with fourth and fifth grade students to assess the causal relations between children's structural and procedural knowledge of equivalence. The children were asked to solve standard equivalence problems of the form $a+b+c=a+$___ to assess their ability to solve such problems prior to receiving specific instruction related solving such problems. After the students received the instruction—in the form of procedural or conceptual instruction—the students were asked to solve standard equivalence problems that differed from the pretest problems. The differences reflected changes made in the operation used in the equation or the position of the blank in the equation. Along with solving equivalence problems, they were also asked to evaluate and rate three correct and three incorrect proposed procedures for solving such problems. This study suggests that most children of this age group understand what it means for quantities to be equal, but there is still an incomplete understanding of the equals relation and the structure of equations.

Even when students encounter the equals relation in multiple contexts as they progress through their formal education, they do not fully grasp the complete, relational, meaning of the equal sign (Rittle-Johnson & Alibali, 1999). The lack of an appropriate relational understanding of the equal sign can become a handicap to students as they transition from arithmetic to algebra. A study by Knuth, Stephens, McNeil, & Alibali (2006) found that almost half (141 out of 300) of the sixth, seventh, and eighth grade students proposed an operational definition for the meaning of the equal sign in an algebraic expression. Much fewer than half of those same students (106

out of 300) proposed an appropriate relational definition. The study also showed that students who did have an appropriate understanding of the equal sign were more likely to utilize an appropriate strategy to solve a basic linear equation.

As students progress through middle school and high school they begin to encounter the equality relation in non-arithmetic contexts. These contexts include algebraic equations and scientific relations that require a relational interpretation of the equal sign. These encounters often contradict understanding of the equals sign and force students to change their interpretation of equals (Kieran, 1981). McNeil and Alibali (2005A) found that most junior high school students, when exposed to an equal sign in a "typical addition context", held onto an operational interpretation of the symbol (p. 290-291). When the context changed, where a relational interpretation of the equal sign was required, the students were able to interpret the equal sign appropriately. The study also found that college students who had completed at least one semester of calculus were able to give a relational interpretation of the equal sign in all three of the contexts that were studied. The authors concluded that with increased exposure to the equal sign in contexts that require a relational interpretation, students eventually supplant the operational interpretation of equals with an appropriate relational interpretation. This study, however, only assessed student's interpretation of the equal sign in the three basic contexts and did not provide evidence that these same students correctly interpreted the equals relation in all contexts they may be exposed to within even a basic algebra course.

There is also evidence that high school and college students misunderstand the equals relation as they solve equations or evaluate expressions in algebra and calculus (Byers & Herscovics, 1977; Clement, 1982). Students erroneously use the equal sign as they write out procedures to solve story problems or calculate the derivative of a function. Even after students have acquired a basic relational interpretation of the equal sign, there are still tendencies to view the equal sign as an operator symbol that indicates the result of an operation (Kieran, 1981).

Even with some debate concerning the cognitive ability for children at different ages to obtain a relational interpretation of the equal sign, these studies all suggest that there remains a tendency for students of all ages to hold to an operator interpretation within at least some contexts. McNeil and Alibali (2005A) contend that this is due in large part to early mathematical experiences dominated by an emphasis on arithmetic operations. Their study showed a negative correlation between adherence to operational patterns prevalent in arithmetic and their ability to learn procedures for solving algebraic equations. The study also tested a group of college students who were randomly selected to receive a computer mediated stimulus that activated their knowledge of arithmetic operational patterns. The study found that students who received the stimulus were less likely to utilize appropriate strategies when solving a set of equations than the students who did not receive the stimulus.

To summarize, research suggests that the equals relation is difficult for students to understand. Many studies have demonstrated the inability of elementary students to correctly interpret the equal sign as an indication of an equivalence relation. Studies do indicate that as students get older they begin to interpret the equal sign correctly within certain contexts, but the contexts that have been studied are far fewer than those experienced by even a beginning algebra student. Research provides little information regarding college student's interpretation of the equal sign and their understanding of the equals relation in contexts of algebraic identities, graphs of equations, function notation, set theory, and the difference between equals and equivalent. The goal of this paper is to present evidence concerning college students who have little experience in college level mathematics and their understanding of the equals relation—to

show that such students' understanding of the equals relation is dependent on the context in which the relation is presented.

## Method

All participants in the study were selected from a single land grant university. Every student from this institution that was enrolled in an intermediate algebra class during Spring 2010 semester had the opportunity to volunteer as a participant in this study. Each student was randomly placed into one of two groups. One group of students, hereafter referred to as group A, would take an online form of the primary instrument. Each of the items included on the primary instrument are shown in Appendix A. A total of 242 out of 696 students from group A volunteered to participate, and there were a total of 222 students from group A who completed the instrument. The other group of students, hereafter referred to as group B, would take an online form of an alternative instrument. A total of 204 out of 667 students from group B volunteered to participate, and there were a total of 191 students from group B who completed the alternative quiz.

Another group of students who were enrolled in a mathematics course at the same university were selected to participate in a think-out-loud interview session with the researcher. These four students were selected to form a representation of different mathematical experiences and abilities. One student had just completed an introductory statistics course and rated herself as a poor mathematics student. Another student was enrolled in an intermediate algebra course and reported doing B grade-level work in the course. The third student was enrolled in a calculus course reported doing C grade-level work at the time of the interview. The fourth student was enrolled in the same calculus course and reported doing A grade-level work.

The students from group A, and B participated by responding to the prompts on the instruments as they were presented one at a time in an online format. The format did allow for participants to go backward and forward and revisit any of the items before they submitted the quiz. Before deciding to participate, the group was informed that they would be taking an online quiz that was part of a research project. They were instructed that the quiz consisted of items aimed at measuring student understanding of basic algebra concepts. They did not know before participating that the quiz was an attempt to measure their understanding of the equals relation.

After the students had completed the quizzes, the researcher scored each of the instruments. The data that was collected consisted of a total score for each student from group A as well as a score for each item for each student from group A. Only the score for items #4 on the alternative instrument was scored for students from group B.

The four students who participated in the think-out-loud interviews were presented with the prompts from the primary instrument. All items were presented one at a time and participants were instructed to respond to the prompts and to think out loud as they responded. After the participants responded to all of the prompts, the researcher pointed out any mistakes, identified appropriate responses, and requested further insight as to why participants responded the way they did.

## Results

The first three items on the primary instrument were included to serve two purposes: 1) to determine if students are able to interpret equals as a relation about a mathematical structure in a context where a value must be determined in order to satisfy the equals relation, and 2) to encourage students to recognize equal signs as expressions of relations rather than as prompts to evaluate expressions in preparation to responding to item #4. These items required students to consider the equals relation expressed in the equation in order to determine an unknown value.

The first three items on the alternative instrument were simple equations where students had only to calculate binary operators from left to right in order to satisfy the equations. Students could respond to these prompts without recognizing the equals relations designated in the equations.

Item #4 was included on both the primary and secondary instrument. Students were asked to fill in the blanks of an equation that contained many operators and equal signs. The purpose of this item is determine if students would recognize the equals relation designated by the equation or ignore the equals relation and erroneously compute operators from left to right. After responding to three simple prompts that encouraged students to consider equals relations in equations, 121 out of 222 (54.5%) students from group A correctly responded to the prompt in item #4. After responding to three simple prompts that only required left-to right-calculations, only 14 out of 191 (7.3%) students from group B responded correctly to the prompt in item #4. A t-test comparing these proportions was performed and showed a significant difference in these proportions ($p=0.00$).

The proportion of correct responses to individual items from group A are included along with the actual items and shown in Appendix A. The scores on the individual items on the primary instrument obtained from group A suggest that the vast majority of students in this group were able to correctly interpret the equals relation in contexts that should be familiar to any student with a minimum amount of experience in algebra courses. However, for those items where the context for which the equals relation is expressed was not as likely familiar to the student, the proportion of correct responses dropped significantly. Also, the items that presented an equation can that could be interpreted relationally or procedurally, the students in the group were more likely to give a procedural interpretation.

## Conclusions

In field tests and in the think-out-loud session, the vast majority of students interviewed did understand that equals is an indication that two representations are the same structure. But they often failed to implement this understanding when confronted with equations in different contexts. This study illuminates two significant reasons for this finding. First, students fail to recognize the extent of the sameness suggested by an equation. Second, when students focus on solving, evaluating, or coming up with “the answer” they fail to recognize the contribution of the equals sign or other indications of the equals relation in a given context. This could be a conditioned response from their previous experience with situations featuring math problems.

These findings are consistent with the theory offered by Sfard (1991), and suggest that students do not instinctively offer a structural understanding of equations. While most remedial-level students have the ability to transition from a procedural understanding of equations to a structural understanding, they do not bring that knowledge to bear without prompting or encouragement. These findings indicate that student mistakes on prompts involving the equals relation are often a result of the students’ failure to pay sufficient attention to the equality designation, especially in specific contexts. Analysis of both quantitative and qualitative data obtained in this study suggests that when remedial-level students are confronted with an equation, they proceed according to what they think they are “supposed to do,” and the equals sign or other equals designations do little to discourage their response patterns. This indicates students’ misinterpretation is rooted in their conditioning that the purpose of math is to evaluate expressions; and has less to do with misunderstanding the equals relation itself.

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## Appendix A
*Items Used on the Primary Instrument*

Item #1: Fill in the blank so that the equation below is true.
12 + _____ = 13
Scoring Rubric: +1 for writing some form of the number 1 in the blank.
Total Correct Responses: 222 out of 222 (100%)

Item #2: Fill in the blank so that the equation below is true.
8 = ____ - 5
Scoring Rubric: +1 for writing some form of the number 13 in the blank.
Total Correct Responses: 214 out of 222 (96.4%)

Item #3: Fill in the blank so that the equation below is true.
8 + 4 = ____ + 2
Scoring Rubric: +1 for writing some form of the number 10 in the blank.
Total Correct Responses: 212 out of 222 (95.5%)

Item #4: Fill in the blanks so that the equation below is true.
3 + 7 = ____ + 2 = ____ - 2 = ____ + 1 = ____
Scoring Rubric: +1 for 8, 12, 9, 10.
Total Correct Responses: 121 out of 222 (54.5%)

Item #5: Is the equation below true or is the equation below false?
$5 \times 2 = 4 \times 2 + 2$
Scoring Rubric: +1 for True
Total Correct Responses: 216 out of 222 (97.3%)

Item #6: Is the equation below true or is the equation below false?
$8 - 2 = 6 - 3$
Scoring Rubric: +1 for False
Total Correct Responses: 222 out of 222 (100%)

Item #7: Is the equation below true or is the equation below false?
$7^{10} = (2+5)^{10}$
Scoring Rubric: +1 for True
Total Correct Responses: 184 out of 222 (82.9%)

Item #8: Is the equation below true or is the equation below false?
*4 + 3 = 14 ÷ 2*
Scoring Rubric: +1 for true.
Total Correct Responses: 221 out of 222 (99.5%)

Item #9: Which of the following <u>best</u> describes the meaning of the equation
$15 \div 3 \;=\; 5$ ?

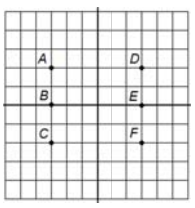
a) Given the expression 15 divided by 3, the solution to the equation is the number 5.
b) (15 ÷ 3) and the number 5 are the same number.
c) When the number 15 is divided by the number 3 then the result is the number 5.
Scoring Rubric: +1 for B
Total Correct Responses: 19 out of 222 (8.6%)

Item #10: The distributive property states: $a\,(b + c) = ab + ac$.
Which of the following statements <u>best</u> describes the meaning of the distributive property?

a) When solving a problem related to the expression $a\,(b + c)$, the correct solution is $ab + ac$.
b) The value of the expression $a\,(b + c)$ is calculated by adding the product $ab$ to the product $ac$.
c) $a\,(b+c)$ and $ab + ac$ are the same mathematical entity.

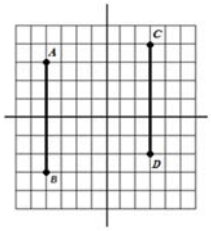
Scoring Rubric: +1 for C
Total Correct Responses: 119 out of 222 (53.6%)

Item #11: Is the statement below true or is the statement below false?
The set of points {A, B, C} shown below is equal to the set of points {D, E, F}.




Scoring Rubric: +1 for False
Total Correct Responses: 166 out of 222 (74.8%)

Item #12: *A line segment $\overline{AB}$ is defined to be the set of points on a line that include A and B and all points between A and B.*
Look at the diagram below and then choose the statement below the diagram that best describes the relationship between the line segment $\overline{AB}$ and the line segment $\overline{CD}$.




a) The line segments are equal because they have the same length.
b) You can't determine the lengths of the line segments so you can't determine if they are equal.
c) The line segments are not equal because they are not the same line segment.
Scoring Rubric: +1 for C.
Total Correct Responses: 86 out of 222 (38.7%)

Item #13: Let $A$ be the number of apples in a basket and let $P$ be the number of peaches in the same basket. If there are 10 apples in the basket and if $P + 4 = A$ then how many peaches are in the basket?
Scoring Rubric: +1 for 6
Total Correct Responses: 208 out of 222 (93.7%)

Item #14: Which of the following is suggested by the equation $x + 10 = 10x$ ?
a) The left side is larger than the right side.
b) The right side is larger than the left side.
c) They are the same.
d) You cannot determine which side is larger unless you know what $x$ is.
Scoring Rubric: +1 for C
Total Correct Responses: 77 out of 222 (34.7%)

Item #15: Let B be the number of boys at Joe's party and let G be the number of girls at Joe's party. If there are twice as many boys at Joe's party than there are girls at Joe's party, then which of the equations below describe the relationship between B and G.
a) 2B=G
b) B=2G
Scoring Rubric: +1 for C
Total Correct Responses: 95 out of 222 (42.8%)

# The Evolution of Classroom Mathematical Practices in a Mathematics Content Course for Prospective Elementary Teachers

## Contributed Research Report

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Abstract

We report on the classroom mathematical practices that developed in a mathematics content course for prospective elementary teachers. The course focused on number and operations and was intended to promote number sense development. Instruction was guided by a local instruction theory for number sense development, which we have described previously. The present report focuses on the classroom mathematical practices that emerged in the class involved in a recent teaching experiment. The actual learning trajectories identified inform elaboration and refinement of the local instruction theory and shed light on prospective teachers' number sense development.

Keywords: Classroom mathematical practices, local instruction theory, number sense, prospective teachers

We report on results of an analysis of collective activity (Rasmussen & Stephan, 2008) in a mathematics content course for prospective elementary teachers. Our previous research showed that students involved in an earlier teaching experiment developed improved number sense, particularly in the form of flexible mental computation (Whitacre, 2007). The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory (Nickerson & Whitacre, 2010). The present study concerns a recent iteration of the classroom teaching experiment, in which the local instruction theory guided instructional planning. In this report, we shift focus from instructional design to empirical analysis. We describe actual learning trajectories for prospective elementary teachers' number sense development in terms of a chronology of classroom mathematical practices.

### Theoretical Perspective

We view learning as a inherently situated process (Cobb & Bowers, 1999). This perspective is consistent with sociocultural theory, which is concerned with understanding "how mental action is situated in cultural, historical, and institutional settings" (Wertsch, 1991, p. 15). Learning occurs in the doing of activities within a culture. The nature of those activities and the culture in which they are situated profoundly shape what is learned. Knowledge becomes meaningful and useful in the practice of *authentic activities*, which "are most simply defined as the ordinary practices of the culture" (Brown, Collins, and Duguid, 1989, p. 34).

Greeno's (1991) environment metaphor informs our conceptualization of number sense from a situated perspective. Greeno characterized number sense as situated knowing in a conceptual domain – the domain of numbers and quantities. From this perspective, a person's knowledge and activities are seen metaphorically as situated within a physical environment. Knowing in an environment consists of knowing how to get around, where to find things, and

how to use them. In various conceptual domains, knowing one's way around requires relating concepts and solving problems. Greeno's metaphor relates mathematical properties, such as the distributive property of multiplication over addition, to features of a physical environment. The strategies that an individual uses, then, are ways of making use of those features in order to accomplish one's goals (Greeno, 1991).

## Local Instruction Theory

We are involved in an ongoing design research effort (Cobb & Bowers, 1999), which focuses on prospective elementary teachers' number sense development. This research takes the form of both classroom teaching experiments and theory building, and these are reflexively related. We have developed a local instruction theory for number sense development, which continues to evolve as our research progresses. A *local instruction theory* (LIT) refers to "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). In a recent publication, we have described in some detail our LIT for number sense development (Nickerson & Whitacre, 2010). Here, we briefly list the three major goals around which this LIT is organized: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models. In the proposed session, we focus primarily on the first of these goals.

In broad strokes, we have described the instructional activities and envisioned learning route toward Goal 1 in the form of Table 1. The table describes the route to Goal 1, in relation to instructional activities, chronologically, proceeding from top to bottom, left to right. (Numbers appear here to highlight that ordering.)

*Table 1. Route to Goal 1.* (Table adapted from Nickerson & Whitacre, 2010)

| **Instructional Activities** | **Envisioned Learning Route** |
|---|---|
| 1. Instructor identifies and engineers opportunities for computational reasoning | |
| 2. Students are invited to use computational reasoning and to reason quantitatively | 3. Many students initially rely on standard algorithms |
| 4. Students are invited to carry sense making to solutions with nonstandard strategies | 5. Students use their own nonstandard strategies |
| 6. Students solve problems mentally in a variety of contexts | 7. Students capitalize on opportunities to use number-sensible strategies |

The description that Table 1 offers is at a particular grain size and focuses on a particular strand of activity and learning: students' use of mental computation strategies. We have found this grain size and focus useful in thinking broadly about a progression that we seek to facilitate during the course. At the same time, this grain size and focus leave other important aspects of the instructional activities and envisioned learning route implicit. In particular, significant activity and learning must occur between cells 3 and 5 and between cells 5 and 7. The manner in which we have described our LIT previously has foregrounded certain aspects, thus leaving others in the background. Our purpose in this paper is to describe *specific* instructional activites and an *actual* learning route that occurred during a recent teaching experiment. We do this through an analysis of collective activity (Rasmussen & Stephan, 2008).

## Methods

We state our research questions as follows: In a mathematics content course for prospective elementary teachers, which is guided by a local instruction theory for the development of numbers sense,

1. What classroom mathematical practices emerge and become established?
2. How do the findings concerning the evolution classroom mathematical practices relate back to the local instruction theory?

In this brief proposal, we focus on the first of the two research questions.

Data collection took place during Fall Semeser 2010 in a mathematics content course taught at a large, urban university in the southwestern United States. There were 39 students enrolled in the course, and 38 of the students were female. The majority of the students were freshmen Liberal Studies majors. The instructor of the course was a mathematics educator and experienced teacher of mathematics courses for prospective teachers.

The data corpus for the study reported here consisted of videotapes of Days 3 through 12 of the semester-long course. The class met twice weekly, so that this analysis concerns five weeks of instruction. Three cameras were used to record classroom activity. A stationary camera built in to the classroom followed the instructor and recorded whole-class discussions. Another stationary camera, situated within the classroom, focused on students both during group activity and whole-class discussions. A mobile camera was used to survey students' work during group activities and to supplement the other recordings of whole-class discussions by viewing them from different angles and zooming in closely on participants in whole-class discussion and on their written work.

We employed the methodology of Rasmussen and Stephan (2008) for the analysis of collective activity. This is a three-phase process: (1) Whole-class discussions are transcribed. The researchers watch video of each discussion and identify the claims that are made. Then argumentation schemes are constructed for each argument, using Toulmin's model (1969). This analysis yields a chronological argumentation log. (2) Reseachers look across the argumentation log to identify ideas that *functioned as if shared* in whole-class discussion. Criteria for ideas functioning as if shared are (i) warrants or backings dropping off, (ii) an element of an argument shifting roles (e.g., from claim to warrant), and (iii) repeated use of data or warrants in support of different claims (Cole, Becker, Towns, Sweeney, Wawro, & Ramussen, 2011). (3) The ideas that functioned as if shared are then organized according to related mathematical activities to describe classroom mathematical practices. Rasmussen and Stephan (2008) define a classroom mathematical practice (CMP) as a "collection of as-if shared ideas that are integral to the development of a more general mathematical activity" (p. 201). This definition differs from that of Cobb and Yackel (1996) in that a CMP is defined in terms of a set of mathematical ideas, rather than a single idea.

## Results

We focus on the CMPs that developed around place value and whole-number addition and subtraction. We present these chronologically and attempt thereby to tell the story of the actual learning route that was traversed by the class.

The succession of CMPs around place value, addition, and subtraction was as follows:

*$CMP_0$: Appealing to the authority of the standard algorithms*

*$CMP_1$: People acting as place values*

*$CMP_2$: Meaningfully operating with place-value numeration systems*

*$CMP_3$: Meaningfully adding and subtracting with regrouping*

*$CMP_4$: Mentally adding and subtracting using aggregation strategies*
*$CMP_5$: Mentally adding and subtracting using compensation strategies*
*$CMP_6$: Reasoning about differences in terms of distance between*

We briefly describe the collective activity that characterized each CMP.

*$CMP_0$. Appealing to the authority of the standard algorithms*

In discussions of mental calculative work in the first few days of the course, the class behaved as if the authority of the standard algorithms was assumed. Mental computations using the mental analogues of the standard algorithms went unquestioned, whereas nonstandard strategies required mathematical justification. Early written records of mental computations were numeric-algorithmic in nature, with digits arranged in rows and columns, even when nonstandard strategies were used. Gesturing associated with the articulation of these strategies involved finger tracing up and down columns, essentially reenacting written work.

This practice is labeled $CMP_0$ to highlight the fact that this was essentially the starting place for the class. Prospective elementary teachers are familiar with the standard arithmetic algorithms. They tend to be able to use these correctly but be unable to justify why the algorithms work or to solve problems by alternative methods (Ball, 1990).

*$CMP_1$: People acting as place values*

During the second instructional unit, which focused on place value, students engaged in activities involving various bases. Students worked implicitly in base eight when solving problems about an apple farm. Later, multi-link cubes were used for counting and operating explicitly in base three. In the course of these activities, $CMP_1$ became established. This refers to a set of ideas and activities involving counting and representing numbers in a given base *b*: Ones are counted until a group of *b* is formed and passed on to the place to the left, and so on. $CMP_1$ includes the activities of physically counting, grouping, and passing the multi-link cubes, as well as creating drawings to represent numbers in terms of little cubes, longs, flats, and big cubes.

*$CMP_2$. Meaningfully operating with place-value numeration systems*

Beyond physically grouping objects or creating drawings of those objects, the class came to use place-value numeration systems to record numbers as numerals in various bases. $CMP_2$ includes the following set of ideas and activities: Numerals are formed using digits, and the meaning of each digit in a numeral is determined by its place value. Related to this use of notation, the class also unpacked numerals explicitly as linear combinations of powers of *b*. During the emergence of $CMP_2$, students used place-value notation informally, as in the apple farm context. As students came to work explicitly in various bases, formal place-value notation became established.

*$CMP_3$. Meaningfully adding and subtracting with regrouping*

Building on $CMP_1$ and $CMP_2$, $CMP_3$ combined place-values ideas with the operations of addition and subtraction: Addition came to involve an aggregating and regrouping process, grounded in counting in the given base. The addends and sum were recorded as numerals in the given base. Regrouping moves were notated by writing a *1* above the digit in the next place to the left. Likewise with subtraction, "borrowing" took on the meaning of unpacking a group of size *b*. The minuend, subtrahend, and difference were recorded as numerals in the given base. Regrouping moves were notated in one of two ways: (1) by writing a *1* to the left of the digit of the minuend in the place that received the extra items, or (2) by writing *10* above that digit.

In the route from $CMP_0$ to $CMP_3$, the class moved from appealing to the authority of the standard algorithms to meaningfully relating these to place-value relationships and regrouping.

Moving forward, nonstandard mental computation strategies, independent of the standard algorithms, came to function as if shared.

*$CMP_4$. Mentally adding and subtracting using aggregation strategies*

Aggregation strategies (Heirdsfield and Cooper, 2004) were established on the basis of backings that came to function as if shared, and the class gave names to these nonstandard strategies. Addition aggregation was named "Borrow to Build," and subtraction aggregation was named "Separate-Subtract-Subtract." Addition and subtraction activities during the place-value unit had reflected aggregation meanings but followed the conventions of the standard algorithms (operating one place at a time, from left to right). Now, students moved beyond Standard/Transition strategies by adding or subtracting in convenient chunks of their choosing, rather than in a manner prescribed by convention.

*$CMP_5$. Mentally adding and subtracting using compensation strategies*

Following aggregation, compensation strategies became established ways of computing sums and differences mentally. Justifications for addition compensation were "undoing" or inverse-operations arguments: adding (or rounding up) must be undone by subtracting, and subtracting (or rounding down) must be undone by adding. In terms of subtrahend compensation, students argued that increasing or decreasing the subtrahend had the opposite effect on the difference. The common backing was "taking away too much" in cases when the subtrahend had been rounded up. The empty number line was the notational form associated with the establishment of $CMP_5$, and class members gestured in ways that related to this representation.

*$CMP_6$. Reasoning about differences in terms of distance between*

Once the full-fledged empty number line had come into use, reasoning about differences in terms of distance between quickly became established. Differences were represented as distances between number-locations on the number line, and the validity of "Shifting the Difference" (e.g., 364 – 79 = 385 – 100 = 285) was established on the basis of maintaining the distance between these number-locations. Shifting the Difference, in turn, was used to justify the equal additions algorithm. Empty-number-line inscriptions were integral to this classroom math practice, as was gesturing that illustrated distances spanned and shifted.

In the route from $CMP_3$ to $CMP_6$, a variety of nonstandard mental computation strategies became established ways of computing sums and differences. Once students had made sense of the standard algorithms, they moved beyond them, using increasingly sophisticated addition and subtraction strategies. The ways of reasoning that characterized $CMP_4$ through $CMP_6$ are unusual for prospective elementary teachers, and these strategies are regarded as indicative of good number sense (Markovits & Sowder, 1994; Yang, 2007; Yang, Reys, & Reys, 2009).

## Discussion

In the earlier description of our LIT for number sense development (Nickerson & Whitacre, 2010), we focused on students' activities of naming strategies and using tools like the empty number line. The previous emphasis reflected a focus on the instructional design aspects of our work. In service of Goal 1, we sought to move prospective elementary teachers from dependence on the standard algorithms to capitalizing on opportunities to use number-sensible strategies. However, we had not documented such a learning route. The results presented here reflect a focus on the instructional sequence *as enacted* and highlight the development of students' ways of reasoning. The progression of classroom math practices that our analysis revealed complements the previous description of the LIT. It enables us to relate the envisioned learning route described in the LIT to an actual learning route, with students' activities providing the bridge between these.

## Implications

The motivation for our research program stems from the troubling reality that prospective elementary teachers in the United States and elsewhere tend to be poorly prepared to teach mathematics effectively (Ball, 1990; Ma, 1999; Newton, 2008; Tsao, 2005; Yang, Reys, & Reys, 2009). In mathematics content courses like the one that we studied, mathematics educators have the opportunity to facilitate prospective teachers' number sense development and thus help them become better prepared to foster children's learning of mathematics. For this reason, analyses that illuminate processes by which prospective teachers develop improved number sense are valuable to the field.

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# An Analysis of the Effect of the Online Environment on STEM Student Success

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## Abstract

Both online and STEM courses have been shown to have lower student retention; however, there is little research indicating what effect the online environment may have on retention in STEM courses specifically. This study compares retention rates for online and face-to-face STEM and non-STEM courses to determine if the online environment affects STEM courses differently than non-STEM courses. In addition, different subcategories of STEM courses are compared to see if the effects of the online environment are different for different course subtypes. Each online course is matched with the same course taught face-to-face by the same instructor in the same semester to control for possible confounding effects.

This study found that retention rates in STEM courses were more negatively impacted by the online environment than in non-STEM courses. In particular, the course types which had significantly lower retention online were lower level STEM courses taken as electives or distributional requirements.

## Keywords

Online learning, STEM, retention, observational study

## Purpose of the Study

A report from the Sloan Foundation affirms that online learning is growing substantially, far exceeding the growth of higher education in general (Allen & Seaman, 2010); millions of students nationally are enrolled in online courses, particularly at community colleges. However, retention of students in distance education courses is often 10-20% lower than campus-based courses (Carr, 2000; Morris & Finnegan, 2008; Hachey, Wladis & Conway, In Press).

At the same time that online enrollments are exploding, there exists a concern that fewer students are succeeding in STEM disciplines (Anderson & Kim, 2006). However, there seems to be little research specifically connecting online retention to course discipline, particularly for STEM courses. The focus of this study is to determine to what extent the online environment affects retention in STEM courses when compared to comparable courses given face-to-face, and more specifically, to see what types of STEM courses are most vulnerable to higher attrition online. The results of this study may aid in identifying needed support for students in specific online courses, enabling a more focused use of resources.

## Literature Review

Community colleges host almost half of all E-learning programs in the U.S., and have the highest enrollment rates of all higher education institutions offering online courses (Parsad & Lewis, 2008; Ruth, Sammons, & Poulin, 2007). At the same time as online learning is becoming a core method of instruction at community colleges, community colleges are playing a critical role in STEM fields; up to 40% of bachelor's and master's degree recipients in science, engineering and health initiate their studies at a community college (Mooney & Foley, 2011). Despite this confluence, little data is available on the number of STEM courses offered online, particularly at community colleges. A Sloan Foundation study found that the proportion of institutions offering a fully online program in a STEM field ranged from 17% to 33% (Allen & Seaman, 2008). For example, at the community college in this study, as much as one quarter to one third of the courses offered online each semester is within STEM disciplines[1].

Some research has found differences in online course design based on discipline (Smith, Heindel & Torres-Ayala, 2008). Additionally, research has revealed lower retention rates in mathematics-related versus non mathematics-related online courses (Smith & Ferguson, 2005). Further, Finnegan, Morris & Lee (2008-9) report differences in student engagement behaviors which affected retention in general education course types as characterized by academic fields. Given the findings noted in a few studies, potential differences across disciplines in the online environment could be an important issue to consider. However, discipline effects on student retention, particularly related to STEM courses, have not been widely assessed.

The few research studies which have focused specifically on STEM online learning had very small sample sizes and did not distinguish between different types of STEM courses. Students in a social statistics course at California State, Northridge were randomly divided into two groups, one taught in a traditional classroom and the other taught virtually. The online students scored an average of 20% higher on examinations. Further, post-test results indicated that the virtual class had significantly higher perceived peer contact, and time spent on class work, with a perception of more flexibility, better understanding of the material and greater

[1] At the college in this study, mathematics is the most strongly represented STEM subject online, although a number of other subject areas also regularly offer online courses. Typically a majority of the STEM courses offered online are in mathematics.

affect toward math, at semester end, than did the traditional class (Schutte, 1997). Two studies examined students in business statistics classes. In one study, both grades and persistence were independent of the course modality (McLaren, 2004). In the other study, the online students perceived less interaction with their peers, but no difference between course design, grading and work load (Kartha, 2006). A study of online and face-to-face students in a general education soil science course with lab and field components found no difference in overall grade or lab assignment grades between course formats. However, the online students showed a larger grade improvement from pre- to post-assessment (Reuter, 2009).

### Conceptual Framework

This study analyzes one set of factors as a part of a larger project that aims to build a model for online student retention. The overarching framework which motivated this study was Tinto's longitudinal model of student departure (1975) with a specific focus on the student's integration into the formal academic system (academic performance). Tinto suggests that student retention is impacted by a student's academic performance and that the institution plays an important role by placing students in appropriate first year courses and assessing students for counseling and advising purposes. Kember (1995) developed a causal model of student persistence in distance learning applying Tinto's concept of academic integration to positive impressions of the course, counseling support, and intrinsic motivation.

Much of the recent research on retention has focused on external variables, pressures outside of the academic environment which place competing demands on a student's time, but these models also acknowledge that a key component of student success is academic integration, whether in the form of the perceived value of the coursework toward future employment or the level of institutional support and commitment (Bean & Metzner, 1985; Braxton & Hirschy, 2005). A student that is taking a course as a major requirement rather than an elective might perceive the course to have greater value in his/her goal of degree attainment.

This study was also informed by research on student self-efficacy as a factor contributing to academic success (Bandura, 1997; Zimmerman, 2000). Where a student has already had success in a lower level course, one would expect the student to have greater self-efficacy, and thus greater success, in a higher level course in the same subject area.

## Methodology

The community college in this study is located in the largest urban area in the U.S and has approximately 23,500 degree-seeking students and over 10,000 in continuing education programs. The College has an online Associate's Degree in Liberal Arts and about 120 online courses. The College has a diverse student body, with enrollees from over 150 countries around the world. Eighty percent of the College's student population belongs to groups historically underrepresented in higher education, and about two-thirds of the student body is female.

This study utilizes two sets of data provided by the College's Office of Institutional Research. Dataset 1 (see *Table 1*) included 122 course sections, half taught online and half taught face-to-face. The sample was chosen to include only those course sections for which an instructor taught the same course both face-to-face and online in the same semester (where there were more than three sections for a single instructor, a series of coin flips determined which sections would be included in the sample, so that no one instructor was overrepresented in the sample). The sample was limited to instructors who had taught online for at least three semesters to remove possible confounding effects from instructor inexperience. A wide distribution of courses that covered both upper and lower level courses in STEM and non-STEM disciplines were included in the sample, with the aim of making this study as broadly applicable as possible.

For every student enrolled in a course in the sample, the G.P.A. and major at the beginning of the semester in which they were enrolled in the course and the final course grade (including withdrawal status) was obtained. This resulted in a total dataset of 2,330 participants. We excluded students who received INC or ABS grades (both indicating an incomplete for the course), so the actual *N* for the analyses conducted was 2,247.

Some initial analysis was inconclusive, so a larger dataset was obtained which included all online STEM sections from spring 2004 through fall 2010 which could be paired with a face-to-face section taught by the same professor in the same semester. This new dataset, Dataset 2 (see *Table 2*), included 208 sections of STEM courses, half taught online and half taught face-to-face. For some analyses in which a larger dataset was needed to clarify results, Datasets 1 and 2 were combined, and any duplicate sections which were included in both datasets were counted only once. This produced a third larger dataset, Dataset 3 (see *Table 3*).

This research uses *z*-tests for comparison of proportions to compare retention rates in different course types and binary logistic regression to assess interaction effects. Cohen's *d* is calculated for several cases to determine effect size. Experiment-wide significance levels of $\alpha=0.05$ for statistically significant and $\alpha=0.01$ for highly statistically significant were used. In cases where multiple pairwise comparisons were performed on the same data, the Bonferroni procedure was used to adjust the alpha levels to much more conservative per test levels.

## Results and Discussion

### Do different types of courses have different retention rates?

The focus of this research is to determine whether or not retention rates[2] differ for STEM courses online vs. face to face. For all the students included in the sample, retention rates were computed for both course delivery type (online vs. face-to-face) and class type (STEM vs. non-STEM); then *z*-scores and *p*-values were calculated for each comparison (see *Table 4*). These results suggest that attrition is lower (or retention is higher) in face-to-face courses compared to online courses, and this difference is highly statistically significant ($\alpha=0.01$), which is what one would expect given the research literature (Patterson & McFadden; Hachey, Wladis & Conway, *In Press*). STEM courses had higher overall retention regardless of delivery method, and this difference was highly statistically significant ($\alpha=0.01$) when a large enough dataset was used (Dataset 3). Since the research focus of this study is to determine precisely how the online environment might impact STEM attrition, the next step is to see if there is a significant difference in STEM attrition online vs. face-to-face, and to compare this to the effect of the online environment on non-STEM courses. While STEM courses have higher retention overall, they may be more susceptible to factors which increase attrition in the online environment.

### What effect does the online environment have on retention in STEM courses?

To determine whether STEM course retention is affected more strongly by the online environment than non-STEM courses, it is necessary to assess interaction effects. First, online and face-to-face retention for course type and level are computed (see *Table 5* and *Figure 1*). In *Table 5*, the results show that both STEM and non-STEM courses had a higher attrition rate online than face-to-face, with STEM courses having a much larger and more highly significant

[2] Retention rates and Attrition rates are complements of one another: the Attrition rate is the percentage of students who earned a "W" or "WU" designation for the course, whereas the Retention rate is the percentage of students who did not earn a "W" or "WU." A "WU" designation at the College is given to students who stop attending classes before the college's official withdrawal deadline (at the end of the ninth week) but fail to formally withdraw from the class. Students who stop attending class after this date receive an "F" grade instead. Students who officially withdraw from a course after the third week of classes receive a "W" designation.

difference in retention rates. In addition, lower level, or introductory, courses had a higher attrition rate online than introductory face-to-face courses. All of these differences were highly statistically significant ($\alpha$=0.01). Higher level courses also had a higher attrition rate online than face-to-face, but this difference was only statistically significant ($\alpha$=0.05).

The gap between face-to-face and online course retention appears greater among STEM than non-STEM courses. To determine if this interaction is significant, a binary logistic regression with an interaction term is performed on these data, using retention rates as the dependent variable and course delivery method and STEM status as the independent variables. The results of this analysis do not show a significant interaction for the original sample (Dataset 1), probably because the sample size was too small; however, when the analysis is rerun with a larger sample (Dataset 3), the results do show that this interaction is statistically significant (see *Table 7*). The Nagelkerke $R^2$ for this test is 0.033 (See *Table 6*), suggesting that 3.3% of the variance in the chosen model for course retention could roughly be attributed to: course delivery type, STEM status, and the interaction between these two factors. This is a medium effect size.

*Table 7* shows that both course delivery type and STEM status are highly statistically significant predictors of course retention ($\alpha$=0.01) and that the interaction between course delivery type and STEM status is a statistically significant predictor ($\alpha$=0.05). In other words, the increased risk of dropping out when moving from a face-to-face to online environment is even greater for students in STEM courses than in non-STEM courses, even though STEM courses typically have higher retention overall in the samples at the college in this study.

Looking at *Figure 1*, it is clear that the gap between STEM online and face-to-face courses is greater than the gap for non-STEM courses. However, it is important to note that retention in STEM courses is still higher online than in non-STEM courses. These results indicate that courses that do not have higher levels of attrition overall may still have significantly higher levels of attrition online than face-to-face, suggesting that specific types of courses are particularly susceptible to factors in the online environment.

**Are particular types of STEM courses more vulnerable to factors of the online environment which decrease retention?**

Now certain subtypes of STEM courses are considered by looking at course level (lower vs. upper) and student motivation for enrolling (elective vs. distribution vs. major requirement) - see *Figures* 2 and *3*. In a previous study (Wladis, Conway & Hachey, *in press*), the authors found that attrition in online courses taken as electives or distributional requirements were much more strongly impacted by the online environment than in courses taken as major requirements.

Retention rates are practially identical for STEM major requirements but go down dramatically in online for STEM courses taken as distributional requirements and electives (*Figure 2*). In order to determine if this interaction between course delivery method and student motivation for taking the course is significant, it is necessary to perform a binary logisitic regression with an interaction term, with retention rates as the dependent variable and course delivery method and student motivation for taking the course as the independent variables. An initial analysis using only STEM courses in the original sample (Dataset 1) did not yield significant results, probably because the sample size was small, so the analysis was repeated with a larger sample (Dataset 2) which did show a significant interaction (see *Tables 8* and *9*). The Nagelkerke $R^2$ is 0.084, suggesting that 8.4% of the variance in this model for STEM course retention could roughly be attributed to course delivery type, student motivation for taking the course, and the interaction between these two factors. This is a medium effect size.

*Table 9* shows that retention rates in STEM courses were not significantly affected by either course delivery method or student motivation for taking the course when the interaction between these two factors is included in the model; this interaction was statistically significant ($\alpha$=0.05), indicating that the stark differences in slope in *Figure 2* are significant. So while all students taking STEM courses all had roughly similar risks of dropping out face-to-face, in the online environment, students taking STEM courses as electives and distributional requirements were significantly more likely to drop out than those taking major requirements. Course level was also analyzed, but the interaction between course level and course delivery method in STEM courses was not significant for either dataset under binary logisitic regression analysis.

Finally, the previous results were refined by combining all three factors (STEM status, student motivation for taking the course, course level) and performing a last set of significance tests. In *Table 10*, lower level STEM electives and distributional requirements both had statistically significantly lower retention online than face-to-face, with medium effect sizes of 0.56 and 0.41, respectively. No other course combinations yielded significant results.

## Implications

**For Practice:** This research suggests that online retention can be improved by targeting support to students in online STEM courses. Students taking STEM courses as elective or distributional requirements in particular are at a greater risk of withdrawing than their face-to-face peers, and therefore need special support in the online environment. Institutions hoping to improve online course retention might establish advisement and counseling programs specifically courses in STEM disciplines where students are at greatest risk of drop-out. Better guidance should be provided in selecting appropriate elective courses and consideration should be given to identifying key STEM distributional requirements and providing those courses with additional resources. A student making an inappropriate course choice, either due to content or course difficulty, is much more likely to dropout than a student enrolled in an appropriate course (Gibson & Walters, 2002; McGivney, 1996; Yorke, 1999).

**For Research:** This study shows that the type of online STEM course in which students enroll can have a drastic effect on the likelihood of withdrawal. However, before larger generalizations can be made about which types of courses lead to higher attrition online in the general college population, this type of analysis should be repeated with larger samples containing a wider range of courses across different college campuses. Some of these results may be institution specific.

We do not yet know the reasons for the higher rates of attrition in lower level elective and distributional requirement STEM online classes, and further research could shed some light on this. Further investigations could explore if the results have something to do with the online technology itself or is it related to the difficulty that students may already have in STEM courses when they often come into them underprepared and lacking confidence in quantitative fields.

## Conclusions

In this study, attrition rates in STEM courses were significantly more strongly affected by the online environment than non-STEM courses, even though STEM courses had lower attrition overall, suggesting that there may be factors in the online environment that impact STEM courses particularly strongly. In addition, STEM courses taken as electives and distributional requirements were particularly vulnerable to magnified attrition online, particularly when they were lower level courses, suggesting that whatever factors lead to attrition face-to-face in these courses are likely magnified by the online environment. Extra support in the form of advisement, tutoring, or online technical assistance may be required to improve retention rates in these courses to rates on a par with comparable face-to-face STEM courses.

## Figures

*Table 1. Dataset 1 Overview.*

| Categories | *N* | % |
|---|---|---|
| **face-to-face** | 1285 | 57.2 |
| **online** | 962 | 42.8 |
| **lower level** | 1297 | 57.7 |
| **upper level** | 950 | 42.3 |
| **non-STEM** | 1493 | 66.4 |
| **STEM** | 754 | 33.6 |

*Table 2. Dataset 2 Overview.*

| Categories | *N* | % |
|---|---|---|
| **face-to-face** | 1172 | 56.7 |
| **online** | 895 | 43.3 |
| **lower level** | 1203 | 58.2 |
| **upper level** | 864 | 41.8 |

*Table 3. Dataset 3 Overview.*

| Categories | *N* | % |
|---|---|---|
| **face-to-face** | 2074 | 57.1 |
| **online** | 1560 | 42.9 |
| **STEM** | 1830 | 50.4 |
| **non-STEM** | 1804 | 49.6 |

*Table 4. Retention by course delivery & STEM classification, with significance tests.*

| **Dataset 1** | **retention** | ***N*** | ***z*** | ***p*** |
|---|---|---|---|---|
| **face-to-face** | 81.00% | 1107 | 5.46 | **<0.0001** |
| **online** | 70.60% | 887 | | |
| **non-STEM** | 75.90% | 1338 | -0.78 | *ns* |
| **STEM** | 77.40% | 656 | | |
| **Dataset 3** | **retention** | ***N*** | ***z*** | ***p*** |
| **face-to-face** | 83.21% | 2103 | 7.52 | **<0.0001** |
| **online** | 73.00% | 1589 | | |
| **non-STEM** | 74.91% | 1861 | -5.87 | **<0.0001** |
| **STEM** | 82.80% | 1831 | | |

Results for *p*-values in bold are highly statistically significant (α=0.01, two-tailed), even when the Bonferroni procedure is used to control for Type I error. The abbreviation *ns* means not statistically significant.

*Table 5: Retention online & face-to-face for by course type, with significance tests (Dataset 1).*

| | **face-to-face retention** | ***N*** | **online retention** | ***N*** | ***z*** | ***p*** |
|---|---|---|---|---|---|---|
| non-STEM | 79.4% | 741 | 71.5% | 597 | 3.33** | 0.0004 |
| STEM | 84.4% | 366 | 68.6% | 290 | 4.81** | <0.0001 |
| lower level | 75.5% | 593 | 61.9% | 499 | 4.86** | <0.0001 |
| upper level | 87.4% | 514 | 81.7% | 388 | 2.35* | 0.0094 |

* and ** indicate α=0.05 and α=0.01 (one-tailed), respectively, for overall set of tests (0.0125 and 0.0025, respectively, per test using the Bonferonni procedure)

*Table 6. Goodness of fit statistics (Dataset 3) for Binary Logistic Regression testing significance of the interaction between course delivery method and STEM status.*

| | |
|---|---|
| Observations | 3634 |
| DF | 3630 |
| -2 Log(Likelihood) | 3553.139 |
| $R^2$(Nagelkerke) | 0.033 |

*Table 7. Type III analysis (Dataset 3) for Binary Logistic Regression in Table 6.*

| Source | DF | Chi-square (Wald) | Pr > Wald | Chi-square (LR) | Pr > LR |
|---|---|---|---|---|---|
| course delivery type | 1 | 13.788 | 0.0002 | 13.843 | 0.0002** |
| STEM status | 1 | 19.143 | < 0.0001 | 19.424 | < 0.0001** |
| course delivery type* STEM status | 1 | 6.025 | 0.0141 | 6.037 | 0.0140* |

* and ** indicate significance levels of α=0.05 and α=0.01

***Figure 1. Retention Rates for STEM and non-STEM Courses by Course Delivery Method (Dataset 3).***

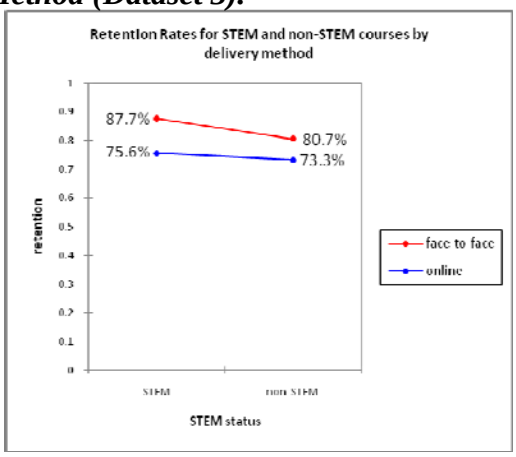


***Figure 2. Retention Rates for STEM Elective, Distribution Requirements & Major Requirement Courses by Delivery Method (Dataset 2).***

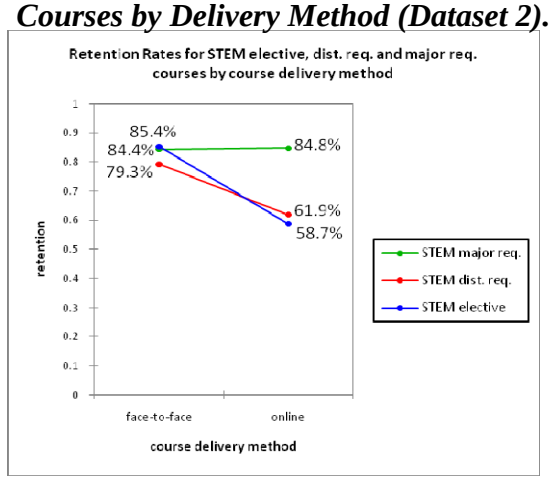


***Table 8. Goodness of fit statistics (Dataset 2) for Binary Logistic Regression on STEM courses testing significance of the interaction between course delivery method and student motivation for taking the course.***

| | |
|---|---|
| Observations | 744 |
| DF | 738 |
| -2 Log(Likelihood) | 766.790 |
| $R^2$(Nagelkerke) | 0.084 |

***Table 9. Type III analysis (Dataset 2) for Binary Logistic Regression in Table 8.***

| Source | DF | Chi-square (Wald) | Pr > Wald | Chi-square (LR) | Pr > LR |
|---|---|---|---|---|---|
| course delivery method | 1 | 0.007 | 0.934 | 0.007 | 0.934 |
| student motivation | 2 | 2.109 | 0.348 | 2.103 | 0.349 |
| course delivery method* student motivation | 2 | 8.392 | 0.015 | 8.757 | 0.013* |

* indicates significance level of α=0.05

***Table 10. Comparison of retention for all possible STEM course type pairs online and face-to-face, with tests for significance and effect size (Dataset 1).***

| | **F-to-F retention** | ***N*** | **online retention** | ***N*** | ***z*** | ***p*** | ***d*** |
|---|---|---|---|---|---|---|---|
| LL STEM dist. req. | 80.5% | 154 | 62.3% | 114 | 3.32 | 0.0005** | 0.41 |
| LL STEM elective | 77.5% | 40 | 50.0% | 64 | 2.79 | 0.0026* | 0.56 |
| UL STEM elective | 100.0% | 26 | 82.4% | 17 | 2.22 | 0.0132(*ns*) | 0.69 |
| UL STEM major req. | 93.6% | 47 | 82.4% | 34 | 1.59 | *ns* | 0.36 |
| LL STEM major req. | 74.5% | 51 | 68.8% | 16 | 0.45 | *ns* | 0.13 |
| UL STEM dist. req. | *n/a* | 0 | 50.0% | 4 | *n/a* | *n/a* | *n/a* |

* and ** indicate significance levels of α=0.05 and α=0.01 respectively (one-tailed) for overall set of tests (adjusted to 0.0083 and 0.0017 per test respectively, using the Bonferonni procedure)

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# IDENTIFYING DEVELOPMENTAL STUDENTS WHO ARE AT-RISK: AN INTERVENTION USING COMPUTER-ASSISTED INSTRUCTION AT A LARGE URBAN COMMUNITY COLLEGE

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**Abstract:**

Nationally, developmental mathematics courses can have completion rates as low as 25%, which can be a major barrier to degree completion. This article argues that specific institutional interventions can do much to ameliorate this situation by describing a particular intervention implemented in remedial courses at an urban community college over three semesters. Changes to the developmental mathematics course structure included using a mandatory departmental midterm to identify at-risk students and implementing a series of required intervention assignments using an online homework system in conjunction with regular class time for those students identified as at-risk. Significant gains in retention rates were obtained, with retention in some semesters as high as 50% greater than in the semester prior to the intervention. In addition, in this study, at-risk students who spent at least twenty hours on intervention assignments obtained retention rates that were approximately twenty-two percentage points higher than the average remedial student.

**Keywords:**

developmental mathematics

student motivation

classroom research

computer-assisted instruction

## Research Question:

Developmental mathematics courses give students who come to college with inadequate preparation the chance to succeed in college level courses and therefore to obtain a degree (Brothen & Wambach, 2004; Day & McCabe, 1997). However, the number of students successfully completing mathematics remediation can be quite low, suggesting that developmental education does not always provide the accessibility to higher education that was originally intended. In one study of 107 community colleges only 25% of students completed mathematics remediation (Bahr, 2008). Because of this low completion rate, there is a pressing need to develop and test interventions that can increase the success rate of students in developmental mathematics courses.

In order to address this issue, this study implemented a number of changes to the developmental mathematics course structure over three semesters. These changes included using a mandatory departmental midterm to identify at-risk students, followed by a series of required intervention assignments using computer-assisted instruction for those students identified as at-risk. These interventions were intended to boost student passing rates in remedial courses by motivating students to spend more time on mathematics practice, thereby potentially strengthening students' internal locus of control. This analysis attempts to determine to what extent these changes increased rates of successful course completion in the remedial courses in which they were implemented.

## Background and Motivation for the Study:

*Theoretical Perspective:* A number of studies suggest that students with an internal locus of control have higher levels of academic success and degree attainment (Parker, 1994; Shepherd, Owen, Fitch, & Marshall, 2006). Stage and Kloosterman (1992) found that mathematical self-confidence and beliefs, rather than incoming mathematics skills or past mathematical exposure, were significant predictors of success in remedial mathematics courses. This suggests that interventions which increase students' internal locus of control should lead to higher passing rates in remedial courses, even for populations that are typically at high risk of dropping out.

However, many students who take developmental mathematics courses are more likely to be characterized by an external locus of control and are therefore less likely to see the connection between their own work and the final outcome of a course such as the course grade (Findlay & Cooper, 1983; Weiner, 1979). This can lead to a self-destructive cycle: if students do not believe that there is a connection between the time spent on assignments and their degree of success in a course, they are not motivated to do the course work; and if they do not do the course work, they are not able to see how completion of the work can directly improve their mathematical understanding and course performance. The intervention which is the focus of this study was prompted by a desire to motivate students to spend more time on mathematics problems in remedial courses, even if external motivation was required to do so, so that they could begin to see the connection between the work they put into the course and their final course grade.

In addition, institutional support can also have a significant impact on remedial passing rates. Students with a lower preparedness level, when given the required assistance and support while taking developmental courses, are able to succeed in STEM courses, even at higher levels (Brown, 1988). The institutional academic and administrative support system and the supportiveness of the learning environment play an extremely significant role in students' success (Seymour, 1992).

*The Institution:* The college in this study runs approximately 225 remedial course sections containing approximately 5000 students taught by about 165 instructors each semester. The vast

majority of these instructors are adjuncts teaching nine or fewer hours at the college, and as many as thirty of these may be teaching at the college for the first time each semester. Classes are offered seven days per week, from 7am to 11pm, and as a result of the sheer number of instructional staff and the wide range of schedules, there is a certain lack of unified teaching culture among instructors. The teaching philosophies of different instructors can vary widely, from a traditional lecture format to a game-based collaborative learning structure.

Because the mathematics department at the college is so large and diffuse, it sought a change to the developmental course structure that could reasonably be implemented across hundreds of course sections taught by both full time and adjunct faculty with a wide range of pedagogical approaches and degrees of teaching experience. Technology in the form of computer homework systems seemed to be one possible tool for providing students with an interactive feedback loop which might increase student time spent solving problems and strengthen student awareness of the connection between practice and success, while also providing more uniform institutional support across diverse course sections.

*Technology and Reform Efforts:* Many recent reform efforts in developmental mathematics have focused on how technology, usually through computer homework and learning systems, can help students master developmental mathematics. Several recent articles (for example, Lenz, 2010, and Baker & Diaz, 2010) have shown that computer homework systems can improve student outcomes. Epper and Baker (2009) report that most institutions that participated in mathematics course redesign through technology found significant improvements, although they note that it is not enough to simply add technology to the current curriculum and practices; it is important to leverage the technology to approach the course in a different way.

## Research Methodology:

Since the goal was to get students to practice more, despite the lack of connection they might see between practice and success, this intervention encompassed the following:

1. *Online homework systems were included in all developmental mathematics course sections, and instructors and students were required to use them.* Such systems give students instant feedback, even when they are out of the classroom, so that students know immediately whether or not they have done a problem correctly, and so that they have the opportunity, in most cases, to redo a similar problem to increase their score. This may motivate students to work harder to improve their scores by strengthening students' internal locus of control, as they begin to see more clearly how the work that they do can translate directly into higher scores on the online problems. In addition, the system provides some automatic support for problem solving: Students can get help from the system (through tutorials and videos) to redo the problem, and instructors can see immediately how much time a particular student has spent on the online assignments and even identify the particular topics with which each student is struggling, making targeted support by instructors easier to implement.
2. *A departmental midterm was created and all students were required to take it during the seventh week of classes. Students who did not pass this midterm were then required to do online intervention assignments before they could take the final exam.* The midterm was implemented in order to identify early on those students who were at risk of failing or withdrawing from the course so that support services could be targeted at these students. The department theorized that students would be more motivated to study if they got a clear indication of where they stood early in the semester, and would be more likely to complete assignments that were required in order for them to take the final exam.
3. *An Intervention Lab was provided.* This lab was equipped with computers and peer tutors,

and was open from the time between the midterm and the final exam. Students could go to the lab for assistance in completing the online intervention assignments, or simply to use the computers. The first semester of the study, students who failed the midterm were mandated to spend 20 hours or more in the Intervention Lab; but because of budgetary, space, and staffing pressures at the college, this was modified in future semesters so that attendance at the lab was optional and students could complete assignments at home if they wished.

This study involved changes to remedial course structures that were implemented department wide in fall 2009, spring 2010, and fall 2010. At the college in this study, students are placed into remedial mathematics courses based on scores on the COMPASS placement test, which they take upon admission to the college. To pass a remedial course, a student must satisfy all criteria established by the instructor; pass a paper departmental final (on which they get two tries); and retake the COMPASS exam to meet cutoff scores to exit remediation.

At the beginning of this study, the Remediation Committee developed a complete set of online assignments (including homework, quizzes, tests, and reviews) in the homework management system which accompanied the textbook for each course. Thus a complete course package of assignments was made available to each instructor. Instructors could delete or edit any of the pre-made assignments except for the intervention assignments. Instructors were trained at an orientation a few days before the semester began, and further training was offered several times throughout the semester. In addition, the Remedial Coordinator for the math department regularly contacted instructors who did not seem to be using the system.

The way in which this course intervention was implemented did not require instructors to change their teaching methods in any way. At one end of the spectrum, if an instructor did not want to use the computerized intervention assignments as a part of their regular curriculum, they could simply require students who did not pass the midterm to complete these assignments at home or in the Intervention Lab; on the other end, instructors could rely entirely on the pre-made set of online course assignments (or edit them or create their own), leaving out paper and pencil assignments entirely. During the intervention, there were a number of instructors at both of these ends of the spectrum, with most instructors falling somewhere in the middle.

Student midterm scores, time spent in the Intervention Lab, and course passing rates were collected to assess differences between fall 2008 and fall 2009/fall 2010 and between spring 2009 and spring 2010. Standard $z$-tests for comparing two proportions were used to assess significance, and significance levels of 0.05 for statistical significance and 0.01 for high statistical significance were used.

**Results:**

The success of the intervention was assessed using passing rates for the remedial courses. Students cannot pass the course if they do not first pass the COMPASS exam which was taken to place them into the course at the beginning of the semester, so passing rates in the course also indirectly measured the rate at which students passed the COMPASS pre-algebra and/or algebra exams, depending upon the subject of their developmental mathematics course.

The college has four developmental mathematics courses: 1) MAT 010, a six hour course in arithmetic; 2) MAT 011, a three hour course in arithmetic; 3) MAT 012, a six hour course which combines elementary algebra with arithmetic; and 4) MAT 051, a four hour course in elementary algebra. Passing rates for each of these courses were assessed individually, and also combined to create a total for all remedial courses. The intervention was first implemented in fall 2009, so fall 2008 and spring 2009 semesters were used as control groups to assess the interventions.

*Fall Passing Rates:* We can see the results for the fall semesters in Table 1. From this data we can see that course passing rates improved by a significant margin in all remedial classes from the 2008-2009 school year to fall 2009. Comparing data fall-over-fall (from 2008 to 2009) the passing rate rose from 31.9% to 43.7%, or 34.7% to 51.1% if WU[1] grades are excluded from the analysis. The passing rate for all courses accordingly improved by 37.1% after only a single semester of the curriculum changes, and by 47.1% if WU grades are excluded. This result is highly statistically significant ($\alpha$=0.01). In particular, approximately 500 remedial students passed their courses in fall 2009 who would otherwise have failed based on fall 2008 passing rates.

For Fall 2010, these gains were improved further (see Table 1). Comparing data fall-over-fall (from 2008 to 2010) the passing rate rose from 31.9% to 47.4%, or 34.7% to 54.4% if WU grades are excluded from the analysis. Thus, the passing rate for all courses improved by 48.8% after three semesters, and by 56.6% if WU grades are excluded. This result is highly statistically significant ($\alpha$=0.01). In particular, approximately 600 remedial students passed their courses in fall 2010 who would otherwise have failed based on fall 2008 passing rates.

*Spring Passing Rates:* We can see the results for the spring semesters in Table 2. Comparing data spring-over-spring (from 2009 to 2010) the passing rate rose from 28.1% to 36.7%, or 31.2% to 35.8% if WU grades are excluded from the analysis. So the passing rate for all courses improved by 30.5% after two semesters of the curriculum changes, and by 46.8% if WU grades are excluded. This result is highly statistically significant ($\alpha$=0.01). In particular, approximately 500 remedial students passed their courses in spring 2010 who would otherwise have failed based on spring 2009 passing rates.

*Hours Spent in the Intervention Lab:* In fall 2009, a total of 2009 students failed the midterm. Of these students, 1410 visited the Intervention Lab in fall 2009, and 1418 logged into the Intervention courses online to complete work; many additional students completed Intervention Assignments within their instructor's own online course. Intervention students consisted of only those students who failed the departmental midterm with a grade below 70%. Anecdotal evidence suggests that these students typically had failed the course in past semesters. The passing rates for students who attended the Intervention Lab even once were higher than those for typical students in spring 2009 – this is surprising, since Intervention Students are only those students who already failed the departmental midterm (see Table 3).

The range of time spent in the Intervention Lab was 7 sec to 32 hrs, and so some students included in the data below may not have spent a significant amount of time on Intervention Work. In order to analyze the outcomes for students who actually completed the intervention requirement in the lab (rather than simply attended once or twice), we selected a random sample of 30 students who completed 20 hours or more in the Intervention Lab during the fall 2009 semester (chosen using the random number generator in Microsoft Excel); these results can be seen in Table 4. We can see from this table that students who spent at least 20 hours in the Intervention Lab had a passing rate of 65% for their remedial courses, which is highly statistically significantly higher than the general course passing rate of 43.7% for the semester.

The group of intervention students who had the commitment to spend 20 hours in the Intervention Lab may have been somewhat self-selecting, which could affect the interpretation of these results; however, these students failed to pass the departmental midterm, which suggests that without intervention their chances of failing the departmental final exam (which is similar in

[1] WU grades are given to students who stop attending after the tenth week of classes.

format and difficulty but broader in coverage than the midterm) were likely quite high. Thus, the 70% passing rate for these students on the departmental final exam and the 65% passing rate in the course (which is 28% higher than the prevailing remedial pass rate containing a majority of students who did pass the midterm), suggests that the Intervention Lab requirement had a significant impact on the success of these students who were at-risk of failing at the midterm.

*Limitations:* This study was not conducted using a random sample of students; rather it was implemented department-wide. It seems feasible to assume that students in prior semesters who were placed into remediation were likely comparable to students taking remedial courses during the study; however, without true randomization it is impossible to guarantee total equivalence on all factors that might contribute to student success and retention. Given the large effect of the intervention, it seems reasonable to conclude that the intervention was in fact effective, even if there is the possibility of reduced effect size with a true random sample; however, it is necessary to exercise caution in interpreting the applicability of these results.

**Conclusions:**

This research suggests that students can be motivated to do more practice in mathematics, even if that motivation must be extrinsic. Course structures that contain incentives for students to spend independent time on certain types of computer-assisted learning can increase student retention, and institutional support structures such as providing assignments to instructors and lab help for students can aid in this process. Significant gains in retention rates for developmental mathematics courses can be obtained with the right mix of early identification of at-risk students and required independent work using computer assisted instruction for those identified as at-risk. Retention can be improved by as much as 50% over a few semesters. The greater the number of hours students can be induced to work on mathematics problems, the greater their chances of passing the course, even if they were at high risk of failing the course initially.

Mastery of the material is not necessarily required in order for the assignments given to at-risk students to be effective; in this study, even twenty hours spent over the course of the entire semester on intervention assignments was enough to produce retention rates that were approximately twenty-two percentage points higher than those of the average remedial student.

Even without changing teaching practices, assignments that identify at-risk students early, followed by required online intervention assignments, can significantly improve student performance in developmental mathematics courses. It may be possible to obtain even more dramatic improvements if changes in pedagogy or curriculum were combined with the one studied here.

A caveat must be made about the research in this study: two different changes were made at once -- a mandatory midterm to identify at-risk students, and a mandatory online intervention for those who did not pass. It is not clear what proportion of the results is due to the midterm, and what may be due to the online assignments or the existence of the intervention lab. The faculty at the college in this study believe that both parts are necessary to obtain these results because they work in conjunction with one another to improve student outcomes; however, this study did not include control groups for which one or other component of the intervention was excluded, so it is impossible to draw any firm conclusions about this; such a study may be a good focus of future research. In addition, this study looked only at gains for the entire student population in remedial mathematics, but further research could shed light on the effectiveness of these interventions for different subgroups of students based on such characteristics as placement scores, ethnicity, gender, and repeater status.

**Table 1. Fall Passing Rates**

| **Fall 2008 (pre-intervention)** | | **Fall 2009 (post-intervention)** | | **Fall 2010** | | **% Change from Fall 09** | | **% Change from Fall 08** | |
|---|---|---|---|---|---|---|---|---|---|
| **Passing Rates** | | **Passing Rates** | | **Passing Rates** | | **Passing Rates** | | **Passing Rates** | |
| | **pass** | | **pass** | | **pass** | | **pass** | | **pass** |
| MAT 010 | 29.8% | MAT 010 | 42.0% | MAT 010 | 46.8% | MAT 010 | +11.6% | MAT 010 | +57.3% |
| MAT 011 | 32.2% | MAT 011 | 49.1% | MAT 011 | 58.3% | MAT 011 | +18.7% | MAT 011 | +80.9% |
| MAT 012 | 28.6% | MAT 012 | 42.3% | MAT 012 | 40.7% | MAT 012 | -3.8% | MAT 012 | +42.6% |
| MAT 051 | 34.0% | MAT 051 | 42.4% | MAT 051 | 45.2% | MAT 051 | +6.7% | MAT 051 | +33.1% |
| **TOTAL** | **31.9%** | **TOTAL** | **43.7%** | **TOTAL** | **47.4%** | **TOTAL** | **+8.6%** | **TOTAL** | **+48.8%** |
| **Passing Rates Excluding WU** | | **Passing Rates Excluding WU** | | **Passing Rates Excluding WU** | | **Excluding WU** | | **Excluding WU** | |
| | **pass** | | **pass** | | **pass** | | **pass** | | **pass** |
| MAT 010 | 33.1% | MAT 010 | 47.2% | MAT 010 | 54.0% | MAT 010 | +14.4% | MAT 010 | +63.1% |
| MAT 011 | 35.0% | MAT 011 | 59.5% | MAT 011 | 66.7% | MAT 011 | +12.1% | MAT 011 | +90.3% |
| MAT 012 | 32.3% | MAT 012 | 49.9% | MAT 012 | 47.9% | MAT 012 | -4.0% | MAT 012 | +48.4% |
| MAT 051 | 36.2% | MAT 051 | 49.5% | MAT 051 | 51.3% | MAT 051 | +3.7% | MAT 051 | +41.8% |
| **TOTAL** | **34.7%** | **TOTAL** | **51.1%** | **TOTAL** | **54.4%** | **TOTAL** | **+6.5%** | **TOTAL** | **+56.6%** |

**Table 2. Spring Passing Rates**

| Spring 2009 (pre-intervention) | | Spring 2010 (post-intervention) | | % Change | |
|---|---|---|---|---|---|
| **Passing Rates** | | **Passing Rates** | | **Passing Rates** | |
| | **pass** | | **pass** | | **pass** |
| 010 | 25.9% | 010 | 39.4% | 010 | 52.3% |
| 011 | 29.5% | 011 | 46.9% | 011 | 58.6% |
| 012 | 23.8% | 012 | 39.5% | 012 | 65.9% |
| 051 | 29.8% | 051 | 30.3% | 051 | 1.6% |
| **TOTAL** | **28.1%** | **TOTAL** | **36.7%** | **TOTAL** | **30.5%** |
| **Passing Rates Excluding WU** | | **Passing Rates Excluding WU** | | **Excluding WU** | |
| | **pass** | | **pass** | | **pass** |
| 010 | 29.3% | 010 | 49.6% | 010 | 69.1% |
| 011 | 32.9% | 011 | 60.1% | 011 | 82.8% |
| 012 | 26.7% | 012 | 47.7% | 012 | 78.4% |
| 051 | 32.7% | 051 | 37.8% | 051 | 15.4% |
| **TOTAL** | **31.2%** | **TOTAL** | **45.8%** | **TOTAL** | **46.8%** |

**Table 3. Success of Students Who Went to the Intervention Lab at Least Once**

| Intervention Students who Signed in to the Intervention Lab at least once | |
|---|---|
| Passing Rate | Passing Rate Excluding WUs |
| 31.6% | 34.3% |

**Table 4. Success of Students Who Spent at Least 20 Hours in the Intervention Lab (S=pass, R=fail)**

| Random sample (n=30) of students who completed 20 or more hours in intervention lab | | | | | |
|---|---|---|---|---|---|
| average time in lab: | 21:18:58 | | range of time in lab: | 20:00:36 | - 23:44:47 |
| **Passing rates:** | | | | | |
| **S** | **R** | **Percentage difference between this sample and all Remedial Students** | | | |
| 65.4% | 34.6% | +28.0% | | | |
| **Percentage of these students who passed the dept. final exam:** | | | | **70.0%** | |

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# Increasing Student Success in Intermediate Algebra through Collaborative Learning at a Diverse Urban Community College

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**Abstract:**

There is evidence that cooperative learning can improve student outcomes, but much of the research has been focused on pre-college mathematics or college calculus-level mathematics and above. This project tests the hypothesis that a change from a lecture-based class to one incorporating scripted collaborative discovery-based projects would increase successful course completion and exam results in Intermediate Algebra and Trigonometry at a diverse urban community college.

Twelve pairs of experimental and control sections were chosen so that each pair had the same instructor and assignments. Surveys, pre/post-tests, and success rates were used to assess intervention effectiveness. Statistical analysis suggests that the intervention had a significant effect on student success that was contingent upon a suitable period of instructor training and revision of course assignments. Increases in student exam scores of approximately two-thirds of a letter grade and a thirteen percentage point gain in successful course completion were obtained in experimental sections.

**Key words:**

intermediate algebra

design experiment

classroom research

collaborative learning

cognitive elaboration perspective

## HYPOTHESIS

Treisman (1990, 1992, 1995) has shown that the success of minority students in math courses is strongly dependent upon their integration into a cooperative learning community, and research confirms that individuals learn STEM disciplines best by proactively exploring and engaging in the content (Bredderman, 1984; Champagne & Hornig, 1987; Ahlgren & Rutherford, 1993; Yager, 2005; Yager & Akcay, 2008; Siebert & McIntosh, 2001).

At the college in this study, success among STEM majors is heavily determined by success in the Intermediate Algebra and Trigonometry course, so this study tests the following: Replacing the traditional lecture format with challenging collaborative group projects will increase student grades and test scores in Intermediate Algebra and Trigonometry (MAT 056).

## BACKGROUND AND RATIONALE

Sixty-five percent of math and science majors graduating at the college in this study were initially placed into MAT 056 or below. The success rate for MAT 056 in a recent semester was 56.2%, but about 94% of recent math and science graduates passed the course on their first try, suggesting that STEM majors who pass the course the first time are more likely to graduate.

Research demonstrates that collaborative learning increases student conceptual understanding, ability to tackle difficult mathematical problems, and student self-confidence in mathematics (Reynolds, et. al., 1995). Treisman (1990, 1992, 1995) showed that the reason for the failure of many African American students in calculus courses was their academic and social isolation in these classes, and that in several instances the GPAs and passing rates of these students increased dramatically after cooperative group learning was introduced.

However, there are few controlled studies focused on the effectiveness of collaborative learning at the college level, and most that do are for calculus-level courses and above. A few studies have analyzed formal classroom experiments at the level of beginning or college algebra with promising results (Dees, 1991; Lucas, 1999; Rupnow, 1996); however, these studies either had a relatively small sample size or did not include controls for instructor, and were primarily conducted at four-year colleges. This study aims to fill this gap by providing results for a highly structured classroom experiment using 24 sections of Intermediate Algebra at a diverse urban community college, while controlling for instructor, course assignments and class meeting time.

There are a number of theoretical perspectives that have been used to explain the success of cooperative learning. This study used motivationalist and developmental perspectives to inform the creation of group projects and class structure, but the primary theoretical perspective which was what Slavin (1996) calls the cognitive elaboration perspective. The motivationalist perspective (Johnson & Johnson, 1992; Slavin, 1996) is based on the idea that grading group performance as a whole incentivizes students to work harder in response to peer pressure from group members, and research has shown that when group assessment is based on the "sum of group members' individual learning," significant improvements are obtained 78% of the time (Slavin, 1995). In this study, assessment of collaborative work was structured based on this perspective: group projects grades were the average of the final group product and short individual quizzes modeled after the project; group projects were 25% of the final course grade.

The developmental perspective is based on Vygotsky's "zone of proximal development" (1978) and Piaget's theories of disequilibrium and accommodation (1963); cognitive dissonance introduced by group discussion with peers of similar but slightly different capability level is thought to prompt a learner to construct knowledge. To address this perspective, groups in this

study were chosen so that they consisted of students of slightly (but not significantly) different levels of mathematical preparedness. In addition, group projects were constructed to require high levels of discussion in which all group members were required to participate.

The cognitive elaboration perspective on collaborative learning is based on the idea that cognitive restructuring, required when explaining material to another student, is essential to learning (Wittrock, 1986). A large body of research (Dansereau, 1988; O'Donnell & Dansereau, 1992; Newbern, et all, 1994; Webb, 1989; Webb, 1992) has shown that students who were required to give elaborate explanations to others had significant gains in learning (and that students who listened to these explanations had somewhat smaller learning gains). To address this perspective, the group projects in this study involved students rotating through three distinct scripted roles: *Prover*, *Explainer*, and *Checker*. The *Prover* would work out the steps of the proof or problem; the *Explainer* would write out an explanation of each step; and the *Checker* would read through the work of the first two students to identify any errors and suggest corrections where needed. Each role required that the student discuss each step with the entire group before writing it down; all parts of the project were written on a *Project Report Sheet* that required students to write their answers in a structured way (left column for the *Prover's* steps, middle column for the *Explainer*, and right column for the *Checker*). Students were required to identify who played each role for each problem. At the beginning of the semester students were given a sheet explaining each of these roles, giving "good" examples of successful group work step-by-step, and were guided through a short sample project so they could practice the roles on a basic algebra topic. A short excerpt from one of the first group projects can be seen here:

> For each of the following problems, rotate through the roles of the *Prover*, *Explainer* and *Checker* for each step of the proof. Use Examples 1 and 2 as a model to prove:
> 1. A specific example of rule (c): $(ab)^6 = a^6b^6$ for all $a, b \neq 0$.
> 2. Now the general case of rule (c): $(ab)^r = a^rb^r$ for all $a, b \neq 0$ and all positive $r$ ($r > 0$).

The projects themselves were highly scaffolded and aimed to have students tackle important higher-level conceptual tasks such as proofs.

## RESEARCH DESIGN AND METHODS

To test the effectiveness of the new course structure, six pairs of pilot and control sections were taught each semester. Sections were matched by time of day and instructor so as to minimize any increased variation in the sample that might be contributed by these two factors. The control sections and pilot sections had identical assignments and exams, aside from the group projects which were the focus of the intervention. The designation of a particular section as a pilot or control section was made randomly, using a coin toss.

Class time in pilot sections was restructured to focus approximately one third of class time on cooperative group projects. As a part of the pilot section development, the PIs and pilot instructors created eight original group projects covering each major topic of the course. The PIs and pilot instructors met approximately biweekly during each semester of the project, and pilot instructors underwent training to ensure consistent use of techniques. The instructor training in this study was itself collaborative: each meeting began by asking faculty what was and was not working in their pilot course sections; a few particular issues were chosen as a result of the instructor response to this question; and then all the instructors discussed the different techniques they had used to address the particular problems under discussion. Each meeting ended with a summary of good techniques and solutions to try, and a written summary was emailed to the

group. Sometimes the problems involved a revision of course structure or assignments, and sometimes it involved particular instructor actions in the classroom, in which case the instructor would try out the new techniques and report back at the next training session.

Student and faculty surveys, scores on departmental exams, and success rates were used to assess this intervention's effectiveness. Placement data were obtained from Institutional Research so that the pilot and control groups could be compared on measures of mathematical preparedness and aptitude at the start of the semester, to ensure that the samples were comparable. In addition, pre/post-tests, pre/post-surveys, and course grades were used to assess changes from the beginning to the end of the semester in both groups. ANCOVA tests were used to assess differences in pre/post-surveys, and *z*-tests for comparison of proportions were used to compare course success rates. Because pre/post-test scores were not normally distributed, change scores were computed and student *t*-tests were used to compare them. Alpha levels of 0.05 and 0.01 were used as the threshold for significant and highly significant results, respectively.

*DIFFERENCES BETWEEN THE FIRST AND SECOND SEMESTERS OF THE INTERVENTION:* Instructors encountered a number of challenges during the first semester of teaching the pilot course sections, which were addressed before implementing the pilot courses for a second time. Some of the issues that arose during the first semester of implementation were: clarification of group project structure and instructions; changes in group structure and procedures to address student absenteeism; modification of assessment methods to better balance individual vs. group assessment; and improvement of instructor oversight of group project interactions.

Because the first semester in which the new course structure was implemented was an adjustment period (as projects and procedures were revised and as instructors learned to teach with the new method), it was anticipated that better results would be obtained in the second semester of implementation. Final analysis of the data confirmed this expectation.

## RESULTS

### *EQUIVALENCE OF PILOT AND CONTROL GROUPS*

To determine the comparability of pilot and control groups, several measures of math preparedness were collected: 1) Average score on the COMPASS placement exam in elementary algebra; 2) Time elapsed since the COMPASS exam (a proxy for time elapsed since last math class); 3) Proportion of students exempt from the COMPASS exam (by high Regents, SAT, etc.); and 4) Average score on MAT 056 pre-test given on first day of class. The appropriate *z* or *t* statistics were computed comparing each of these four measures in the pilot group to the control group. A summary of these results can be seen in Tables 1 and 2.

From these data it is clear that none of these measures shows statistically significant differences between the control and pilot groups, in either semester ($\alpha$=0.05, two tailed), suggesting that analysis comparing the pilot and control groups is unlikely to be unduly affected by confounding variables pertaining to prior preparation.

### *MEASURES OF STUDENT SUCCESS AND ATTITUDES IN PILOT AND CONTROL SECTIONS*

The success and attitudes of students in the pilot and control sections were measured by comparing the following three measures: 1) The proportion of students who completed the course with a grade of "C" or higher; 2) The scores on a student attitude pre/post-survey; and 3) The change in score for ten exam questions on a pre/post-test.

SUCCESS RATES: Course grades were collected from Institutional Research at the end of each semester. Only those students who completed the course with a "C" or better were considered to have completed the course successfully. (Students with grades of "INC" were excluded.) The numbers of students who successfully completed the course in the pilot and control sections were compared using a *z*-test for the comparison of two proportions. In the fall, success rates were not statistically significantly different (see Table 3). In contrast, in the spring semester, the pilot group's success rate (61.2%) was significantly higher ($\alpha$=0.05, one-tailed) than the control group (48.1%), by 13.1 percentage points. The *p*-value for this difference was 0.0197, and the effect size was medium-sized at 0.26 (see Table 4).

These results suggest that the pilot course structure was likely effective in improving course success rates, but that its success was largely dependent sufficient time for practice and training of instructors and revision of course materials and structure to reflect classroom experiences.

STUDENT ATTITUDE SURVEYS: Students in both groups were given a twenty statement survey designed to assess attitudes towards mathematics, on the first day of class and again just before final exams. These surveys contained statements like "Anyone who works hard can do reasonably well at math," and students were asked to reply using a five-point Likert scale reporting degree of agreement. One quarter of the items was reverse-worded to limit yea or nay-saying bias. A principal component factor analysis revealed only one major underlying factor, so the reverse-worded items were reverse-scored and the responses from all question were summed. The post-survey results can be seen in Tables 5 and 6. While post-survey scores were slightly higher in pilot sections both semesters, these differences were not statistically significant.

An ANCOVA was run with the post-survey score as the dependent variable, section type (control vs. pilot) as the independent variable, and the pre-survey score as the covariate (see Tables 7 and 8). The pre-survey score was a highly statistically significant ($\alpha$=0.01) predictor of a student's post-survey score (with $p$ <0.0001); however, a student's section type was not (*p*-value of 0.1696 in fall and 0.7619 in spring), suggesting that the pilot course structure did not have a significant impact on student attitudes as measured by the survey, at least in the short term. It is possible that the timing of the post-surveys, given just before the final exam, affected these results; scores on the post-survey were lower than on the pre-survey for both groups each semester, suggesting that the timing of the post-survey may have a significant effect on results.

FINAL EXAM SCORES: Students in both groups were given departmental final exams at the end of the semester, containing questions such as, "Solve for *t*: $\sqrt{t+15} = t+3$." Ten of the questions from these exams, chosen to cover each of the major topics of the course, were given as a pre-test on the first day of class. Instructors graded these exams using a partial credit rubric developed jointly. Because pre-test and post-test data were far from normally distributed, it was not possible to use an ANCOVA method directly on these scores. Instead change scores, which were normally distributed, were computed for each student by subtracting the pre-test score from the post-test score. A summary of change scores can be seen in Tables 9 and 10.

A *t*-test was then computed with the test change scores as the dependent variable and the section type as the independent variable. The difference in change scores in the fall was not statistically significant; however, in the spring the pilot sections had a change score that was statistically significantly higher ($\alpha$=0.05), with a *p*-value of 0.0403 (see Tables 11 and 12). This was equivalent to a gain in exam score that was 6.4 percentage points higher than in the control group, equivalent to about two-thirds of a letter grade. As with the analysis of student success

rates, this suggests that the pilot course structure was likely effective in improving student understanding of course material, but that this improvement was largely dependent upon successful implementation after a period of practice, training and revision.

*LIMITATIONS:* In this study, students were not randomly assigned to sections; rather, sections were randomly assigned to be either control or pilot. While the comparison of several measures of student readiness showed the pilot and control groups to be comparable, and therefore ameliorates many of the concerns regarding non-randomized assignment, it is impossible to measure all significant factors that could impact student success, and therefore one should be cautious before making broader conclusions about the results of this particular study.

## IMPLICATIONS

*FOR PRACTICE:* This study suggests that using structured collaborative learning projects in stable cooperative base groups can be effective in significantly increasing student success in mathematics gateway courses at diverse urban community colleges. The success of such an intervention depends upon a number of things, however. Assignments must be carefully constructed so that students' scripted roles in the collaboration are clearly outlined. Student work must be carefully scaffolded to allow students to tackle challenging assignments in an orderly way. Project rules and structures, such as group discussion and students taking turns playing different roles, must be supervised adequately by the instructor. And assessments must grade both group work and individual student knowledge gained from the projects. A semester or more may be needed before new assignments are adequately revised to be effective and before instructors new to this teaching technique can apply the technique effectively. As a result, implementations of interventions of this kind require adequate time for revision and instructor learning before they can be adequately assessed for their effectiveness, and instructor training and support may also be very important in determining intervention efficacy.

*FOR RESEARCH:* These results suggest that an instructor's experience with this teaching method improved the success of the method over time. Since this study was limited to two semesters only, the full potential improvement in student success may not yet have been reached during that period. Future research could study to what extent this improvement might continue in subsequent semesters. In addition, it is possible that the benefits of this intervention were not the result of collaborative learning per se, but rather of other features of the projects; a follow-up analysis of the actual nature of student interaction in pilot classes could be used to clarify this.

## CONCLUSIONS

The statistical analysis for this study seems to suggest that scripted collaborative learning projects used as a part of a comprehensive course structure can have a significant effect on student success in college intermediate algebra. However, this success is contingent upon a suitable period of instructor practice, training, and revision of course structures and assignments. It seems that collaborative learning can work very effectively, but that there is a learning curve, for both instructors and curriculum developers. However, with a bit of experience, collaborative group work in stable base groups can lead to increases in student performance on exams of approximately two-thirds of a letter grade and about a 13 percentage point gain in successful course completion compared to standard courses using a lecture format as the primary course structure.

## TABLES

| Table 1: Two-tailed *t*-tests (*z*-test in the case of the proportion) comparing the fall 2010 pilot (*N*=141) and control (*N*=143) groups | | | |
|---|---|---|---|
| | ***z*/*t*-score** | ***p*-value** | **significant?** |
| **COMPASS algebra score** | -0.63 | 0.5265 | *ns* |
| **days since COMPASS was taken** | 0.41 | 0.6849 | *ns* |
| **proportion exempt from COMPASS** | 0.61 | 0.2709 | *ns* |
| **pre-test score** | -1.26 | 0.2081 | *ns* |
| *ns* means nonsignificant (α=0.05) | | | |

| Table 2: Two-tailed *t*-tests (*z*-test in the case of the proportion) comparing the spring 2011 pilot (*N*=142) and control (*N*=136) groups | | | |
|---|---|---|---|
| | ***z*/*t*-score** | ***p*-value** | **significant?** |
| **COMPASS algebra score** | 0.41 | 0.6818 | *ns* |
| **days since COMPASS was taken** | -0.70 | 0.4830 | *ns* |
| **proportion exempt from COMPASS** | 0.35 | 0.2709 | *ns* |
| **pre-test score** | 1.71 | 0.0884 | *ns* |
| *ns* means nonsignificant (α=0.05) | | | |

| Table 3: Success rates for fall 2010 | | | | |
|---|---|---|---|---|
| | **success** | ***z*-score** | ***p*-value** | **Cohen's *d*** |
| **pilot** | 48.5% | -0.55 | 0.2912 | -0.07 |
| **control** | 51.9% | | | |
| * statistically significant (α=0.05, one-tailed) | | | | |

| Table 4: Success rates for spring 2011 | | | | |
|---|---|---|---|---|
| | **success** | ***z*-score** | ***p*-value** | **Cohen's *d*** |
| **pilot** | 61.2% | 2.06 | 0.0197* | 0.26 |
| **control** | 48.1% | | | |
| *statistically significant (α=0.05, one-tailed) | | | | |

| Table 5: Descriptive Statistics for Post-survey Results Fall 2010 | | | | |
|---|---|---|---|---|
| Group | Mean | Std Dev. | Std Err | *N* |
| **pilot** | 41.105 | 8.860 | 1.016 | 76 |
| **control** | 39.759 | 8.128 | 0.914 | 79 |

| Table 6: Descriptive Statistics for Post-survey Results Spring 2011 | | | | |
|---|---|---|---|---|
| Group | Mean | Std Dev. | Std Err | *N* |
| **control** | 41.092 | 10.299 | 1.104 | 87 |
| **pilot** | 42.333 | 9.164 | 1.018 | 81 |

| Table 7: ANCOVA with pre-survey total as covariate, comparing post-survey scores for fall 2010 pilot and control sections | | | | | |
|---|---|---|---|---|---|
| **Source** | **Type III SS** | **Df** | **Mean Sq.** | **F** | **Prob.** |
| **Pre-survey score** | 3482.080 | 1 | 3482.080 | 70.033 | <0.0001** |
| **control vs. pilot** | 94.684 | 1 | 94.684 | 1.904 | 0.1696 |
| **highly statistically significant (α=0.01) | | | | | |

| Table 8: ANCOVA with pre-survey total as covariate, comparing post-survey scores for spring 2011 pilot and control sections | | | | | |
|---|---|---|---|---|---|
| **Source** | **Type III SS** | **Df** | **Mean Sq.** | **F** | **Prob.** |
| **Pre-survey score** | 5308.519 | 1 | 5308.519 | 83.176 | <0.0001** |
| **control vs. pilot** | 5.881 | 1 | 5.881 | 0.092 | 0.7619 |
| **highly statistically significant (α=0.01) | | | | | |

| Table 9: Descriptive Statistics for Exam Change Scores Fall 2010 | | | | | | |
|---|---|---|---|---|---|---|
| Variable | Mean | Std Dev. | Std Err | *Lower 95% CL* | Upper 95% CL | *N* |
| **control** | 27.7 | 11.5 | 1.1 | 25.4 | 29.9 | 101 |
| **pilot** | 24.6 | 13.5 | 1.4 | 21.8 | 27.4 | 92 |

| Table 10: Descriptive Statistics for Exam Change Scores Spring 2011 | | | | | | |
|---|---|---|---|---|---|---|
| Variable | Mean | Std Dev. | Std Err | *Lower 95% CL* | Upper 95% CL | *N* |
| **control** | 23.9 | 12.8 | 1.3 | 21.3 | 26.4 | 102 |
| **pilot** | 27.1 | 13.1 | 1.3 | 24.5 | 29.7 | 99 |

| Table 11: One-tailed *t*-test for exam change scores for fall 2010 testing pilot > control | | | | | |
|---|---|---|---|---|---|
| **Ho. Diff** | **Mean Diff.** | **SE Diff.** | **T** | **DF** | **P** |
| 0.000 | 3.1 | 1.8 | -1.71 | 191 | 0.9558 |
| *statistically significant (α=0.05) | | | | | |

| Table 12: One-tailed *t*-test for exam change scores for spring 2011 testing pilot > control | | | | | |
|---|---|---|---|---|---|
| **Ho. Diff** | **Mean Diff.** | **SE Diff.** | **T** | **DF** | **P** |
| 0.000 | -3.2 | 1.8 | -1.76 | 199 | 0.0403 |
| *statistically significant (α=0.05) | | | | | |

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# Student Concept Images of Function and Linear Transformation

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**Abstract**

As part of a larger study of student understanding of concepts in linear algebra, we interviewed 10 university linear algebra students as to their conceptions of functions from high school algebra and linear transformation from their study of linear algebra. Analysis of these results led to a classification of student responses into *properties*, *computations* and a series of five interrelated *metaphors*. We see this classification as providing richness and nuance to existing literature on students' conceptions of function. In addition, we are finding these categories helpful in describing the compatibilities and distinctions in student understanding of function and linear transformation.

**Keywords**

Concept image, function, linear algebra, linear transformation, metaphor

## Introduction

The research reported in this paper began as part of a larger study into the teaching and learning of linear algebra. As we examined student understanding of linear transformations we wondered how student understanding of functions from their study of precalculus and calculus might influence their understanding of linear transformations and vice versa. In order to explore this issue, we found that we needed ways to describe student understanding of functions and linear transformations that might go beyond traditional characterizations of functions from the research literature. This proposal elaborates our new characterization and provides an example of how this characterization can be used to compare student understanding of function and linear transformation.

## Literature and theoretical background

The nature of students' conceptions of function has a long history in the mathematics education research literature. This work includes Monk's (1992) pointwise versus across-time distinction, the APOS (action, process, object, scheme) view of function (e.g., Breidenbach, Dubinsky, Hawkes, & Nichols,1992; Dubinsky & McDonald, 2001), and Sfard's (1991, 1992) structural and operational conceptions of function. A comparison of these views may be found within Zandieh (2000). More recent work has focused on descriptions of function as covariational reasoning (e.g., Thompson, 1995; Carlson, Jacobs, Coe, Larsen & Hsu, 2002). A recent summary with a focus towards covariational reasoning is found in Oehrtman, Carlson, and Thompson (2008).

In addition to work specifically on student conceptions of functions, we were interested in research that explores how one may characterize the conceptions that a student has for a particular mathematical construct. The term *concept image* has been used to refer to the "set of all mental pictures associated in the students' mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). A number of studies delineate students' concept images of particular mathematical ideas (e.g., Artigue, 1992; Rasmussen, 2001; Wilson, 1993; Zandieh, 2000). In addition to work that uses concept image as its framing, we find useful studies that (whether they refer to it by the term concept image or not) detail student concept images of mathematical constructs using the construct of a *conceptual metaphor* (e.g., Lakoff & Nunez, 2000; Oehrtman, 2009; Zandieh & Knapp, 2006). This follows from the earlier work in cognitive linguistics of Max Black (1977), Lakoff and Johnson (1980) and Lakoff (1987). Following from this work, our assessment is that a person's concept image of a particular mathematical idea will likely contain a number of metaphors as well as other structures. Zandieh and Knapp (2006) provide an example of this for the concept of derivative.

The work in linear algebra has tended to focus more on student difficulties (e.g., Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000; Harel, 1989; Hillel, 2000; Sierpinska, 2000). However, there have been a few studies on student understanding of linear transformation (Dreyfus, Hillel, & Sierpinska, 1998; Portnoy, Grundmeier, & Graham, 2006). Our work seeks to add to this research in ways that will highlight the connections or discrepancies between student conceptions of function and student conceptions of linear transformations.

## Methods

The data for this report comes from interviews with 10 students who were just completing an undergraduate linear algebra course. The interviews were videotaped and transcribed and

student written work was collected. The focus of the interview was to obtain information about students' concept image of function and their concept image of linear transformation and to see in what ways students saw these as the same or different. To this end we not only asked the students how they thought of a function or linear transformation, but also questions about characteristics that would be relevant to both functions and linear transformations such as one-to-one, onto, and invertibility. Several sample interview questions are provided below:

1. In the context of high school algebra, explain in your own words what a function is.
2. In the context of linear algebra, explain in your own words what a transformation is.
3. Please indicate, on a scale from 1-5, to what extent you agree with the following statement: "A linear transformation is a type of function."
4. In the context of high school algebra, give an example of a function that is 1-1 and one that is not 1-1. Explain.
5. In the context of linear algebra, give an example of a linear transformation that is 1-1 and one that is not 1-1. Explain.
6. Please indicate, on a scale from 1-5, to what extent you agree with the following statement: "1-1 means the same thing in the context of functions and the context of linear transformations."

We initially used grounded theory (Strauss & Corbin, 1994) to analyze student responses. As we refined our coding we noticed that the responses seemed to fall into three main types – *properties*, *computations*, and various *metaphors*. The details of these categories will be illustrated in the Results section. Coding with these categories followed an iterative cycle of coding by individual researchers, coming to consensus as to coding across individual researchers, and revising or refining the coding scheme as needed to more accurately reflect what we were seeing in the data. The next section documents the results of these deliberations.

**Results**

The main result of this paper comes in the form of a categorization of how students think about function and linear transformation. In order to compare students' concept images of function and linear transformation, we determined three main categories of tools students use to reason about these mathematical concepts: *properties*, *computations*, and *metaphors*. In this section we will provide examples of students reasoning with properties, computations, and each of the metaphors. We will then provide sample results of how this categorization can be used to reveal important distinctions or connections between student conceptions of function and linear transformation.

*Properties*

While reasoning with the interview tasks, many students referenced a property of a function or linear transformation or a property of a feature associated with either such as a graph or a matrix. The property category refers to student statements that do not delve into the inner workings of the function or transformation. In the first example below, Andrew describes a function using a property about equations, and was coded P(equations). In the second example, Dana reasons about why a linear transformation is one-to-one by referring to linear independence P(li), presumably the fact that the columns of the associated matrix were linearly independent.

*Andrew: A function is an <u>equation with a variable</u>.*
*Dana: I said that was one-to-one because it's <u>linear independent</u>.*

*Computations*

Students often drew upon computational language while reasoning through the interview tasks. We differentiated between computations that were done to carry out the function or transformation (labeled as C1), i.e., to get from the starting entity to the ending entity, and side computations done involving the function or transformation (labeled as C2), for example to compute the inverse function. In the first example, Ryan uses computational language (multiplication) to discuss how a linear transformation acts, which is indicative of C1. The second example shows Dana describing how to find the inverse of a 2x2 matrix. Her language (switch, make negative) is procedural and algorithmic, and *involves* the linear transformation but does not describe *how* the linear transformation acts.

*Ryan:* *A transformation is a <u>multiplication</u> of matrices that leads to a new image produced from the original matrix or vectors in the matrix.*

*Dana:* *Oh, I think you <u>switch</u> these two [points to entries on the off diagonal] and then probably <u>make this negative</u> [points to entries on the diagonal]. <u>Switch</u> those negatives.*

*Metaphors*

We identified five different metaphors that students called upon when reasoning about function or linear transformation: *input/ output, traveling, morphing, mapping,* and *machine.* These five metaphors share the common structure of a beginning entity, an ending entity, and a description about how these two are connected (see Fig. 1). Note that not all three parts of a metaphor must be stated by a student for the statement to be classified as this metaphor.

*Metaphor: Input/ Output*

The input/ output metaphor involves an input, which goes into something, and an output, which comes out. This can be viewed from the point of view of the person 'putting in' the input and 'taking out' the output, and/or from the point of view of the function or transformation 'accepting,' 'receiving' or 'taking' an input and 'returning' or 'giving' an output. The first example shows Jordan using both of these perspectives in the same sentence. In the second example, George uses the metaphor from the point of view of the function.

*Jordan:* *A function f of x = y means that <u>putting</u> x <u>inside</u> would <u>give</u> you a specific output, y.*

*George:* *... a function is an equation that <u>accepts</u> an input and <u>returns</u> an output based on that input.*

*Metaphor: Traveling*

The traveling metaphor involves a beginning location being sent or moving to an ending location. Some phrases that we found to be indicative of this metaphor were the use of 'gets sent', 'goes to', 'moving, 'reach', 'go back', and 'get to.' This metaphor was used almost exclusively when reasoning about linear transformation. We saw this metaphor used to describe a pointwise change in location as well as a global move. In the first example, Andrew describes a transformation as a pointwise change in location, and in the second example George uses the metaphor as a way to describe how transformations act more globally.

*Andrew:* *A transformation is <u>moving</u> a point or object in a <u>certain direction</u>.*

*George:* *When you're in transformations, you'll always be able to <u>get back</u>. If a matrix is invertible, you should be able to <u>go both ways</u>.*

*Metaphor: Morphing*

The morphing metaphor involves a beginning state of an entity that changes or is morphed into an ending state of the same entity. There must be a clear sense that the beginning entity did not simply move to the new location (ending entity), nor was it replaced by the new output (ending entity), but that there was actually a metamorphosis of the beginning entity into the ending entity. The morphing metaphor may be used pointwise by imagining one object changing, or globally by imagining a collection of objects changing. We found the phrases 'become', 'transform', and 'change' to be indicative of this metaphor. In the following example, Dana uses the morphing metaphor to explain what a transformation does to individual 'things'.

*Dana: Linear transformations to me are more or less something that <u>changes</u> something from one thing to another.*

*Metaphor: Mapping*

The mapping metaphor involves a beginning entity, an ending entity, and a relationship or correspondence between the two. This metaphor is most closely related to the Dirichlet-Bourbaki definition of function, and was not commonly used by students. We found the phrases 'map', 'rule', and 'correspondence' to be indicative of this metaphor, as well as 'per' and 'for', as in there is one input for/per every output. This metaphor was more commonly used in connection to function, but was used in relation to linear transformation as well. The following utterance is one of these uses:

*Lawrence: [A linear transformation is] a <u>rule</u> that assigns a given input to a certain output or image of the input.*

*Metaphor: Machine*

The machine metaphor includes a beginning entity or state, an ending entity or state, and a reference to a tool, machine or device *that causes* the entity to change from the beginning entity/state into the ending entity/state. A necessary component to this metaphor is language that indicates that the function or transformation is performing the action on the entity. We found the phrases 'acts on' and 'produces' to be indicative of this metaphor. In the first example, Noah indicates that the function is performing an action, and in the second example George uses the machine metaphor to discuss how a linear transformation acts.

*Noah: A function is an <u>operation</u> on something.*

*George: Pretty much anything you toss in here, this is still that transformation should be able to <u>act on</u> it.*

*Combined metaphors*

There are several general things to note when comparing across these metaphors for function and linear transformation. Each of the metaphors has the same general structure and they are often used in combination in student reasoning. In particular since the input/output metaphor focuses more on the beginning and ending entity, it can most easily be combined with each of the other metaphors. However, students often flow from one metaphor to another even in the same sentence. Below Brian combines the machine, input/output and morphing metaphors, while Landon combines the mapping, machine and input/output metaphors.

*Brian: I just remember when I was in middle school or elementary school or whatever, learning about functions, and learning about them as a machine, you put something in, and it transforms it to something else.*

*Landon: Because it essentially does the same thing. So it's like, how I have here a rule that assigns, essentially a function is the same thing, you put in an input, and it manipulates that input and turns it into an output.*

*Comment on metaphors in relationship to the process-object dichotomy*

The mapping metaphor is closest to Sfard's (1992) or Breidenbach et al's (1992) object conception of function. The other metaphors provide interesting nuances to our understanding of the process view of function.

**Discussion: Using the categories to analyze student understanding**

In the Results section we provided details of the categories that came out of our analysis. Here we discuss some further results that illustrate the usefulness of a categorization of this type. The first two questions of the interviews directly addressed students' concept images of function and linear transformation (see questions 1 and 2 in the Methods section).

By comparing each student's responses to these questions, we can see that certain metaphors are called upon more frequently than others when reasoning about function or linear transformations (see Fig. 2). These results provide an interesting resource in understanding how students see function and linear transformation as similar or different mathematical concepts.

When discussing function, the input/output metaphor (7 students) and the property of being an equation (3 students) were the most prevalent. By contrast when answering the same question for linear transformation, the morphing metaphor (5 students) and the machine metaphor (4 students) were most common. The traveling metaphor (2 students) was only used by students answering this question for linear transformations. Notice also that all but one of the students used different metaphors to answer this question for function than they did for linear transformation. However, when asked to indicate, on a scale from 1-5, to what extent you agree with the following statement: "A linear transformation is a type of function," all ten students marked 4 or 5 to indicate their agreement with that statement. Thus, these students may believe that function and linear transformation are related mathematically, they hold different metaphors about how these two concepts act.

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*Tables:*

| *Table 1:* Structure of metaphors | | | |
|---|---|---|---|
| Metaphor | Entity 1 | Middle | Entity 2 |
| Input/Output (IO) | Input(s) | Entity 1 goes/is put into something and Entity 2 comes/is gotten out. | Output(s) |
| Traveling (Tr) | Beginning Location(s) | Entity 1 is in a location and moves into a (new) location where it is called Entity 2. | Ending Location(s) |
| Morphing (Mor) | Beginning State of the Entity(ies) | Entity 1 changes into Entity 2. | Ending State of the Entity(ies) |
| Mapping (Map) | First Entity | Entity 1 and Entity 2 are connected or described as being connected by a mapping (a description of which First entities are connected to which Second entities). | Second Entity |
| Machine (Mach) | Entity(ies) to be processed | Machine, tool, device acts on Entity 1 to get Entity 2. | Entity after being processed |

| *Table 2:* How students initially explained function and linear transformation | | |
|---|---|---|
| **Student** | **Function** | **Linear Transformation** |
| Andrew | P(equation) | Tr |
| Brian | IO, Mor | IO, Mor |
| Dana | IO | Mor |
| George | IO, Mach | Tr, Mach |
| Jordan | IO | Mach |
| James | C1 | Mor, Mach |
| Lawrence | IO | Map, IO |
| Noah | Mach | Mor |
| Nadine | P(equation), IO | Mach |
| Ryan | P(equation), IO, Map | C1, Mor |

# MUSINGS ON INFINITE SAMPLE SPACE

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*We examine the responses of secondary school teachers to a probability task with an infinite sample space. Specifically, the participants were asked to comment on a potential disagreement between two students when evaluating the probability of picking a particular real number from a given interval of real numbers. Their responses were analyzed via the theoretical lens of reducing abstraction. The results show a strong dependence on a contextualized interpretation of the task, even when formal mathematical knowledge is evidenced in the responses.*

Consider the conversation between two students presented in Figure 1 and a teacher's potential response. The scenario is a familiar one – two students grappling with opposing views on a probability task. The task itself is less familiar – the likelihood of choosing a particular event from an infinite sample space.

> The following conversation occurred between Damon and Ava, two high school students. Imagine you are their teacher and that they have asked for your opinion. They approach you with the following:
>
> *Damon*: I asked Ava to pick any real number between 1 and 10, write it down, and keep it a secret. Then we wanted to figure out what the probability was that I would guess right which number she picked.
>
> *Ava*: Right. And we did this a few times. The first time I picked 5, and Damon guessed it right on the first try. The next time, since he said "any real number", I picked 4.7835. He never got that one.
>
> *Damon*: So, we tried to figure out the probabilities. I think that the probability of picking 5 is larger than the probability of picking 4.7835. Ava thinks the probability is the same for both numbers. Who's right?
>
> Please consider and respond to the following questions:
>
> 1. What is the probability that Damon would guess correctly the real number Ava picked between 1 and 10 when that number was: 5? 4.7835? How do you know?
> 2. Going back to Damon's question… Who is right? And also: Why are they right?

Figure 1: The Task

### *The study*

The task was presented to six secondary school mathematics teachers. Unlike conventional probability tasks, such as tossing a coin or throwing a die, a special feature of the presented task is that the embedded experiment – picking "any real number" – cannot be carried out. We examined different aspects of probability tasks, the context in which they are presented and the associated interpretations. We then considered the specific mathematics embedded in the task and analyzed participants' responses of as they addressed the scenario.

### *On platonic vs. contextualized*

Chernoff (2011) distinguished between platonic and contextualized sequences in probability tasks related to relative likelihood of occurrences. He suggested that platonic

sequences are characterized by their idealism. For example, when considering a sequence of coin flips, it is assumed to be generated by an "ideal experiment – where an infinitely thin coin, which has the same probability of success as failure, is tossed repeatedly in perfect, independent, identical trials" (p.4). In contrast, contextualized sequences are characterized by their pragmatism. For example, a sequence of six numbers when buying a lottery ticket was considered as contextualized. In fact, most probability tasks, found in textbooks or discussed in educational research pertaining to probability – such as tossing a coin, throwing a die, spinning a spinner – refer to contextual scenarios. However, it is a convention in mathematics, as well as in mathematics classrooms, to think of the events described in these tasks as platonic, as if they concern an infinitely thin coin, a perfect die, or a frictionless spinner. This convention is also accepted in educational research. Chernoff (2011) noted that early probability studies in mathematics education clarified "platonic assumptions" in accord with mathematical convention, such as "fair coin, equal probability for Heads and Tails". However, such conventions have been taken for granted in subsequent research and specific assumptions are omitted.

***On tasks, experiments and interpretations***

While Chernoff (2011) labeled sequences as platonic or contextualized, in what follows we refine this distinction as it applies to probability tasks in general. The event of "tossing a coin and getting heads" is presented in a context, and as such the task – in which we are asked to determine the probability of this event – is contextual. However, the experiment itself and the resulting event can be seen as platonic under the assumptions listed above. This means that it is the *interpretation* of the experiment and of the event that is platonic, rather than the experiment itself. As such, we consider standard tasks used in probability classrooms and in probability research as "platonicized by convention". To reiterate, the "platonicity" of an experiment is a feature of the individual's interpretation rather than a feature of the experiment itself.

However, we also note that there are contextual events to which a platonic interpretation is not applicable. Consider for example tossing an uneven 11-sided solid with numbered sides and landing it on a 7, or meeting a high school friend in a foreign country. Since the solid is asymmetric, and the factors in "meeting a friend" are not pre-determined, the probability of such events can be determined only experimentally or statistically.

We consider the task that is of interest in our study – that relates to picking a real number at random from a given interval – as contextual. As many other probability tasks, the scenario is described in a context, though unlike a coin toss, it cannot be carried out. As such, the experiment can only be imagined. The platonic interpretation of such an experiment considers the infinite set of real numbers as a sample space, where each number has the same probability of being picked. Chernoff (2011) has shown that when considering tasks that fall outside of those platonicized by convention (e.g. considering answer keys to a multiple choice test), students' interpretation is contextualized or embedded in their experience with the context of the scenario.

**Our objective** was to investigate whether individuals with strong mathematical preparation have a similar tendency towards contextualization, that is, whether their interpretation will be pragmatised when an "ideal experiment" is considered. Before exploring participants' interpretations, we take a closer look at the mathematics of our task.

***Probability and an infinite sample space***

Recalling the scenario presented in Figure 1, Damon and Ava argue about the likelihood of picking one real number versus another in the interval between 1 and 10. Mathematically, the probability of picking the number 5 is the same as the probability of picking the number 4.7835, even though pragmatically, the number 5 might be a more common choice. This hinges on the

fact that each number occurs exactly once in the interval, and thus each has the same chance of being picked. But exactly what is this chance? The sample space in question is the set of real numbers between 1 and 10, written as [1, 10]. As there are infinitely many real numbers in that interval, the chance of picking 5, and the chance of picking 4.7835, is 1 out of infinity.

To be more specific, we first must take a brief diversion to the realm of infinity. By definition, the sample space [1, 10] contains $\aleph_1$ many elements, where the symbol $\aleph_1$ represents the transfinite cardinal number associated with any interval of real numbers, including all of them. Transfinite cardinal numbers were defined by Cantor (1915) as analogues to the natural numbers. They describe the "sizes" of infinite sets, of which there are infinitely many. Cantor established corresponding definitions and algorithms for arithmetic. As such, we may say that by definition, 1 out of infinity, or more precisely, $1/\aleph_1$, is equal to zero. Thus, the probability of Damon picking 5, and the probability of him picking 4.7835, is $1/\aleph_1 = 0$.

As with other aspects of transfinite cardinal numbers (see e.g., Mamolo & Zazkis, 2008), dealing with probabilities and infinite sample spaces is paradoxical and counter-intuitive. As we allude to in our scenario, there is a paradox in the idea that the number 5 has a "zero chance" of being picked, and yet, it *was* picked. To give credence to this claim would take us beyond the scope of this paper, but we mention it here as interesting mathematics trivia, and also as an illustration of why, in our view, probability questions that involve an infinite sample space must be interpreted "platonically" – not only is it impossible to carry out any such experiment, but the reality (if we can call it that) and the mathematics are not in accord with one another.

Paradoxical elements aside, there are other conceptual challenges associated with the statement "the probability of picking the number 5 is zero". Chavoshi Jolfaee and Zazkis (2011) observed that for a group of prospective secondary teachers the sample space of probability zero events was predominantly comprised of "logically impossible" events, such as rolling a 7 with a regular die. They further noted confusion between an infinite sample space and a "very large" sample space, and as such between probability zero and probability that is "very small". Confusion regarding the distinction between a "very large, unknown number" and infinity is well documented (e.g., Sierpinska, 1987). This distinction coincides with what Dubinsky, Weller, McDonald and Brown (2005) interpreted as process and object conceptions of infinity via the lens of the APOS Theory (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996). For example, Dubinsky et al. juxtapose the process of counting numbers forever with the totality of a set with infinitely many elements. In the context of our scenario, this distinction is significant as it impacts how the sample space, and thus the probability of the event, is treated – that is, whether the probability *approaches* zero, or *is* zero. The existence of a "realized totality" of infinitely many elements is strictly conceptual, and hence necessarily platonic. As such, to address probability questions with an infinite sample space, both the experiment and the event must be interpreted as platonic.

### *Theoretical Framework*

The framework of "reducing abstraction" introduced by Hazzan (1999) is applicable to our study. As individuals engage in novel problem solving situations, their attempts to make sense of unfamiliar and abstract concepts can be described through different means of reducing the level of abstraction of those concepts. Hazzan elaborated on three ways to interpret abstraction level:

(1) Abstraction level as the quality of the relationships between the object of thought and the thinking person.
(2) Abstraction level as reflection of the process–object duality.
(3) Abstraction level as the degree of complexity of the mathematical concept.

We provide further elaboration of this framework as we present the analysis.

***Results and analysis***

Recalling Figure 1, we investigated participants' responses to two particular events:

- Picking 5 at random from the set of real numbers between 1 and 10
- Picking 4.7835 at random from the set of real numbers between 1 and 10

While mathematically both events have the same probability of occurring, and participants acknowledged this fact, they also added explanations and constraints that suggested, either explicitly or implicitly, that the first event was more likely than the second. This conclusion was rooted in participants' contextualized interpretation of the task. Some participants considered what numbers people might pick and how they interpret the game, others focused on popular, rather than mathematical, interpretations of "real" numbers or on the impossibility of actually carrying out the experiment. Further, while participants noted that the probability of both events "should be" equal, the value assigned to this probability was "almost" zero, rather than "zero".

Prominent trends that emerged as participants attempted to interpret the task may be grouped under four main themes: Randomness and people's choices; Real numbers – what numbers are "real" for students; Mathematics vs. Reality; The infinite and the impossible. We found in participants' responses to our task different attempts to contextualize the problem, that is, to impose pragmatic considerations on both the experiment and the event. In particular, participants attempted to make sense of the task via contextualizing the experiment by considering, e.g., what numbers individuals were likely to choose and why, or by considering what would happen if the experiment were actually carried out, and also via contextualizing the event by considering, for example, a context of infinity with which they were familiar. Table 1 presents a summary of the trends in participants' responses within each of the aforementioned themes. In what follows we analyze these trends through the lens of reducing abstraction.

| | | **Ernest** | **Albert** | **Alice** | **Sylvia** | **Kurt** | **J.D.** |
|---|---|---|---|---|---|---|---|
| **Randomness and people's choices** | | | | | | | |
| | What A&D know about D&A | ✓ | ✓ | ✓ | | | |
| | What A&D would do | | ✓ | ✓ | ✓ | | |
| | Likely numbers to choose | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| **"Real" Numbers** | | | | | | | |
| | For "average person" | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | In reality or the classroom | | | ✓ | ✓ | | ✓ |
| | 1/10 | | ✓ | | ✓ | | ✓ |
| **Math vs. Reality** | | | | | | | |
| | Experimental vs. Theoretical | ✓ | | | | ✓ | |
| | "Should be" equal | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | Realistically P(5) > P(4.7..) | | | | ✓ | ✓ | |
| | Experiment impossible | | | | ✓ | ✓ | |
| **The infinite and the impossible** | | | | | | | |
| | Almost impossible | | ✓ | ✓ | ✓ | | |
| | Approaches (or is?) zero, limits | ✓ | ✓ | ✓ | ✓ | | |
| | $1/\infty$ or almost | ✓ | ✓ | ✓ | ✓ | | |
| | Infinite sample space | ✓ | ✓ | ✓ | ✓ | | |

Table 1: Themes and Trends in Participants' Responses

*(1) Relationships between the object of thought and the thinking person*

Hazzan noted that the same object can be viewed as abstract by one person and concrete by another, that is, the level of abstraction depends on the person rather than the object itself. A powerful illustration of this idea is provided by Noss and Hoyles (1996) who suggested that "To a *topologist*, a four-dimensional *manifold* is as concrete as a *potato*" (p. 46). Hazzan and Zazkis also clarified "that some students' mental processes can be attributed to their tendency to make an unfamiliar idea more familiar or, in other words, to make the abstract more concrete" (p.103).

Contextualization – which is embedding the experiment in a (familiar) context – can be seen as reducing the level of abstraction by moving from the unfamiliar (or less familiar) to a familiar situation. Participants' responses that mention the impossibility of carrying out the experiment, that refer to possible relationship between the two students Ava and Damon, that consider what people usually do when choosing numbers can be seen as illustrations of reducing abstraction in accord with interpretation (1). Further, we suggest that embedding the experiment in a familiar context can refer either to a "realistic" interpretation of the situation, or to a previous mathematical experience. The former is exemplified by response of Kurt's, wherein he suggested the experiment was flawed "because we do not have a bag large enough to hold all slips of paper (each with a real number written on it)". With respect to the latter we consider participants' responses that referred to familiar mathematical contexts in which they dealt with infinity, specifically the context of calculus and limits. Such a calculus-based contextualization (combined with a lack of exposure to measure theory or transfinite cardinalities) resulted in determining that the probability "approaches" zero, rather than is zero. This consideration manifests explicitly in Alice's response that "the probability of picking the favourable outcome, tends towards 0… [but] it can't be 0", and also underlies distinctions such as the one made by Albert that "the probability technically is not zero, it is infinitely close to zero".

*(2) Process–object duality*

Researchers (e.g. Asiala et al., 1996; Sfard, 1991) agree that process conception precedes object conception of mathematical notions and in such process conceptions can be viewed as less abstract. As mentioned above, process and object conceptions of infinity are juxtaposed as distinctions between how the sample space of real numbers may be interpreted. The interpretation of the infinite set of real numbers as a process – e.g. a set with indeterminate size and numbers that go on forever – or as an object – e.g. a completed set that contains infinitely many numbers – influences how the probability of the event of choosing a specific number from that set is described – either as *approaching* zero, or as *equal to* zero. Several responses of our participants demonstrate process conception of infinity and therefore are in accord with interpretation (2). Further, an object conception of infinity goes hand in hand with a platonic interpretation of our task. As such any contextualization of the task which attempts to situate the experiment in terms of a process that could actually be carried out suggests an attempt to reduce the level of abstraction, and is also in accord with interpretation (2).

*(3) Degree of complexity of the mathematical concept*

Infinity is a complex concept. Embedding infinity-related ideas in probability situation adds further complexity. Hazzan (1999) relates the complexity of a mathematical entity to how compound it is, stating that "the more compound an entity is, the more abstract it is" (p.82). As such, an individual may attempt to reduce the level of abstraction of a compound entity by examining only part of it. Hazzan exemplified that students employ this kind of reducing abstracting when thinking of a set in terms of one of its elements, as a set of elements is more

compound than any particular element in the set. In our case, participants demonstrated thinking of a sample space of real numbers by referring to subsets of the real numbers, such as natural numbers or numbers with finite number of digits in their decimal representation. Thinking of a subset (and this particular case, a subset of lesser cardinality) is dealing with a less compound and more tangible object, and is in accord with interpretation (3).

***Conclusion***

Hazzan and Zazkis (2005) note "that these interpretations of abstraction are neither mutually exclusive nor exhaustive" (p. 103). This observation is definitely applicable to our data. For example, referring to a familiar game of picking a number among natural numbers can be described in terms of interpretation (1) as well as interpretation (3). Similarly, relying on calculus/limit interpretations of infinity corresponds to (1) as well as (2). Hazzan developed the framework of reducing abstraction and showed its applicability to interpret undergraduate students' thinking when they struggle with difficult-for-them, at least initially, mathematical concepts. What is partially surprising, that in the case described here, participants with rather strong mathematical background, who demonstrated their ability to approach the task on mathematical/theoretical level, also regressed to reducing abstraction and adding contextual considerations that were at times inconsistent with their formal mathematical solution. However, this finding is in accord with Chernoff (2010) study, that showed prospective elementary school teachers' tendency toward contextualized interpretation.

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# CHALLENGES AND TOOLS IN THE FACILITATION OF COMBINED PROFESSIONAL DEVELOPMENT AND RESEARCH SESSIONS: THE CASE OF COMMUNITY COLLEGE TRIGONOMETRY INSTRUCTORS

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*Facilitation in professional development and research contexts is a delicate craft. In the proposed paper we describe the facilitation of study group sessions among community college trigonometry instructors. The study groups were designed to collect data about instructors' practical rationality (Herbst, 2006; Herbst & Chazan, 2003). While these sessions took place to fulfill a particular goal and involved a particular population we believe that our facilitation methods can benefit any group in which the facilitator is responsible for managing public reflection. To this end, we describe questioning strategies for facilitation of the sessions that support productive conversations—that is, conversation that support both professional development and research goals. The notion of a productive conversation is developed in the paper.*

*Keywords*: trigonometry; community college; professional development

## Introduction

Facilitation in professional development and research contexts is a delicate craft (Borko, 2004; Elliott, Kazemi, Lesseig, Mumme, Carroll, Kelley-Petersen, 2009; Koellner, Schneider, Roberts, Jacobs, & Borko, 2008; Suzuka, Sleep, Ball, Bass, Lewis & Thames, 2009). In mathematics education these two contexts are often combined which implies that the facilitator often has two competing goals. The first goal is to ensure that the participants feel comfortable enough to share their ideas, that each participant is heard and respected, and that participants' individual comments join to form a cohesive conversation. The second goal involves uncovering and addressing some knowledge, skill, or disposition that is the target of professional development or eliciting some knowledge or information that is the target of the research (Nachlieli & Herbst, 2010). It is crucial that the first goal is met so that participants will make their reflections public so that both researchers and other participants can learn from them. The second goal ensures that these reflections are of a quality that is valuable to both the participants and the researchers.

In this paper we describe the facilitation of study group sessions among community college trigonometry instructors. The study groups were designed to collect data about instructors' practical rationality (Herbst, 2006; Herbst & Chazan, 2003), in particular, or the practical knowledge that instructors use to guide their instructional decisions. While these sessions took place to fulfill a particular goal with a specific population we believe that our facilitation methods can benefit any group in which the facilitator is responsible for managing public reflection. To this end, we describe questioning strategies for facilitation of the sessions that support productive conversations—that is, conversation that support both professional development and research goals. The notion of a productive conversation will be further developed in the paper. We recognize that there are many features of a session that support productive conversations besides questioning; we address here only those related to questions proposed by the facilitator.

## Information about the sessions

The sessions analyzed are part of a larger research study that seeks to investigate the nature of mathematics instruction at community colleges (Mesa, 2010, Accepted; Mesa, Celis, & Lande, 2011; Mesa, Suh, Blake, & Whittemore, 2011). Twenty instructors (ten full time and ten part time) were recruited from 12 different community colleges in Michigan and Ohio. Participants meet once per month for five months. Each session is three hours long and includes a mathematical activity (e.g., defining angles, constructing a protractor) and analysis of representations of instruction (e.g., a video of an online tutoring session, video of students' responses to an interview prompt, an animation of a classroom episode). We seek to fulfill social, mathematical, pedagogical, and research goals with each session. The research goals revolve around seeking information about the rationales that instructors have for doing or not doing certain things as they teach trigonometry.

## Methods

We analyzed the facilitator's questions and the participants' responses to those questions in the sessions described above. Facilitation questions are coded after the challenges that they address, as well as the research or learning goal they advance. Participants' responses are coded after their usefulness in answering research questions or evidence of participant reflection (Hatton & Smith, 1995). The aim of these methods is to empirically ground the development of a framework for productive facilitation advanced in the paper.

## Challenges

We have identified at least five challenges to productive conversations that the facilitation needs to overcome to produce productive conversations. The five challenges are: participants are disinclined to discuss mathematical ideas; participants tend to talk about instruction in general terms; participants avoid talking about the mundane features of instruction; participants are disinclined to provide justification for actions that are not supported by reform documents; and participants talk about individual instructors instead of instruction. Below we briefly describe the last three of these challenges.

### *Avoid talking about the mundane*

Participants are not inclined to share mundane details; instead they are inclined to talk about instructional events that are out of the ordinary. However, we are interested in learning about the work that participants do everyday in their classrooms so the challenge to the facilitation is to get participants to share the features of their instructional practice that are unremarkable. We believe that the day-to-day actions and decisions of instruction are the most productive site for making lasting and sustainable changes to participants' instructional practice.

### *Hesitant to provide reasons for actions that would be frowned upon*

Reform documents contain strong support for student-centered approaches to instruction; however, we have seen that instruction in community college mathematics classrooms is often content-centered. Because content-centered actions are frowned upon, participants are reluctant to admit that they perform them and reluctant to discuss reasons that support them. The challenge to the facilitation is to uncover and document the reasons that instructors have for performing these actions since we believe that instructors have valid reasons (real constraints) for using these forms of instruction.

### *Talk about features of instructors instead of instruction*

In the sessions participants may produce long monologues about their own instruction, learning experiences, or in cases where there is a representation of instruction, the participants may talk about the individual instructor in the representation. In our work we are interested in studying the

work of teaching, so we are interested in hearing about individual instructors insomuch as the experience of the instructor informs us about the work of teaching. It is essential for participant learning and for the research that comments be connected to the work of teaching in general, not just about individual idiosyncrasies of teaching.

These challenges to the facilitation of leading productive conversations could potentially interfere with the learning of the participants and with our research goals. While these examples came from our particular situation we believe that these challenges, or very similar ones, could affect other researchers and teacher educators who endeavor to facilitate sessions in which participants are asked to publicly reflect on their own teaching and decision-making. In the full paper we will provide a more comprehensive list of challenges and examples of how they manifest in the sessions.

**Facilitation questions**

Here we describe questions that the facilitator used as tools to overcome the challenges described. There are other design considerations that can contribute to productive conversations but we limit this discussion to the use of facilitation questions. In Table 1 we list the challenges to facilitation and the types of questions that address the challenge along with examples.

The set of questions listed in the first column of Table 1 address the challenge of avoiding talking about the mundane. These questions overcome the challenge by asking participants to consider specific moments in a representation of instruction. For example, in our sessions we found that our participants found it difficult to explain why they used examples to illustrate mathematical procedures or techniques. We could ask teachers to share other ways in which they might illustrate a mathematical procedure to highlight the benefits of working through examples.

The set of questions listed in the middle column of Table 1 address the challenge of hesitation to provide reason for actions that could be frowned upon. The proposed questions overcome the challenge by making the tacit assumption that there are conditions under which these actions are appropriate and asking participants to provide these conditions. Other questions ask participants about other sources of constraints on their instruction, such as students or administrators, and ask how these stakeholders encourage these actions. For example, in our sessions we found that participants initially claimed that they would never ignore a student question, however this is an action that we have seen happen repeatedly in community college trigonometry classrooms. We could ask teachers when it might be appropriate to ignore a student question to find the reasons that they engage in this action.

The set of questions in the third column of Table 1 address the challenge of talking about features of instructors instead of instruction. The proposed questions overcome the challenge by inviting other participants to share their experiences, expanding the conversation beyond one instructor. Another set of questions asks participants to consider the generality or specificity of the instruction being discussed. This strategy also takes the focus off of a single instructor and moves it to the setting in which the instructional move takes place. For example, one participant in our sessions was inclined to talk at length about her own learning of trigonometry. We could ask other participants if they have students who had similar experiences to find out more about the usual experience for community college trigonometry students.

**Discussion**

The proposed paper addresses a general question of what tools can facilitators use to address the challenges of managing public reflections on instruction in research and teacher education settings. We use the context of study group sessions with community college trigonometry instructors as a setting for exploring the work involved in this type of facilitation. These sessions

are unique because of their participants, learning goals, and research agenda, but the issues of sharing reflections for the purpose of education and research are shared across contexts.
We situated the work of asking questions to facilitate productive discussions in the "facilitation triangle" (Figure 1). In this triangle the vertices are the facilitator [F], the participants [P], and the representation of teaching [R]. We also recognize that these discussions take place in environments. Important features of the session that promote productive conversations can be located in this triangle, but we are focusing on the arrow between the facilitator and the connection between the participants and the representation of teaching. We see the proposed paper as contributing to understanding these interactions and therefore improving our capabilities to design and enact productive conversations among participants.

**Questions for the audience**

1. What are other challenges to facilitating sessions where public reflections are managed? 2. Could different research or learning goals lead to a different type of facilitation? 3.What do you think participants could learn from a session like this? 4. How can the nature of the artifacts used (video, animations of teaching) shape these conversations?

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Table 1: Questions addressing challenges to productive conversations

| *Avoiding talking about the mundane* | *Hesitation to provide reasons for actions that could be frowned upon* | *Talking about features of instructors instead of instruction* |
|---|---|---|
| • Questions about specific moments in a representation of instruction<br>  ○ "Would you do something different from this instructor at this moment?"<br>  ○ "What else could have the instructor done?"<br>  ○ "Why did you think the instructor acted the way she did in this moment?" | • Questions about the conditions of appropriateness<br>  ○ "When is it okay for wrong answers to be left uncorrected?"<br>  ○ "Is such a thing ever a reasonable action for a instructor?"<br>• Questions about constraints<br>  ○ "Do your students expect you to [act in a way that could be constructed as negative]?"<br>  ○ "Do your administrators expect you to [act in a way that could be constructed as negative]?" | • Questions about other participants' thoughts<br>  ○ "Do others agree that you would do the same thing?"<br>  ○ "Have others been in that same situation?"<br>• Questions about generality/specificity<br>  ○ "Are there other situations where this action would be appropriate?"<br>  ○ "What about this situation makes that action appropriate?" |

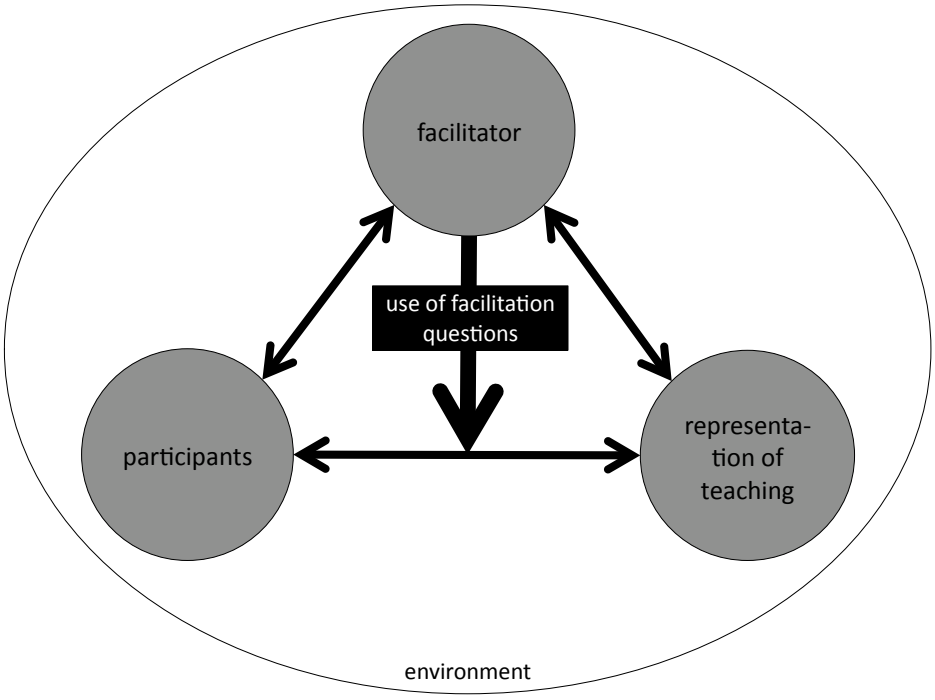


Figure 1: The facilitation triangle (Adapted from Cohen, Raudenbush, & Ball, 2003)

# Inverse, Composition, and Identity: The Case of Function and Linear Transformation

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**Abstract**

In this report we examine linear algebra students' conceptions of inverse and invertibility. In the course of examining data from semi-structured clinical interviews with 10 undergraduate students in a linear algebra class, we noted a proclivity for students to identify 1 as the result of the composition of a function and its inverse. We propose that this may stem from the several meanings of the word "inverse" or the influence of notation from linear algebra. In addition, we examined how students attempted to reconcile their initial incorrect predictions with their later computational results, and found that students who succeeded in this reconciliation made heavy use of what we termed "do-nothing function" ideas. The implications of this work for classroom practice include a possible method to help students develop object conceptions of function, as well as the need to pay more explicit attention to often-backgrounded notational issues.

**Keywords**
linear algebra, function, linear transformation, process/object pairs

## Background

The nature of students' conceptions of function has been well-studied (e.g., Sfard, 1991, 1992; Dubinsky & McDonald, 2001; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Sfard (1992) suggests that there are both structural (object-like) and operational (process-like) facets to the function concept, that structural conceptions are the result of reification of operational conceptions, and that processes are reified into objects that are then operated on by yet other processes. Sfard observes further that students' conceptions are usually closer to operational than structural, and that many students develop pseudostructural conceptions – that is, object-like conceptions that they cannot unpack to obtain the underlying process.

Much work has also been done examining students' understanding in the field of linear algebra in general (Dorier, Robert, Robinet, & Rogalski, 2000; Hillel, 2000; Sierpinska, 2000) and of the concept of transformation in particular (Dreyfus, Hillel, & Sierpinska, 1998; Portnoy, Grundmeier, & Graham, 2006). This study contributes to these bodies of research by examining the link, or lack thereof, made by students between the closely-related concepts of function and transformation, and the influence that knowledge from the one context has on the other.

## Methods

Data for this analysis comes from semi-structured clinical interviews with 10 undergraduate students in a linear algebra class at a large public university in the southwestern United States. Interviews were videorecorded and transcribed. In addition, students' written work was retained. Grounded analysis (Strauss & Corbin, 1994) was employed to analyze the data.

The interview covered a wide range of topics relating to students' understanding of the relationship between two mathematical contexts, functions in high-school algebra and transformations in linear algebra. As we began examining the data, we became particularly interested in how students reasoned about inverse and invertibility. In particular, we noted that all ten students predicted that the composition of a function with its inverse would yield 1. This surprising result informed our research questions: How can we account for these predictions? What reasons do students give that the composition of a function or transformation with its inverse should be 1? Also, how do students reconcile their incorrect predictions with the correct answer they later obtain? Accordingly, this analysis focuses on students' responses to the last few questions of the interview:

- Find the inverse of $f(x) = 3x - 9$.
- Find the inverse of $\mathrm{T}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \mathbf{x}$.
- What will you get when you compose $f(x)$ with its inverse that you found earlier?
    - Perform the composition. Does the result match your prediction? If not, is there some reason your result makes sense?
- What will you get when you compose $\mathrm{T}(\mathbf{x})$ with its inverse?
    - Perform the composition. Does the result match your prediction? If not, is there some reason your result makes sense?

## Results

When asked to predict the result of composition of $f(x)$ with its inverse, every one of the ten students answered 1 rather than the correct $x$. For several of the students, this is likely linked to conflating algebraic inverses and functional inverses. For example, when asked to give an example of an invertible function, Nicholas confused algebraic and functional inverses: "So say you have $x$, the inverse is $x$ to the negative 1, or 1 over $x$." Nicholas's mistake appears to stem

from the confusion between two concepts with very similar names that use the same notation, a superscript -1. Similarly, Naheem attributed properties of the algebraic inverse to the functional inverse. When asked to predict the result of composition of a function with its inverse, she stated that "if you take this one [$f(x)$] and multiply it by this one [the inverse], it's supposed to give you 1." This statement, while entirely incorrect in the realm of functions, is entirely accurate in the context of algebraic inverses.

Other students did not conflate algebraic and functional inverses, but symbolized their answers incorrectly. For instance, Grant described the identity function fairly clearly, but chose 1 to represent the result of the composition:

> Int: If I do f of f inverse of x, what do you expect it to come out with?
> Grant: Input, the input that you put in there. It shouldn't modify it.
> Int: If I haven't put in any input though, I'm just doing a calculation?
> Grant: [writes] It's just 1.
> Int: It would be 1?
> Grant: It's not going to change what you put in there, because if you do something and then you undo it, has it really changed?

This is attributable to backward transfer (Hohensee, 2011): the influence of the notation of linear algebra, where the notation representing a linear transformation, $T(\mathbf{x}) = A\mathbf{x}$, is often abbreviated to the matrix A alone. In particular, students may think that the identity *matrix* represents the identity *transformations* and overextend analogies. After all, as Grant reasoned, "this [circles 1] means this [circles identity matrix] when you're dealing with matrices," so since the identity matrix represents the identity transformation, 1 must represent the identity function.

Of the ten students, six were able to resolve the discrepancy between their prediction and their result. These six were exactly the six who expressed what we called "do-nothing function" (DNF) ideas, describing the result of the composition of a function (or transformation) and its inverse as being the function (transformation) that does nothing to the input. Joseph, for example, explained that he originally saw the function and its inverse as canceling to yield 1. Then, however, he decided that $x$ is a more reasonable answer, because "whatever you put in there, is whatever you're getting out." Joseph then used these DNF ideas to inform his correct prediction that the result of the composition of a transformation and its inverse should be $\mathbf{x}$, because "you're pretty much transforming it into something else, and … transforming it back to what it originally was." We conclude that DNF ideas provide students with a helpful lever to reason about functions, their inverses, and the composition of the two. In addition, we theorize that DNF ideas may indicate a robust process conception of function, as well as providing a bridge to object conceptions of function.

In our talk, we will present a case study of one student who was able to resolve the discrepancy between their prediction and their result, and a case study of one student who was not; these students have been chosen to be more or less typical of their respective categories. In addition, we will present as a third case that of Liam, who appeared to transition from a less sophisticated to a more sophisticated understanding with the help of the interviewers. We will also discuss several implications for classroom practice, warn of possible consequences of common notational abuses for students' conceptions of function, and outline what these data suggest teachers may be able to do to help their students develop object views of function.

**Discussion Questions**

1. How can students effectively distinguish between functional and algebraic inverses? At what point in students' education should we expect this not to be an issue?
2. What other instances of "backward" transfer might there be in students' undergraduate mathematical studies?
3. How might DNF thinking relate to process and object conceptions of function?

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# THE STATUS OF CAPSTONE COURSES IN THE PREPARATION OF SECONDARY MATHEMATICS TEACHERS

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*Capstone courses have been recommended as a way to connect the mathematics pre-service secondary mathematics teachers learn in college to the school mathematics they will teach in their own classrooms. Yet little is known about the status of these courses across the U.S, in whether they are offered, the topics that are covered, the curriculum used, and the pedagogical approach, among other aspects of the course. We will present findings from a 2011 survey of U.S. colleges and universities that investigated the status of such capstone courses at these institutions. Discussion will be centered around the importance and future of such courses in teacher preparation programs.*

*Keywords:* capstone course, teacher preparation, secondary, mathematics

## Research Issue

Hodgson (2001) recognized that pre-service secondary school teachers "have no explicit occasion for making connections with the mathematical topics for which they will be responsible in school, nor of looking at those topics from an advanced point of view" (p. 509). Such an experience is important as these future teachers need a "deep conceptual understanding of the school mathematics content which falls under their responsibility" (Hodgson, 2001, p. 512), and this should occur before their entry into their profession. Addressing this same concern, the Conference Board of the Mathematical Sciences (CBMS) recommended that pre-service high school teachers complete "a 6-hour capstone course connecting their college mathematics courses with high school mathematics" (2001, p. 8). Since that time, there have been a handful of reports on implementations of individual courses that fit this description (e.g., Hill & Senk, 2004; Loe & Rezak, 2006; Shoaf, 2000; Van Voorst, 2004). However, the status of the mathematics capstone course in the United States is largely unknown; there has thus far been no systematic study of the extent or characteristics of its varied implementations. The goals of this research study are to uncover the status of capstone courses across the United States, to understand what is offered to pre-service high school mathematics teachers, and to investigate whether CBMS recommendations are being followed by programs that prepare future high school mathematics teachers.

## Methodology

In 2011, we conducted a survey of universities that may offer an upper-level capstone course either in the mathematics department or in the college of education for mathematics majors pursuing secondary certification. From the 1,713 institutions listed by the Carnegie Foundation for the Advancement of Teaching (Carnegie Classifications, 2011) we

selected a stratified random sample of 200 institutions, weighted appropriately for each of nine classification groups (e.g., PhD granting institutions with high research activity). For the purposes of the survey, we defined a *capstone* as a course taken at the conclusion of a program of study for pre-service secondary mathematics teachers that places a primary focus on providing at least one of the following: (1) bridges between upper-level mathematics courses, (2) connections to high school mathematics, (3) additional exposure to mathematics content in which students may be deficient, or (4) experiences with communicating with, through, and about mathematics (Loe & Rezac, 2006).

The survey investigated the prevalence and nature of courses fitting this description. In particular, it included questions about course logistics such as the department, title, duration, textbook(s), technology, and other resources used in the course. The survey also included questions relating to the nature of the course; specifically, data were collected about the description of the course in the universities' course catalogs, the course goals, the instructional style, and the content. To provide a more complete picture of the current state of capstone courses, data were also collected about instructors' backgrounds and their levels of academic freedom. Data collection was completed in November 2011.

## Questions to be considered by the audience

The discussion portion of the presentation will be framed by an initial presentation of the general findings of our study. Specifically, we will share findings about commonalities and differences of capstone courses across the various types of institutions. Then, we will pose two questions to the audience for discussion:

1. **What are your experiences with capstone courses in relation to the national landscape, and what more would you like to learn about capstone courses, instructors, and students?** During future phases of this project, we will be soliciting institutions for a follow-up interview that will collect data to help us look more deeply at the methods used and nature of content taught in capstone courses. The discussion will provide direction and context for the next phase of the study.
2. **What resources or collaborations have the potential to support institutions wanting to offer new capstone courses or to improve the existing capstone experience?** The limited research on and discussion about capstone courses are cause for concern that institutions are building courses from the ground up without a sense of how their efforts fit with others. This discussion may lead to a sharing of ideas about capstone resources and, potentially, the formation of networks of support or collaboration.

The presenters will document the discussions and will share session notes (via email) with attendees and interested parties.

## Implications for the preparation of pre-service secondary mathematics teachers

Ten years after the recommendation for capstone courses by the CBMS, mathematics education researchers continue to emphasize the need for pre-service mathematics teacher training programs to make connections between university-level mathematics, teaching methods, and high school content (e.g., Artzt et al., 2011). This preliminary report will help uncover the extent to which this need is being addressed. Furthermore, the results of the survey may offer direction to mathematics departments wishing to create or to improve capstone courses. The discussion portion of the session will guide future phases of

our research agenda and will potentially foster impactful collaborations. Capstone courses offer great promise for enhancing pre-service teacher training; the research presented in this session, and the discussion it provokes, will provide insight into the popularity of this relatively new course, the variety of implementations, and the future of the capstone course.

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**Title**
Student Understanding in the Concept of Limit in Calculus: How Student Responses Vary Depending on Question Format and Type of Representation

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**Abstract**
Research indicates that calculus students have difficulties with limit. However, underlying reasons for those difficulties and possible influences of question format have not been examined in detail. Since limit is foundational to calculus it would help the mathematics education community to know not only the difficulties students have, but also how questions used to assess knowledge affect responses. Data for this study came from surveys administered to 111 first semester calculus students. Survey questions focused on limit in multiple representations including graphs, mathematical notation and definitions. Questions were multiple choice and free response. Student difficulties documented in previous research were evident in this population. Findings also indicated that difficulties students exhibited in one question were sometimes different then the difficulties those same students exhibited when asked about the same idea in a different representation. Students in general had less difficulty with graphical representations than mathematical notation or definition questions.

**Keywords**
Undergraduate students' thinking
Multiple representations
Survey question design

**Introduction**
Knowledge of how students understand mathematical topics can help inform and improve instruction. Because the limit concept is a foundational concept in calculus it would help the mathematics education community to know not only the difficulties that students have, but also how the questions used to assess their knowledge affect their responses. While research into student ideas and thinking has flourished, research into how students interact with the questions they are given is lacking. This study extends existing work on student thinking about limits by examining how students respond to questions given in different formats. In particular, students were asked questions that involved mathematical notation/symbols and ones that were based on graphs to investigate whether students demonstrated different levels of success on the differently formatted questions.

Other researchers have found that question format can significantly influence student responses. Some of this research has been performed in the context of attitudinal surveys (Tanur, 1992; Schuman & Presser, 1981) and the areas of confirmation bias (Nickerson, 1998), answer confidence (Koriat, Lichtenstein & Fischhoff, 1980) and response elicitation (Garthwaite, Kadane & O'Hagan, 2005). However this issue of links between question format and what data on student thinking is generated has not been examined for student thinking about limits. Knowing whether students perform differently on questions in different formats could be of

importance to researchers examining student thinking of limit as well as instructors who use written tasks to assess student learning. Knowledge about student thinking from this study will be used in future research on college mathematics instructors' knowledge of student thinking about the concept of limit in calculus.

**Student Thinking About Limit**

Research indicates that calculus students have difficulties with the concept of limit (Oehrtman, 2002; Oehrtman, 2008; Bezuidenhout, 2001; Williams, 1991). Researchers have found that first-year university students' knowledge and understanding are based on isolated facts and procedures (Bezuidenhout, 2001). Research has shown that students see limits as a boundary that cannot be passed (Williams, 1991; Davis & Vinner, 1986). Limit is also seen as an approximate value obtained through an evaluative process or by imagining points on a graph getting closer to the limit (Williams, 1991). Some students believe the limit is as an infinite process (Williams, 1991; Orton, 1983). Others see limit as a value reached at the end of a process (Orton, 1983; Davis & Vinner, 1986). Some of these conceptions are combined in a dynamic viewpoint where the limit can be deduced by finding function values closer and closer to a given point (Williams, 1991). Students also tend to show conflicting conceptions of limit, continuity, and differentiability (Benzuidenhout, 2001). These conflicting conceptions may be reflective of the informal mental models students have formed from prior experiences, including nonmathematical intuitions of limit (Williams, 1991; Oehrtman, 2002). Prior experience appears to play a role in the choice of finding a limit as well. Students' faith in the use of graphs and formulas may be due to hours of experiences using them (Williams, 1991). However, students often fail to apprehend the concepts involved when using graphs and formula.

A fair amount of what we know about student thinking about limits was generated with data from written surveys. This study focuses on the representation of the questions asked to illicit student conceptions. This study focused on student understanding of limit using data generated from tasks from multiple sources (see below for details about the research design) and using multiple question formats. The goals were to examine student responses to differently formatted questions and investigate interactions between question format and the knowledge of limit students displayed in those question formats.

**Research Design**

Survey data was collected mid-semester from 111 students in a first semester calculus course at a public university in the northeastern United States. Some of the questions were adopted or adapted from other researchers' studies on student thinking of limit (Benzuidenhout, 2001; Oehrtman, 2002; Williams, 1991) and some were created. Students were asked to explain their definition and meaning of limit in various context and representations. Students were asked to describe what limit means at the beginning of the questionnaire and also at the end as well. Two similar multiple-choice/multiple-answer mathematical notation questions were given using different limits to see if students would give consistent answers. Two graphical representations of limit that addressed the same concepts as the multiple choice questions were given in order to see if students could answer consistently across various representations. Lastly, a true/false multiple-answer question was given to see what definition of limit students' hold and with a final question asking which definition would best describe their definition of limit. Responses were coded using categories from other researchers' studies where possible (Benzuidenhout, 2001)

and using a Grounded Theory (Strauss & Corbin, 1990) approach in other cases. Responses were examined for correctness, inconsistencies between answers and between questions asked with different representations. Definition questions were checked for consistency throughout the questionnaire.

**Data**

Students showed a much higher correct response rate for graphical tasks than mathematical notation or definition tasks. The questions are appended to this report.

| Mathematical Notation Tasks | Graphical Tasks | Definition Tasks |
|---|---|---|
| 19.8% - correct (Q3) | 79.3% - correct (Q5) | 21.6% - correct (Q8) |
| 25.2% - correct (Q4) | 85.6% - correct (Q6) | |

Students answering both mathematical notation tasks (Q3 & Q4) correctly were coded for their response to the graphical tasks (Q5 & Q6). Similarly the students answering the graphical tasks correctly were coded for their response to the mathematical notation tasks. The results are given below:

| Both notation tasks correct = 14 | Both graphical tasks correct = 66 |
|---|---|
| 12 students correctly answered graphical tasks | 12 students correctly answered mathematical notation tasks |
| 2 students did not correctly answer graphical tasks | 54 students did not correctly answer mathematical notation tasks |

Responses to the mathematic notation tasks were coded for contradictory responses. Of the 111 students, 20 (18%) students had contradictory responses between questions 3 and 4. The response to question 4 was examined for mutual contradiction similar to the Benzuidenhout (2001) study where the researcher claimed that these contradictory responses indicated that students have an underdeveloped concept of limit. The contradictions from the current study are listed below:

- 19 students selected A but not B;
- 19 students selected A but not C;
- 29 students selected C but not B;
- 24 students selected E but not A;
- 20 students selected E but not B;

**Conclusions & Implications**

There was a significant difference in the number of correct responses to limit questions based on the representation of the question. This raises the question of which question type provides the more accurate information about student thinking. Additional studies are needed to further investigate these patterns and links between student thinking and question format. There were also some interesting patterns apparent in the more detailed analysis of student responses. In particular, the student response to mathematical notation and definition tasks were low even though those students had correctly responded to graphical tasks about limit. This difference could be due to student prior experience with graphical questions. Further research is needed to determine what it is that students who answer the graphical questions correctly understand about limit and why that understanding is not being demonstrated on the notation-type tasks. It could be that students may know how to respond to graphical tasks without having a solid conceptual foundation about limit or it could be that students are able to demonstrate their solid understanding of the ideas when interpreting a graph but are, for some reason, unable to do so

when reading the notation-type questions. There were inconsistencies among student answers in the mathematical notation representation questions. The contradictions between questions 3 and 4 were significant. There were also many mutually contradictory answers on question 4. Many researchers have suggested that students have multiple models of limit. The responses to question 7 seem to indicate that they do have a wide range of ideas about limit and have not learned a more formal definition of limit. Many students did select the formal model as a true definition for limit, but they did not exclusively select the formal definition. The choice of types of questions and representations used by researchers and instructors may have a significant impact on what knowledge of limits we ascribe to students.

**Questions for discussion during preliminary report**

1. If interviews were to be conducted with students who took the survey, what questions might help uncover the sources of the discrepancies of how they respond to questions?
2. Are there additional questions or question formats that should be included in future surveys if the goal is to further examine these patterns in student responses?
3. The next phase of this project is to examine college mathematics instructors' knowledge of student thinking about limit. What questions might be asked of these instructors to tap into their knowledge of the student thinking, including their knowledge of the impact of these format differences on students' performance on tasks?

**Appendix – (Survey)**

3) Given an arbitrary function $f$, if $\lim_{x \to 3} f(x) = 4$, what is $f(3)$?

a. 3
b. 4
c. It must be close to 4.
d. $f(3)$ is not defined.
e. Not enough information is given. ANS:_____

4) In this question circle the number in front of your choice(s).
Which statement(s) in A to E below must be true if $f$ is a function for which $\lim_{x \to 2} f(x) = 3$?
Circle letter F if you think that none of them are true.

A. $f$ is continuous at the point x = 2
B. $f(x)$ is defined at x = 2
C. $f(2) = 3$
D. $\lim_{h \to 0}\{f(2+h) - 3\} = 0$
E. $f(2)$ exists
F. None of the above-mentioned statements.

5) For this question, refer to the following graph:

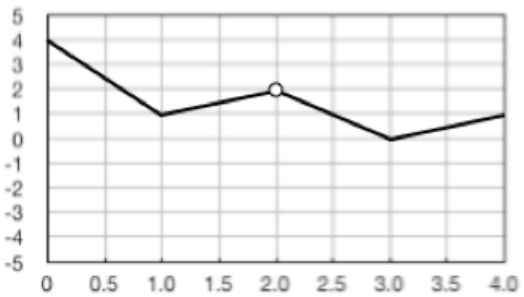


Figure 1

a) What is the value of the function at x = 2?
b) How did you figure out your answer to (a)?
c) Does the function have a limit as x approaches 2?
d) How did you figure out your answer in (c)?

6) For this question, refer to the following graph:

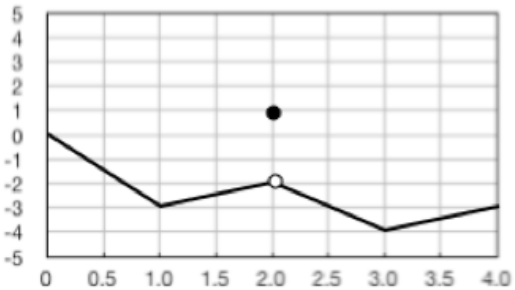


Figure 2

a) What is the value of the function at x = 2?
b) How did you figure out your answer to (a)?
c) Does the function have a limit as x approaches 2?

d) How did you figure out your answer in (c)?

7) Please mark the following six statements about limits as being true or false.

A. T F
A limit describes how a function moves as $x$ moves toward a certain point.

B. T F
A limit is a number or point past which a function cannot go.

C. T F
A limit is a number that the $y$-values of a function can be made arbitrarily close to by restricting $x$-values.

D. T F
A limit is a number or point the function gets close to but never reaches.

E. T F
A limit is an approximation that can be made as accurate as you wish.

F. T F
A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

8) Which of the above statements best describes a limit as you understand it? (Circle one)
A B C D E F None

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# Doing Mathematics: Perspectives from Mathematicians and Mathematics Educators

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Preliminary Research Report

**Abstract:** Learner-centered teaching strategies such as inquiry-based learning ask students to actively engage in the material they are learning, to do mathematics in order to learn mathematics. A teacher's interpretation of the meaning of "doing mathematics" is related to his or her beliefs about mathematics and about mathematics teaching. In this exploratory study, we report the results of interviews with sixteen university level mathematics and mathematics education faculty regarding their perspectives on the meaning of doing mathematics within the context of a calculus course, a proof-oriented course, and their own mathematical experiences.

**Key words:** teacher beliefs, mathematical tasks, communication

## 1 Introduction

One of the foci of the recent mathematics education reform effort has been to shift students' classroom experience to a more learner-centered model. In terms of undergraduate mathematics education, recent research has focused on the impact of inquiry-based learning. Inquiry-based learning refers to "teaching and learning approaches that engage undergraduates in learning new mathematics by exploring mathematical problems, proposing and testing conjectures, developing proofs or solutions, and explaining their ideas" (Hassi et al, 2011, p. 73). Proponents often contrast this approach with lecture, pointing out that "sitting still, listening to someone talk, and attempting to transcribe what they have said into a notebook is a very poor substitute for actively engaging with the material and hand, for doing mathematics" (Bressoud, 2011). Notice the phrase "doing mathematics." In inquiry-based learning, students are active participants that do mathematics in order to learn mathematics. In this study, we explore different faculty perspectives on "doing mathematics." In particular, is there a consensus among university level mathematicians and mathematics educators regarding the meaning of doing mathematics? Further, does the notion of doing mathematics depend on the course, or is it independent of the mathematical content?

## 2 Previous related research

Faculty perspectives on the notion of doing mathematics are connected to teacher beliefs regarding mathematics and mathematics teaching. Philipp (2007) provides an overview of research involving mathematics teachers' beliefs; we will highlight a few key points from that chapter. First, beliefs are fairly stable and resistant to change. Beliefs act as a filter for what we see, making change difficult without observation and reflection on practice. Second, teacher beliefs regarding mathematics and mathematics teaching correlate with instructional practice. For example, if a teachers' beliefs about mathematics have a calculational orientation, their classroom practice will tend to focus on developing procedural skills. On the other hand, researchers have also observed apparent inconsistencies between a teacher's stated beliefs and their actual classroom practice. In some cases, these inconsistencies can be explained by closer examination of the context. Finally, we should point out that the research regarding teacher beliefs summarized in Philipp (2007) involves preservice and

inservice K-12 teachers. We are unaware of similar research regarding the beliefs of college level mathematics instructors.

The phrase "doing mathematics" implies some type of activity. From this perspective, a variety of theoretical research articles attempt to categorize and describe mathematical tasks. For instance, Stein, Smith, Henningsen, and Silver (2000) group mathematical tasks into four categories. Tasks with lower level cognitive demand include memorization tasks and procedural tasks without connections. Tasks with higher level cognitive demand include procedures with connections as well as "doing mathematics" tasks. More specifically:

> The category *doing mathematics* includes many different types of tasks that have the shared characteristic of having no pathway for solving the task explicitly or implicitly suggested and therefore requiring nonalgorithmic thinking. This category includes tasks that are nonroutine in nature, are intended to explore a mathematical concept in depth, embody the complexities of real-life situations, or represent mathematical abstractions (p. 23).

Taking this a step further, Cuoco, Goldenberg, and Mark (1996) suggest that particular habits of mind, developed through a variety of tasks, should be an organizing principle of mathematics curricula. They argue that students should learn mathematics by engaging in activities similar to the activities mathematicians do. These include searching for patterns, experimenting, communicating, exploring ideas, inventing notation, visualizing relationships, and making conjectures. From this perspective, the overriding goal of the curriculum is to help students develop habits that enable them to be mathematically proficient, blending strands such as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001).

## 3 Methods

Sixteen university mathematics and mathematics education faculty members participated in the study. Faculty from both public and private liberal arts colleges and research institutions in the western United States were included. All participants had PhD's and had taught full time in a university setting for between one and 39 years. Participation in the study involved completing a written survey with some background information as well as a phone interview discussing their perspectives on "doing mathematics." The phone interview questions are listed below; the first four questions involve their expectations of students while the last three questions involve their own experiences with mathematics.

1. You are teaching a university level calculus course. What does it mean for one of your students to do mathematics in that setting?
2. What role do you think applications have in doing mathematics?
3. You are teaching an upper division proof-oriented mathematics course. What does it mean for one of your students to do mathematics in that setting?
4. What role do you think group work has in doing mathematics?
5. What does it mean for you to do mathematics?
6. What kind of activities do you do when you do mathematics?
7. What role do other people play in your doing mathematics?

Phone interviews were recorded and transcribed. Participant responses were reviewed and similar responses were placed together in some initial categories. A more thorough review of the transcripts with a corresponding revision of categories is currently underway.

## 4 Results

In this preliminary report, we will only discuss participants' responses to the questions regarding "doing mathematics" within the context of a calculus course, a proof-oriented course, and their own experience. While there was a great deal of variability in participants' responses, common phrases given in response to these questions, in decreasing order of frequency, are outlined below.

1. Mathematicians
    - Calculus: computation, application, conceptual understanding, problem solving
    - Proof-oriented: conceptual understanding, recognize logical structure, communicate
    - Own experience: developing content knowledge, original research, communicate
2. Mathematics educators
    - Calculus: conceptual understanding, making connections, reasoning, application
    - Proof-oriented: proving, conceptual understanding, making connections
    - Own experience: exploring concepts, proving, problem solving

For the majority of the mathematicians, there was a definite progression in terms of expectations between the different contexts, moving from a computational focus to a conceptual focus. Their own experience often involved learning new content (through papers, presentations, and communication with colleagues) in order to do original research. Interestingly, while proof and conceptual understanding are key for students and are essential in original research, mathematicians did not mention these terms when describing their own work. On the other hand, the responses from mathematics educators were much more consistent across the different contexts. Exploring and understanding concepts, making connections, and logical reasoning were common responses to all three questions. In fact, several mathematics education faculty indicated that doing mathematics was essentially the same at any level. It is important to note that mathematics education faculty reported rarely if ever teaching either calculus or a proof-oriented course.

## 5 Discussion

While there was some overlap and similarities between individual responses, our data indicates that there is not a general consensus among mathematicians and mathematics educators regarding the meaning of the phrase "doing mathematics." Further, for many individuals the meaning was highly dependent on the context; for others the meaning was quite consistent with small changes in focus. Returning to the broader context of beliefs about mathematics teaching and learning, Sfard (1998) distinguishes between two metaphors for learning. In the *acquisition metaphor*, learning is viewed as acquiring or accumulating conceptual knowledge. This contrasts with the *participation metaphor*, where learning is conceived as a process of becoming a member of certain community in which individuals use a common language and act according to certain social norms. One way of interpreting our data is that mathematicians tend to have a more aquisititionist perspective, expecting students to acquire specific knowledge and skills as they progress, while mathematics educators lean towards a participationist perspective, expecting students to participate in increasingly sophisticated ways of exploring and reasoning about mathematical ideas. In discussing these metaphors for learning, Sfard (1998) argues that, "Naturally, the discussion between the participationist and acquisitionist is bound to be futile ... It takes a common language to make one's

position acceptable - or even just comprehensible - to another person" (p. 9). Similarly, when discussing perceptions about "doing mathematics" there is a danger that individuals might use the same words to mean different things. This potential communication issue may interfere with attaining our common goal of improving mathematics teaching and learning.

## 6 Discussion questions

- Are you aware of previous research regarding university mathematics teachers' beliefs?
- Are there other questions or types of questions that we should ask to get a better sense of faculty perspectives on "doing mathematics"?
- What are the implications of this research?

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# Calculus Student Understandings of Division and Rate[1]

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Abstract: We have conducted a preliminary investigation of university Calculus students' conceptions of division and rate of change because these ideas are used to define the derivative. We conducted exploratory interviews focused on building models of student understandings of division and rate. Retrospective analysis revealed the students interviewed had a variety of meanings for these concepts. Difficulty thinking about division as multiplicative comparisons of relative size was observed in multiple students. Additionally a student who explained rate as an amount added in equivalent x-intervals struggled to determine if a quantity was changing at a constant rate over unequally spaced x intervals. We hypothesize that difficulty conceptualizing division as quotient, and quotient as a measure of relative size[2] of two quantities, obstructs students' understandings of average and instantaneous rate of change. This research will further our goal of understanding student difficulties with derivatives.

Key words: *calculus, derivative, rate of change, division, student thinking, multiplicative thinking*

## Introduction and Background

As Thompson and Saldanha urged, we take seriously the idea that "how students understand a concept has important implications for what they can do and learn subsequently" (Thompson & Saldanha, 2003, p. 1). Understanding is "what results from a person's interpreting signs, symbols, interchanges or conversations-assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same" (Thompson & Saldanha, 2003, p. 12). We believe students build particular meanings for mathematical ideas by building on preexisting understandings (Steffe & Thompson, 2000a). Based on a conceptual analysis (Thompson, 2008) of the concepts of constant and average rate of change, we believe that conceptualizing division and rates as a multiplicative comparison of relative size is essential to understanding the derivative as a rate of change function. We interviewed university Calculus students to create models of their meanings for division and rate so that we can address the question "How do Calculus students understand division and rate?"

Our inquiry into Calculus students' meanings for division and rates of change emerged from observations of our own Calculus students and research on rates of change, division and derivatives. Asiala et al. (1997) summarizes a variety of studies that show that most Calculus students do not have a strong conceptual understanding of the derivative and struggle to solve non-routine problems. In Orton's (1983) study of student understanding of the derivative, he found that the rule where one divides the difference in $y$ by the difference in $x$ to obtain a rate

[1] Research reported in this article was supported by NSF Grant No. MSP-1050595. Any recommendations or conclusions stated here are the authors and do not necessarily reflect official positions of the NSF.

[2] It is more appropriate to say "relative magnitude" instead of "relative size" to account for comparisons of quantities of different physical dimensions (e.g., distance, time) but space is insufficient to explain this fully.

was not elementary for a large number of students. Orton (1983) alluded to the possibility that "one of the problems of learning about rate of change is that the ideas are basically concerned with ratio and proportion" (p. 243).

Carlson et al.'s (2002) study of 20 high-performing Calculus students revealed that most students struggled on tasks involving average and instantaneous rate of change. Although most students "were frequently able to coordinate images of the amount of change of the output variable while considering changes in the input variable", students were typically unable to coordinate changes in a function's average rate of change with uniform changes in the input (Carlson et al., 2002, p. 372). Most students did not understand situations where rates must be considered as multiplicative comparisons of changes in two variables. They were successful in describing rates of change as additive changes in the output.

Castillo-Garsow (2010) provided a model of one high performing secondary student's meaning for rate that could explain why students find understanding rates of change in Calculus challenging. For this student, an interest rate told her how much money to *add* to a bank account each year. Thinking of a rate as an amount added results in correct interpretations of situations as long as one always considers uniform changes in the independent variable. The student reworked problems with fractional amounts of one year into whole numbers of months so that the denominator of her division problem (change in money)/(change in time) was one unit. This allowed her to ignore division and consider additive changes in account balances. Simon and Blume (1994) cite studies indicating that many other students think additively when multiplicative thinking is more appropriate.

Coe (2007) conducted an in-depth study of three secondary math teachers' understandings of rates of change and revealed experienced teachers were not always able to articulate coherent connections between ideas of division, rate, and slope. For one teacher, Peggy, "the slope of a tangent gives a steepness that connects to speed in some contexts" (E. E. Coe, 2007, p. 176). Coe (2007) reported that in more than one instance Peggy "did not use her thinking of a ratio as a comparison of values" to understand slope (p. 195). Considering slope as an index of slantiness allowed this teacher to correctly answer many questions without thinking about division. Coe (2007) concluded that none of the teachers "could clearly explain the use of division to calculate slope" and "there was no evidence of quantitative understanding of the ratio" (p. 237).

The transcripts of students in Castillo-Garsow's (2010) and Carlson's et al.'s (2002) studies suggests that the students thought about rates of change additively. In problems that would prompt multiplicative thinking, the students invoked "workaround" strategies including only considering rates of change on increments of equal size (usually 1), and thought of speed and slope as indices instead of as ratios. Since understanding division as relative size is an essential mathematical component in many problems identified as obstacles for students, we investigated our students' meaning for division and rate to see if they had meanings for these topics that would allow them to understand derivatives.

## Methodology

To build models of students' meanings for division we used Simon's (1993) descriptions of partitive and quotative meanings for division. These two meanings for division do not require multiplicative reasoning. A third model for division, relative size, requires students to reason multiplicatively; the relative size model for division calls upon a comparison between the size of one quantity with respect to another quantity (Thompson & Saldanha, 2003). Division as

relative size allows students to be able to reason about non-integer divisors. If division is viewed partitively, it only makes sense to divide a number into n equal parts if n is an whole number.

In order to investigate the understandings/meanings that calculus students might have for division and rate of change, we conducted exploratory interviews with seven undergraduate calculus 1 students, guided by the theoretical perspectives of Steffe and Thompson (2000b). Our interview protocol contained tasks and questions that had been used in class or in other research on understandings of division (See Ball, 1990; Simon, 1993). For example, "Describe a situation where you would need to divide 6 by 3/4ths." or "How can you tell if your puppy is growing at a constant rate?" We conducted retrospective analysis to create models for students' understandings of division and rate. In our exploratory interviews, we attended to the idea that phrases students used such as "constant rate" do not necessarily mean the same thing to them as to us.

## Preliminary Results

Preliminary results from our research confirm that individual students held various (and sometimes unproductive) meanings for division. Additionally, students with partitive meanings for division struggled to interpret answers to division problems involving decimals and struggled to provide a context where division by a fraction is needed to solve a problem.

Jack had strong quotative meanings for division but struggled to interpret the quotient as a measure of relative size. When asked to determine if a puppy was growing at a constant rate he explained that if it is measured on equally spaced intervals of time you can compare the changes in height using subtraction. He proposed if the changes in height are equal the puppy is growing at a constant rate. When asked what he would do if he had measurements corresponding to unequally spaced intervals of time, Jack could not use a multiplicative comparison to show the puppy was growing at a constant rate. Eventually he guessed that division might be an appropriate operation, but was unable to identify the expression "four units of height divided by two days" as a rate of growth. Jack's definition for proportionality referred to quantity A growing by *a* units every time quantity B grows by *b* units, which was consistent with his additive thinking about rate of change but distinct from thinking that changes in A are a/b times as large as changes in B.

Another student, Arlene, had been successful on high school Calculus assessments but had additive and procedural meanings for division. Arlene saw division as a command to perform a calculation. She also struggled to explain how 29.66 related to 0.236 when given the statement $7 \div 0.236 = 29.66$. Consistent with the findings of Ball (1990), Arlene's quotative meaning for division broke down when prompted to give a scenario where one would need to divide six by three-fourths. When asked to explain what $6 \div (3/4)$ meant, she invoked the rule of "skip-flip-and-multiply", explaining that this "is what we learned to do" and then gave a numerical answer instead of a meaning or a sensible scenario. Later on, Arlene could not explain why one divides in the slope formula, exclaiming, "I don't really see it as division…I see that there is division but when I think of it in terms of slope I don't, I don't see that." Like the teachers in Simon and Blume's (1994) study, Arelene was inexperienced in representing a physical situation with a mathematical relationship.

Don, who planned to teach high school math, revealed a dominant partitive scheme for division. Don stated that he would emphasize using the long division algorithm to his future students. As a real world example for 37 divided by 3, Don suggested to partition 37 pencils into 3 groups, and later modified his example to each pencil being a bag of 10 M&M's so that he

can divide the M&M's into three equal groups. (Don didn't notice that multiplication by 10 doesn't make 37 divisible by three.)

Another mathematics education student, Cindy, possessed strong quotative meanings for division. She was able to correctly determine when division was an appropriate operation and construct situations where division by fractions was necessary. However, when explaining what an idea like proportional meant she used additive descriptions and struggled to explain why we divide when we find a slope. This strong student was able to correctly solve many problems but still offered primarily additive explanations.

## Early Conclusions

Given our preliminary interviews we believe that it is possible that many Calculus students do not understand quotient as a measure of relative size and will be unable to make sense of average and instantaneous rate in the ways needed to understand derivatives. For example if one thinks of rate as an amount added, common explanations of the derivative which ask students to envision the numerator and denominator of a difference quotient becoming arbitrarily small do not make sense. If a student believes a rate is the amount added to the output instead of a multiplicative comparison, the rate is getting smaller and smaller in the limiting process because the change in y values is getting smaller and smaller. If they understand rates as an index of slantiness of a line, then the derivative is a way to measure a geometric property of a graph and they might not attend to the changing quantities being compared. We plan to conduct individual teaching experiments with pre-service secondary teachers to build models of how they understand division and associated concepts such as multiplication, rate, measure and fractions. We aim to understand why thinking of quotients as a measure of relative size appears to be so challenging.

## Questions for the Audience

How can we promote understandings of division as relative size?
In the research that you do, are there any concepts related to division that students struggle with?
Can you think of any alternative explanations/models for our data?
Why do you suppose articulating meanings for seemingly elementary topics is so difficult?

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# A Study of Abstract Algebra Textbooks

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**Abstract**: This study will use reader-oriented theory and the analysis of example spaces to understand abstract algebra textbooks. Textbooks can lay the foundation for a course, and greatly influence student understanding of the material. Multiple undergraduate abstract algebra texts were studied to investigate potential audiences of the books, the level of detail in explanations, examples, and proofs, and the overall material included in the book. Conclusions were drawn regarding some discrepancies between the intended reader and the actual reader and the appropriateness and differences among example spaces.

**Keywords**: Textbooks, Abstract Algebra, Reader-Oriented Theory, Example Spaces

**Theory**: Although there has been significant research on mathematics textbooks, much of it has focused on the K-12 level (K-12 Mathematics Curriculum Center, 2005). The calculus reform movement motivated an extension of the study of textbooks into the collegiate level, but still the focus remained on lower-level mathematics or calculus books. Little work has been done to investigate the use, purpose, strengths, and disadvantages of upper-level mathematics textbooks, especially for an abstract algebra course. Many teachers, even in abstract algebra, use the textbook as a foundation, if not an outline, of the course material. As Robitaille and Travers (1992) stated, "Teachers of mathematics in all countries rely heavily on textbooks in their day-to-day teaching, and this is perhaps more characteristic of the teaching of mathematics than of any other subject in the curriculum. Teachers decide what to teach, how to teach it, and what sorts of exercises to assign to their students largely on the basis of what is contained in the textbook authorized for their course."

Authors, even within the field of undergraduate abstract algebra textbooks, have different intentions for the content and use of their texts. Also, generational differences on how mathematics should be presented and learned can affect the language and style of the text. Modern theories of learning indicate the need for student-oriented teaching methods and reader-oriented textbook methods (Weinberg & Wiesner, 2011). Teachers, and textbooks, are no longer meant to simply "cover" material, but should facilitate a learning environment that inspires curiosity, speculation, inference, and quantitative literacy. Student thinking, and the multiple strategies that it may involve, should be valued (Reys, B. J., Reys, R. E., & Chaves, O. (2004).

Reader-oriented theory, although not a new concept in general, was recently applied to the specific area of mathematics textbooks by Weinberg and Wiesner (2011). Within this theory, the use of textbooks moves beyond considering them as a static collection of ideas from which meaning is extracted, and instead considers a student's active engagement with the material and the processes of reading and understanding. In other words, "the meaning of a text does not reside in the text itself, but rather is generated through a transaction between the text and the reader…" (Weinberg & Wiesner, 2011). This theory takes into consideration the *intended*, *implied*, and *empirical* reader. In other words, the author's intended audience, the audience that would truly understand the text, and the actual audience. When the three readers do not match, or even when just the implied and empirical readers do not correspond, the success of the book in terms of student comprehension and engagement is lessened.

Another aspect of textbooks that can influence reader understanding, and which also can illustrate the intended, implied, and empirical reader, is that of example spaces. The creation of

examples is essential in the teaching and learning of mathematics. They are used for reference and as a means to generate other examples, conjectures, and perceptions (Bills & Watson 2008; Alcock & Inglis, 2008; Michener, 1978). Examples, and non-examples, of a theorem can aid in the process of proving the theorem and understanding the conditions involved. Example spaces are similarly needed for definitions, because they can demonstrate the importance and use of particular aspects of the definition. To achieve clarity, the examples should also differ along a narrow set of parameters (Fukawa-Connelly, Newton, & Shrey, 2011; Goldberg & Mason, 2008). Interestingly, the knowledge gleaned from being presented with examples does not seem to be as great as when students generate examples on their own (Dahlberg & Housman, 1997). Zazkis and Leiken (2008) emphasize the importance of students creating their own examples, both to the students and the instructor who is trying to evaluate student comprehension. Textbooks obviously present the reader with examples, but are the example spaces appropriate? Do they texts include essentially the same examples, leading to a *conventional example space* that teachers then expect their students to become familiar with (Watson & Mason, 2005)? The reader should be given a range of illuminating examples, but also should be led to generate personal examples through the text or exercises. The combination of the two ways to enhance an example space seems to be the best way to increase initial understanding of a concept.

**Methods**: In this study, over a dozen abstract algebra textbooks were considered, some of which were later editions of another text in the collection. The years of publication ranged from the 1960s to 2010. Many popular texts were used, such as Fraleigh's *A first course in abstract algebra* (2003, 1976), Gallian's *Contemporary abstract algebra* (1994, 2010), Herstein's *Abstract algebra* (1986) and *Topics in algebra* (1964), and the classic textbook, *A survey of modern algebra,* by Birkhoff and Mac Lane (1965). Sometimes, specific content areas like rings and groups, which could be found in all the textbooks, were examined. Other questions led to a consideration of the book as a whole.

One method of analysis that was used in this study involved reader-oriented theory. Within this framework, I tried to find characteristics of the intended, implied, and empirical readers. Many times information given in the preface of the book served as an indicator of the intended reader. Other factors under scrutiny were the language used by the author, the example spaces, the style of proof, and the level of detail given in explanations. For instance, when the author uses the pronoun "we" or imperatives such as "suppose", then he or she indicates that the reader is part of the mathematical community and a peer of the author (Rotman, 2006). On the other hand, when over a dozen examples are given for one definition an author implies that the reader requires more guidance and need not develop their own examples. This indicates a discrepancy between the implied reader and the intended reader, and could lead to a limited level of discovery and understanding by the student. The style of proof can be revealing as well. Differences such as paragraph style versus list style, or more details versus fewer, give evidence of what knowledge the reader is expected or needs to possess in order to comprehend the proof.

Of particular interest in this textbook analysis were the example spaces of the textbooks. The examples for rings, groups, and equivalency classes have been examined. Some assessments under consideration include: number of examples, types of examples, and difficulty of examples. Also taken into account were the examples that were given or asked for in the exercises.

**Preliminary Results**: Results thus far point to some discrepancies between the intended, implied, and empirical reader of abstract algebra textbooks in terms of maturity of language and

style. However, the type of examples seems to match nicely with the intended reader. The prefaces indicate that the authors are well aware that their reader is a student, but the language and level of detail are often appropriate for an experienced person from the mathematical community. For example, most authors seem to use a paragraph style of proof with the minimum number of steps or details. Some of the proofs refer to previous theorems or lemmas by number without any description, even though it is likely that most students do not memorize theorem numbers and may not look them up. Although the intended reader is usually a student in their first abstract algebra class, the implied reader is a mathematician comfortable with sophisticated proving techniques.

One aspect about example spaces that stood out, coming from sections on equivalence classes, was the contrast in how many real world examples were used, both within the section and in homework exercises. Gallian, who wants students to see that the "concepts and methodologies are being used by working mathematicians, computer scientists, physicists, and chemists," listed many applied exercises and motivated the topic with examples in a physics setting (2010). Birkhoff & Mac Lanes's motivating example was the classic modulo 12 description of how we measure time, which corresponds to their desire to use "as many familiar examples as possible" (1965). Herstein, who aimed for a "chatty" presentation and to "put the readers at their ease," has the first example set in a grocery store (1986). The most recently published textbook that I examined, by Bergen, included sixty-six exercises with no applied problems. Bergen, in the preface, explains that abstract algebra can especially help those who plan to teach mathematics at the high school level by clarifying the concepts encountered in high school (2010). It seems that the number of applied examples correlates with the goals and objectives of the authors in terms of their intended reader.

Despite the differences in real-world examples of equivalence classes, after comparing the other examples for equivalence classes as well as groups and rings, preliminary results indicate that the example spaces of abstract algebra textbooks are remarkably similar. Often, as new editions or new books are published more examples are added to the texts, but even those examples have distinct parallels. This indicates that the authors tend to agree on which examples best demonstrate a definition or theorem, creating the *conventional example space* that Watson & Mason describe (2005). The large number of examples and exercises in the texts, however, may not be beneficial to students. There is little to no motivation for the reader to generate their own examples and hypotheses. For instance, the definition of a ring may be followed by examples that are commutative, non-commutative, with unity, without unity, fields, or not fields. The reader has no need to think deeply about the definition or theorem to create such examples since they are immediately given.

**Questions**:

1. The quantity of textbooks that I have available make the study time-intensive and it is hard to succinctly describe the differences and similarities. Would the benefits outweigh the disadvantages of considering *every* textbook for *every* question?
2. Should I narrow the focus to look only at one example space, such as equivalence classes?
3. I wanted to consider the changes that abstract textbooks have taken over time (1960s to 2010), and did find some interesting patterns. How can I figure out why certain examples began to take precedence over others, and why certain theorems became more or less important?

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# AN EVOLVING VISUAL IMAGE OF APPROXIMATION WITH TAYLOR SERIES: A CASE STUDY

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*This paper will take a close look at the construction of a graphical image for reasoning with approximation in the context of Taylor series. In particular, it is a comprehensive case study of the genesis and evolution of an image created by one student, who draws extensively on other images and knowledge from calculus and physics to supplement gaps in his understanding of Taylor series and reason with Taylor series approximation tasks. His process resulted in a graphical representation that was leveraged to build knowledge and reason with the situation, even while lacking key considerations that are central to an understanding of Taylor series. The preliminary report sets the stage for a paper that will speak to considerations both of students' understanding of particular content, as well as a detailed examination of the processes of constructing a visual image used for problem solving and obtaining and utilizing evidence to amend that visual image.*

*Keywords:* Taylor series, graphical representation, calculus

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Taylor series are a wildly valuable tool in many professions, but students' reasoning with this topic is vastly understudied. What sense do students make of Taylor series? Do they have any image at all for Taylor series and what they're used for? The little research on students' understanding of Taylor series speaks mostly to broad themes of characterizing expert/novice strategies (e.g. Martin, 2009), tendencies for reasoning with them (e.g. Alcock & Simpson, 2004 and 2005), grappling with formal definitions of convergence (e.g. Martin et al, 2011), or use of technology in instruction (e.g. Yerushalmy & Schwartz, 1999; Soto-Johnson, 1998). That is, most of these important studies on students' use and understanding of Taylor series take a more global perspective, examining general themes, post hoc. But to develop a robust knowledge of the concept of Taylor series requires the synthesis of many previous calculus content topics, woven together and used appropriately, to form a more complete image. The question of *how* students synthesize their prior knowledge and arrive at *their* image has not been studied. That is, we have little idea about how students *construct* an understanding of Taylor series from less formal prior calculus notions, and how they attribute meaning to particular aspects of whatever representation of a Taylor series they espouse.

Moment-to-moment analyses about the construction of a mathematical topic are crucial not just to uncover 'misconceptions' that particular students have about the topic, but also to be able to put their responses to tasks about those topics in context. Habre (2009) discovered that even multiple exposures to the topic of Taylor series, at varying levels of mathematical sophistication, are often insufficient for even a broad comprehension of the material. Thus, knowing how students build their understanding can put into perspective some of the issues that persist around this topic.

Though it is not always students' tendency to produce visual images for Taylor series tasks, many do (including the student in the case discussed in this preliminary report). Access to students' visual images, supplemented by their descriptions and explanations, can provide

additional insight into how they are constructing an understanding of topics such as Taylor series, as visualization is "a fundamental aspect to understanding students' constructions of mathematical concepts" (Habre, 2009). Martin (2009) showed unsurprisingly that mathematicians were more fluent than novices in using graphical representations, both in their construction and interpretation, in the context of Taylor series. His dissertation made clear that "many students do not have a good visual image, if they have any visual image at all, of the convergence of Taylor series" (p. 288). Biza, Nardi, and Gonzales-Martin (2008) agree, citing an additional lack of useful imagery in textbooks chapters that students may use for reference. In our experience and works in progress, which align with Alcock and Simpson (2004), many students do in fact turn to visual images to explain and reason with Taylor series tasks. In fact, Alcock and Simpson (2005) also demonstrated that even "non-visualizers" may have a reliable graphical image, but tend to not call on it. So, in this paper, we endeavor to study, with a moment-to-moment analysis, the creation of one student's graphical image that *he* chose to use to play out his reasoning with Taylor series approximation tasks.

## Research Questions

The strength of the case to be presented in this report is two-fold. As it is a detailed examination of the development of a student's reasoning, as it plays out *graphically*, following this student's process with a moment-to-moment analysis can allow for an examination of what he takes as calculus-based and physical evidence for claims he is making in his reasoning, and how those claims are manifested in his graphic. Second, and much more content-specific, Taylor series literature largely examines students' (graphical, and other) reasoning or presentation at the *completion* of a problem, rather than as it is being *built,* negotiated from one moment to the next (exceptions include Martin et al, 2011, which focuses on formal definitions). Therefore, the exploration of this case will speak to the following:

(1) How is additional evidence germane to a problem gathered and used to amend a visual image that serves to represent a particular concept for a student?

(2) In what ways are prior calculus concepts negotiated to construct and attribute meaning to a representation of Taylor series?

With the case presented here, these questions can *only* be addressed for one particular student, but can be used as a model both for future analyses, and to highlight ways in which calculus-based reasoning *can* (and does) influence students' understanding of Taylor series.

## Data Collection and Methods

The study makes use of a particular 1.5-hour semi-structured interview with sophomore physics major Joe, who was participating in a larger, related study. Though it will not be discussed in this paper, the purpose of the larger study was to investigate students' consistencies (and inconsistencies) in reasoning around a set of approximation tasks in calculus and physics contexts. The tasks discussed in this paper are the only two in the larger task set that elicit students' thinking about approximation in the context of Taylor series (the text of which appear in Figs. 1 and 2). They are intentionally vague, and written to allow participants great freedom in what they choose to attend to as they respond. The interview was videotaped and transcribed for analysis. At the time of the interview, Joe had taken three semesters of calculus and two semesters of physics, and earned grades of "A" in all of them. He was identified by instructors as very competent in the subject matter. Upon completion of data collection for the larger study, Joe's interview stood out for several reasons, of interest here are those related to his construction

of Taylor series images. An in-depth, microgenetic analysis of this interaction between the student and the calculus content at his disposal seemed a promising way to examine *change* in his notions of Taylor series approximation on a more fine-grained level than would be ascertained in other assessment situations (Calais, 2008).

The Taylor series about $x$=0 for arctan($x$) is given by: $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$

How big a value can $x$ be, before stopping after the second term is a bad approximation?

**Figure 1: Task 1**

You have a pendulum made of a metal ball on a string. The string is 1 meter long and the metal ball has a mass of 1 kg. You might know that the approximation for the period of a pendulum for small oscillations is

$T = 2\pi\sqrt{\frac{l}{g}}$ where $T$ is the period of the pendulum, $l$ is the length of the pendulum, and $g$ is acceleration due to gravity (9.81 m/s$^2$). This equation only holds for small angle oscillations of the pendulum. For larger angles, the period of a pendulum can be found with the following equation:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^{\,2} + \frac{11}{3072}\theta_0^{\,4} + \ldots\right)$$

where $\theta_0$ is the angle of displacement of the pendulum from vertical in <u>radians</u>. You want to calculate the period of oscillation for this pendulum. How big can the angle of displacement of the pendulum be before the equation for small oscillations isn't a good approximation of the period?

**Figure 2: Task 2**

## Results and Analysis.

The analysis that constitutes this preliminary report is ongoing, but an all-too-brief description of one data sample of an early transition in Joe's thinking (below), while working on Task 1, will serve to highlight the nature of the transition points in the analysis that illuminate both how Joe uses his additional evidence to refine his image, and how that image represents the meaning of approximation with Taylor series (according to him). To carry out an analysis of this entire interview, it was broken into episodes during which Joe is appealing to a stable version of his visual image. Within each episode, it is then instructive to trace his thinking and evidence for his claims, both as he discusses them and as he amends his image based on those claims. When he abandons one image for another structurally different version, a new episode begins.

*Data Sample.* While working on Task 1, after drawing a graph of $f(x)$=arctan($x$), Joe decides to draw a band around the horizontal asymptote of $y=\pi/2$ with two horizontal lines, $y$=1.47 and $y$=1.67 (see Fig. 3). Here referred to as "tolerance bands," he emphasizes that it is reasonable to be within roughly a 0.1-band on either side of $\pi/2$, stating:

*"You need to find where [the approximation] first enters the [tolerance band]. I think you can just assume it's a good approximation until then … And then once it enters the [tolerance band], you begin to encounter the possibility of it being a bad approximation, so then once it leaves that [tolerance band] you know that it's become a bad approximation."*

Joe's language and drawing indicate that he believes the series approximation will look like the thicker line in Fig. 3, pointing out where it enters and exits the horizontal tolerance bands.

Though he explains in great detail why he believes this is a good strategy for determining when the approximation (a cubic) would represent a reasonable approximation for arctangent, and shows great skill in graphing and reasoning about end behavior, upon further reflection, Joe recognizes two problems with this representation. First, he recognizes that that it "starts outside the range" - That is, he notices that the point that the two graphs share (the origin) is outside of his band. He chooses to explain this away and not act on it, not recognizing the importance of the 'center' at $x$=0. However, Joe does act on a second problem – plugging the value of $x$=1,000 into the first two terms of the approximation, he realizes that that cubic should "go off to negative infinity." This does not sync with his knowledge that arctangent will level off at $\pi/2$.

It is at this point that Joe shifts his thinking, uses a calculator to graph $y = x - (1/3)x^3$, and produces Fig. 4. Realizing with this new evidence that the approximation will never even reach his tolerance band, Joe's attention is drawn to more local features such as the maximum of the cubic function. Noting that "arctangent is strictly increasing," and that the cubic has a maximum, Joe posits

*"[The cubic] is decreasing after a certain point, so once it passes that point you know it is rapidly becoming a bad approximation"*

While he had originally convinced himself thoroughly that tolerance bands around $\pi/2$ were appropriate, new evidence (both numerical and graphical) prompted Joe to, for the moment, abandon the idea of tolerance bands. No longer concerned with the asymptote, his focus shifts to the increasing/decreasing features of the two functions in question. That is, he gathered evidence that caused an amendment in his graphic, momentarily foregoing end behavior to accommodate what he knows about a more local feature of the graph.

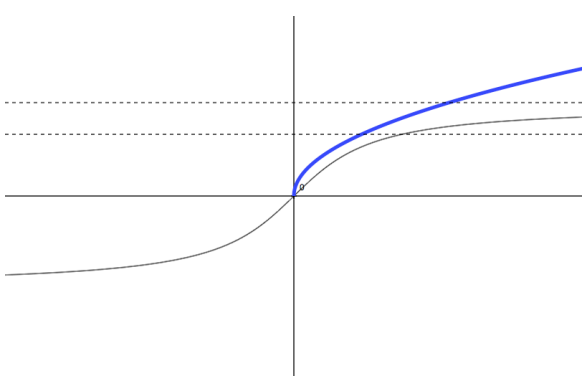

**Figure 3: tolerance banding around $\pi/2$**

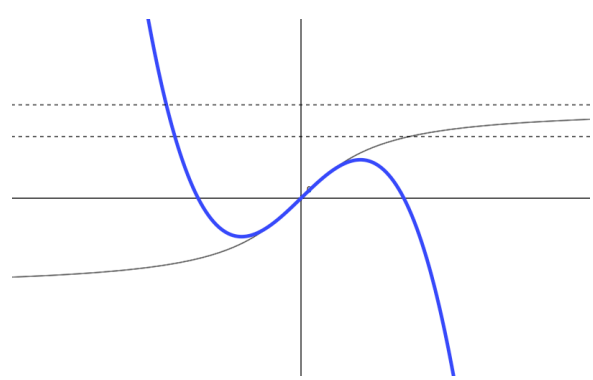

**Figure 4: an accurate graph of the cubic**

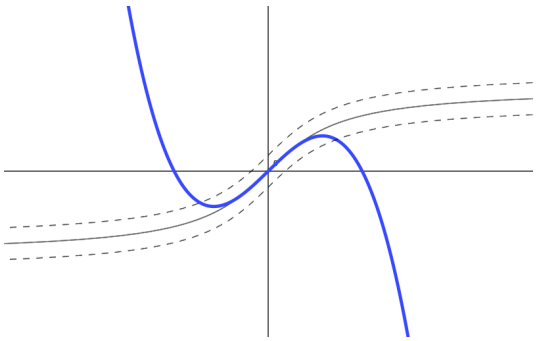

**Figure 5: tolerance banding around arctan**

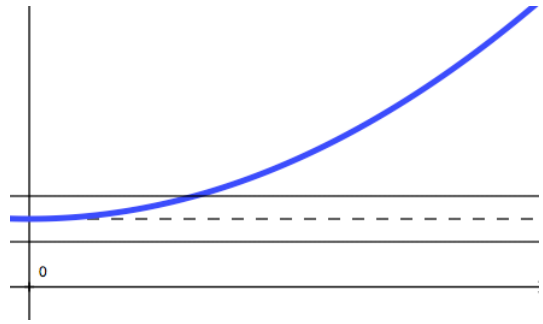


**Figure 6: tolerance banding around $T_0$**

*Breakdown of example.* This short description of one part of Joe's work highlights the sorts of things that analysis of this case will attend to – namely evidence that is used and not used in amending the evolving visual image for Taylor series approximation, and the prior calculus concepts initiated, as well as how they are integrated into Joe's image and understanding.

A complete treatment of the case of Joe, which will appear in a RUME Conference Report in 2012, serves to illuminate how he arrived at his final working visual images in Figs. 5 and 6, corresponding to Tasks 1 and 2. Upon inspection, it would appear that Joe may have a relatively robust understanding of what it means to approximate with Taylor series. Some of his language even supports this. For example, he later states "you're looking at the distance between [the functions] at any given value of $\theta$," which is the more normative way of examining error. However, our analysis shows documentable, systematic gaps in his understanding that are not evident in examining his final products alone, and were not resolved in his construction of those figures. For example, by the end of the interview Joe still does not appreciate the role of 'center,' he persists in attending to infinite behavior instead of local behavior even when the context changes, and more. Most importantly, we have a window into how earlier calculus concepts and understandings mediated the creation of those images, and served as evidence (to Joe) for the evolution of his image.

*Continuing Analysis.* The next steps in this research will be to complete the analysis on Joe's episode, with more emphasis on the first research question. Most of the analysis to date (only a snapshot presented here) has concentrated on the ways that prior mathematics concepts were negotiated to assign meaning to the approximation image for Taylor series, but there are bigger picture issues to be dealt with in the continuing analysis. Namely, what types of claims merit revisions to the image vs. other claims that are discarded, explained away, or deemed less important? How is that additional evidence that eventually causes amendments in the visual image sought?

## Contributions

As Borgen and Manu (2002) emphasize, "an understanding of what images, both correct and incorrect, that students might construct is important if teachers are to help students work toward connected formalizations" (p. 164). Even better – knowing *how* students build those images provides additional perspectives for informing pedagogy around the topic of Taylor series. Returning to Martin's (2009) point, recognizing that graphical representations of Taylor series are one of the most significant factors in separating novices from experts, it is instructive to work

on building students' graphical images for such a topic. However, one cannot responsibly undertake that task without first exploring how students create that understanding for themselves.

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# Teaching Methods Comparison in a Large Introductory Calculus Class

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Keywords: Calculus, design experiment, classroom experiment

## Abstract

We have implemented a classroom experiment similar to a recent study in Physics (Deslauriers, Schelew, & Wieman, 2011): each of two sections of the same Calculus 1 course at a research-focused university were subject to an "intervention" week where a less-experienced instructor encouraged a much higher level of student engagement by design; we employed a modified pseudoexperiment structure for our methods comparison with a Calculus 1 student population and with further steps to improve validity. Our instructional choices encouraged active learning (answering "clicker" questions, small-group discussions, worksheets) during a significant amount of class time, building on assigned pre-class tasks. The lesson content and analysis of the assessments were informed by existing research on student learning of mathematics, in particular the APOS framework.

## Introduction and Research Questions

Our work is motivated by a demand for empirical study of less-traditional but evidence-based instructional methods for introductory Calculus at the undergraduate level. We gleaned structural ideas from the Physics Education Research (PER) community, though instructional decisions in our study were based on research on students in mathematics, with an attempt to situate our analysis in the Action Process Object Schema (APOS) framework (Dubinsky & McDonald, 2001). Our research questions are not unlike those of Deslauriers et al. (2011):

*Question 1: Compared to more traditional lecture-based instruction, will students demonstrate more sophisticated reasoning on an immediate test of learning when high-engagement instruction is implemented for a single topic (100-150 minutes of class time)?*

*Question 2: Will any effects persist to later, more standard tests of learning in the course?*

## Theoretical Perspective

Our framework for the pseudoexperimental design follows that of Deslauriers et al. (2011). To our knowledge, and supported by a recent survey article (Speer, Smith III, & Horvath, 2010), no study of this kind has been reported for this size of college-level mathematics classroom.

Our lesson structures borrowed ideas from Peer Instruction (Crouch & Mazur, 2001) and general principles about learning that are now available (National Research Council, USA, 2000) but are not known to many university mathematics faculty, particularly at research-focused institutions. The key components of the instructional intervention were:

*Pre-class activities*: reading and structured exploration done individually, with some items submitted online for the instructor to read over.

*High-engagement class time*: group discussion and activities using structured notes and

worksheets, driven in part by pre-class results, clicker questions with follow-up discussion among students and/or whole-class directed by instructor, reactive lecture with small portion of the time for (traditional) exposition.

Identical standard exercises were assigned to both sections after the instructional period, similar to previous course years and non-intervention topics. Student exposure in the interventions was thus largely compatible with the Activities, Class, Exercises (ACE) cycle; for previous research on implementation of this cycle, we consulted Weller et al. (2003).

In designing material for the two classroom intervention topics, we considered sources in the literature for APOS-based study of both topics. For the first topic, Related Rates, we considered the work of Engelke (2007) and especially the recent thesis of Tziritas (2011) where a genetic decomposition for related rates problems was performed and tested; our own decomposition is compatible though our data also permit some extension. For the second topic, Linear Approximation, we considered literature on covariational reasoning (Carlson et al., 2002).

**Methodology**

The setting for our study is a research-focused university in a multi-section (11 instructors) Calculus 1 course primarily aimed at business majors, though the course shares most core material with the science Calculus 1 courses at the same institution. For our interventions, we chose sections with 150 and 200 students taught by two tenured faculty with strong teaching records in terms of length of experience, student evaluations and anecdotal department opinion. Both instructors used "clicker" personal response devices to enhance classroom interactivity, asking 1- 2 such questions per hour on average. Otherwise, class time was primarily spent on relatively traditional lecture (concepts introduced at the blackboard, worked examples) with some directed whole-class discussion. Both were receptive to student questions during class.

For our instructional intervention, we employed similar elements as Deslauriers et al. (2011):

- Natural setting of two similar sections in the same course, during the same semester.
- Classroom intervention by an instructor with less experience but recent training on theories of learning and non-lecture pedagogy. In our case, a graduate student (the second author) who has taught 3 courses total, including this course once.
- Single topic intervention over approximately one week of classes.

We extended the experimental design in the following ways:

- Introducing a "crossover" by applying two single-topic interventions, one for each course section in a different week, to account for differing student populations. We claim that the two topics chosen, Related Rates and Linear Approximation, are relatively independent items in the course; in our context, the former draws on the notion of derivative as rate, implicit differentiation, word problems with geometric objects, while the latter is more closely connected to the graphical interpretation and estimation.
- Removing the primary investigator (the first author) further from the classroom intervention: though assisting in the development of instructional materials instruction, the primary investigator was not the instructor (the second author).
- Having the initial post-tests of learning based on agreed-upon learning objectives but written by someone (the third author) not involved in the instructional design.

- Tracking student performance with respect to the two topics on subsequent course exams.
- Using the *Teaching Dimensions Observation Protocol* (TDOP) instrument (Hora & Ferrare, 2010), developed as part of an NSF-funded project at multiple institutions of higher education, where an in-class observer codes instructor behavior and (expected) cognitive demands upon the students in 5-minute intervals. This has permitted a characterization of classroom activity of the control sections and experimental sections.

We have established a baseline of student abilities using three instruments, based on predictive value for course grades in recent years: a *calculus diagnostic*: a 20-minute in-class test of prior calculus knowledge mixing "standard" procedure-based problems and conceptual problems, developed for this project; an *attitudes survey*: online, based on the CLASS Physics survey (Adams et al., 2006), measuring expert-like orientation to the discipline; and a *precalculus quiz*: online, based on a local placement exam, found in the previous year to have the same statistical power as high-school mathematics grades in predicting final grades.

Figure 1 shows a timeline, including the positions of the common assessments.

| Course Week | 1 | 2… | 7 | 8 | 9 | 10 | 11 | 12 | end of term |
|---|---|---|---|---|---|---|---|---|---|
| Sec A Instructor | $A_1$ | $A_2$… | $A_7$ | $\mathbf{X_8}$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | |
| Sec B Instructor | $B_1$ | $B_2$… | $B_7$ | $B_8$ | $B_9$ | $B_{10}$ | $\mathbf{X_{11}}$ | $B_{12}$ | |
| Assessments In Common | $\mathbf{att_1}$ | **D** | | $\mathbf{Q_{RR}}$ | $\mathbf{MT_{RR}}$ | | $\mathbf{Q_{LA}}$ | $\mathbf{att_{12}}$ | **FE** |

Figure 1: Sequence of the pseudoexperiment: instructional interventions ($X_n$) took place in Week 8 in Section A, Week 11 of Section B; assessments were: attitudes ($att_n$), precalculus quiz and calculus diagnostic (D), quizzes for Related Rates ($Q_{RR}$) and Linear Approximation ($Q_{LA}$), common midterm question on related rates ($MT_{RR}$) and the common final exam (FE).

**Results of the research**

Our attitude and precalculus assessments indicated the student populations were similar to those of the previous year. On these and the new calculus diagnostic, the students in both sections achieved similar score distributions. Due to the "crossover", we were not concerned about identical baselines, but this data establishes these as typical sections in this course.

The data from our immediate assessments support a positive answer for our first research question, and the follow-up assessment for the Related Rates material supports a positive result for the second question. In particular, we saw better performance on conceptual parts of the Related Rates assessments (i.e. about 5-15% more of the students demonstrated an Action or Process understanding of various concepts), and a larger number of students able to demonstrate the correct picture for Linear Approximation (66% versus 48% of the class could draw the correct tangent line, while 42% versus 21% could do so and label the relevant points), for the

higher engagement section in each case. Performance in both sections was very close on computational items and concepts more strongly tied to earlier parts of the course. As of the time of writing, the data has not been collected from the common final exam which measures both topics.

**For discussion**

- Do the enhancements to the similar PER study offer improvement? Are they sufficient?
- How broadly convincing are studies involving week-long interventions by "novices"?
- Recommendations on scope of reporting this type of study would be much appreciated; how much detail on the various assessments, instructors, lessons, theory, results are desirable/feasible?

**Acknowledgements**
The authors would like to thank Matthew Thomas (University of Arizona) and the conference referees for their extremely valuable feedback.

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# A Guided Reinvention of the Definitions of Ring, Integral Domain, and Field

Preliminary Research Report

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**Keywords**: abstract algebra, ring and field theory, developmental research, Realistic Mathematics Education, guided reinvention

## Research Problem

The struggles of undergraduate students with their first course in abstract algebra are well-documented (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Hazzan & Leron, 1996; Leron & Dubinsky, 1995). The course is often the first encounter with higher mathematics for many students; in particular, they are exposed to algebraic structures which form unifying threads throughout the rest of mathematics (Edwards & Brenton, 1999). Unfortunately, many students struggle with this transition to higher mathematics and fail to understand even the subject's most basic and fundamental concepts (Dubinsky et al, 1994). As a result, many students who are initially interested in mathematics experience a complete reversal of opinion and become indifferent or disengaged. Leron and Dubinsky (1995) even go so far as to state that "[the] teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures" (p. 227). To this end, alternative approaches to teaching abstract algebra must be explored.

In response to this need, two notable innovative approaches have been developed in recent years. Leron and Dubinsky (1995) developed an instructional method using the programming software ISETL to allow students a more interactive experience with basic algebraic concepts, such as group, subgroup, normal subgroup, coset, and quotient group. Larsen (2004, 2009) used the theory of Realistic Mathematics Education (RME) to develop local instructional theories supporting the guided reinvention of group, group isomorphism, and quotient group. Both of these methods emphasize example-driven approaches which serve to highlight and elucidate the foundational concepts of group theory. Similar research in the area of ring and field theory, however, is exceptionally scarce. Based upon the literature regarding student difficulty in comprehending the definition of a group (Dubinsky et al, 1994), it is reasonable to suspect that many students are just as unsure of the importance of the ring axioms and the subtle differences among such ring-theoretic structures as ring, integral domain, and field. Thus, this research project seeks to address this need by developing an original approach towards increasing student proficiency with the definitions of ring, integral domain, and field.

**Literature.** As mentioned previously, Larsen (2004) developed an innovative method of group theory instruction by testing and revising an instructional theory which supports the guided reinvention of group and group isomorphism. He explicated three iterations of the constructivist teaching experiment (Cobb, 2000) as part of a developmental research design (Gravemeijer, 1998). Larsen's instructional activities employed symmetries of regular polygons as a means by which students are able to interact with the group structure. Gradually, the students harnessed their informal experience with the symmetries of a triangle and square and ultimately were able to reinvent the concepts of group and isomorphism by way of stating a precise mathematical definition. Similarly, Larsen, Johnson, Rutherford, and Bartlo (2009)

developed an instructional theory for the reinvention of the quotient group concept and included results for how such a theory might be implemented in a classroom setting.

The only reference in the literature which directly addresses student learning in ring theory is a case-study of one student's work with the commutative ring *Z99* (Simpson & Stehlikova, 2006). In particular, the student's self-guided explorations of the structure by such devices as equation solving enabled her to recognize and address several fundamental properties of rings with little external prompting, confirming the ideas of Filloy and Rojano (1989) who asserted that equations are a means by which students transition from arithmetic thinking to algebraic thinking. Kleiner (1999) echoed the importance of equation solving by stating that "in the solving of the linear equation $ax+b=0$, the four algebraic operations come into play and hence implicitly so does the notion of a field" (p. 677). Indeed, the informal act of solving basic equations seems to provide a nice context for motivating both the ring and field axioms and other concepts central to ring and field theory, but no research exists which analyzes this claim. Furthermore, Larsen has laid the groundwork for a novel, reinvention-minded approach in group theory, yet no such research exists in ring theory. This research project addresses these gaps in the literature.

**Research questions.** The overarching questions which guide this research project pertain to the reinvention of the definitions of ring, integral domain, and field, as well as how they might be motivated and distinguished from one another: How might students reinvent the definition of a ring? How are students able to motivate the need for the subsequent ideas of integral domain and field? What models or activities will enable students to clearly differentiate between these structures? Supporting research questions include: What activities, models, processes, or ideas are involved in developing these concepts when the students start with their own activity and knowledge? With what types of informal knowledge are the students able to begin the process of reinventing the definition of a ring? What kinds of activity can support the transition of the students' informal knowledge into more robust methods of thinking?

## Theoretical Framework

This study utilizes the ideas of developmental research as a means to evaluate and revise a local instructional theory (Gravemeijer, 1998). The initial local instructional theory and the subsequent instructional activities were guided by the heuristics of Realistic Mathematics Education (RME). The notion of initial local instructional theory can be likened to Simon's (1995) hypothetical learning trajectory, which he defined as a "prediction as to the path by which learning might proceed" (p. 135). Moreover, Gravemeijer (1998) recommended that the initial instructional theory be designed with regards to "informal knowledge and strategies of the students on which the instruction can be built" and "instructional activities that can foster reflective processes which support curtailment, schematization, and abstraction" (p. 280). The RME heuristic which largely guided the design of the initial local instructional theory was that of guided reinvention, the main idea of which is to allow students to discover the desired mathematics for themselves (Gravemeijer, 1998).

## The Initial Local Instructional Theory

Due to the familiarity of most students with solving basic equations, the historical importance of equation solving (Kleiner, 1999), and its potential for motivating the structure of a ring (Simpson & Stehlikova, 2006), I designed instructional activities and the overarching initial local instructional theory with the idea that the ring structure would emerge as a result of solving equations. In particular, I am viewing the general structure of a ring as an emergent model (Gravemeijer, 1998) brought about by the activity of solving equations. Using the model-

of/model-for transition as detailed in Gravemeijer (1999), I anticipated that the ring structure would initially emerge as a model-of the students' informal knowledge of solving equations and would gradually evolve into a model-for more formal mathematical activity to motivate the distinctions between the definitions of ring, integral domain, and field. At the crux of this hypothesized emergence of the ring structure is the students' mental transition from thinking about properties simply as the properties used to solve equations into those properties which explicitly characterize a mathematical structure. Also of significance is the subsequent identification of those properties which make certain equations solvable on some structures but not others; these will be exactly those properties which distinguish general rings from integral domains and fields.

**Research Design**

The research design is comprised of three iterations of the constructivist teaching experiment (Cobb, 2000) that I conducted myself with pairs of undergraduate students. Each iteration consisted of up to 12 sessions of 1.5-2 hour sessions each. The participant pool included students who had recently taken a course in discrete mathematical structures and had not yet had a course in abstract algebra. The multiple iterations of the teaching experiments allowed for the instructional theory to be in a constant state of revision. The data, which consists of both transcribed video data and written work, was analyzed both between sessions within teaching experiments and also between the teaching experiments themselves. The data was analyzed and the instructional theory revised by means of multiple iterative analyses similar to that of Larsen (2004). Other theoretical constructs employed to support the reinvention process and enhance data analysis include Larsen and Zandieh's (2008) *Proofs and Refutations* framework and Zandieh and Rasmussen's (2010) defining as a mathematical activity framework.

**Results and Implications**

As of the submission of this proposal, data collection and analysis is still ongoing, so any statement of conclusive results may be premature. However, based on the literature and my experience with the participants in the teaching sessions, I expect to be able to present preliminary results regarding the revision and evaluation of an instruction theory which supports the guided reinvention of ring, integral domain, and field. These initial results and implications based on the data from the teaching experiments (and the corresponding analysis) will be complete in time for the conference. I hope to engage in conversation with other researchers interested in both my content area (teaching and learning abstract algebra) as well as my research method (RME and guided reinvention) to help me refine the conclusions I am able to harvest from my data.

**Questions**

I will ask the following questions:

- In your experience, what are some other problematic concepts for students in an introductory course on ring and field theory?
- (Continuation of previous question:) How might this study be able to address those problematic concepts given its current design?
- If you were teaching a course in ring and field theory, how might you modify this instructional theory for your classroom?
- What other frameworks, pieces of literature, or research contacts might be relevant to or helpful for my work?
- Do you have any suggestions for future research which could further the work done here?

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# The Use of Dynamic Visualizations Following Reinvention

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Preliminary Research Report

*This research is a part of a larger project to gain insights into how calculus students might come to understand formal limit definitions. For this study, a pair of students participated in an eight-day guided reinvention teaching experiment in which they created a formal definition for sequence convergence even though they had not previously received instruction on formal limit definitions. During the reinvention process they identified and coordinated relevant graphical attributes of sequences as they recognized and resolved problems with their emerging definition. For this paper, we detail the ninth day in which the students participated in an activity using The Geometer's Sketchpad where they had to interpret their understandings of sequence convergence on premade manipulate-able dynamic visualizations of sequences. We hypothesize that by using these dynamic visualizations, the definition and their resolutions to problems were reinforced by strengthening their connections between their definition and these visual representations.*

Keywords: Limit, Definition, Guided Reinvention, Sequences, Dynamic Visualizations

## Introduction and Research Questions

A consensus of research on student understanding of limits has revealed great difficulty in reasoning coherently about formal definitions (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). Recently, some studies have begun to outline how students come to understand formal limit definitions (Cory & Garofalo, 2011; Cottrill et al., 1996; Roh, 2010; Oehrtman et al., 2011; Swinyard, 2011). But even after seemingly successful teaching experiments where students articulate understandings consistent with formal theory, Martin et al. (2012) point out that students can still struggle in recalling their formal limit definitions after short periods of time. Fortunately, dynamic visualizations used by Cory & Garofalo (2011) seemed to effectively increase retention of formal limit definitions by allowing students to manipulate key elements of the definitions within the constraints of relevant relationships to strengthen their connections and their various representations. We recruited a pair of students who had just completed a Calculus II course covering sequences for a teaching experiment to reinvent the formal definition for the limit of a sequence. Following their construction of a formal limit definition, the students participated in activities using a computer-generated dynamic visualization designed to reinforce relationships in the students' definition. Six months after the teaching experiment, the participants will be asked to reconstruct their formal definition. This study attempts to address the following research questions: (a) How did the dynamic visualizations reinforce the students' prior reinvention activities? (b) How did the dynamic visualizations play a part in their reconstruction of their formal definition six months later?

## Theoretical Perspective and Methods

To investigate our research questions, we adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p. 279). *Guided reinvention*, "a process by which students formalize their informal understandings and intuitions," supported the task design (Gravemeijer et al., 2000, p. 237).

Over a month's time, the authors conducted a teaching experiment at a small, southwest university with a pair of students, selected based on their experience with sequences but lack of experience with formal limit definitions. The central objective of the teaching experiment (comprised of nine, 120-minute sessions) was for the students to generate a rigorous definition of *sequence convergence*. The instructional activities, adapted from Oehrtman et al. (2011), engaged the students in an iterative refinement process involving definition creation, definition evaluation against examples and non-examples, conflict acknowledgement of identified problems with the current definition, discussion of potential solutions, and the creation of a modified definition, thus restarting a new iteration. Oehrtman et al. (2011) noted that during the refinement process, the problems identified by the students were the most meaningful and supported the formation of ideas that remained stable through multiple iterations.

During Day 1, Joann and David (pseudonyms) produced and subsequently unpacked details of convergent sequences graphically. By Day 2, they had produced nine graphs of what they viewed as qualitatively different examples of sequences converging to 5 and nine graphs of sequences not converging to 5. During Day 2, the facilitator prompted the students to create a definition for sequences convergence by completing the statement, "A sequence converges to 5 provided…." This was continually qualified by the facilitator as "construct a statement that will keep all of your examples in and keep all of your non-examples out." Days 2 through 8 consisted of the students engaging in the iterative refinement process and unpacking their intended meanings for individual elements within their evolving definition. By the beginning of the 9th session, Joann and David had produced a definition that they felt correctly captured the meaning of sequence convergence (see *Figure* 1). Probes by the facilitator revealed that their understanding of this definition was consistent with formal theory.

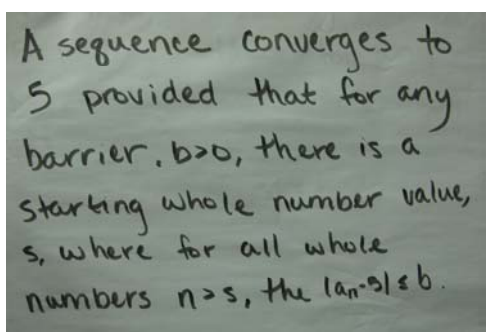


*Figure* 1. The participants' final definition.

During Session 9, the students used *The Geometer's Sketchpad* (Jackiw, 2002) to manipulate dynamic graphs of sequences in hopes of improving upon the lack of retention observed by Martin et al. (2012). This instructional activity's design was adapted from Cory and Garofalo (2011) who found that students strengthened their understanding of sequence convergence by engaging dynamically with a consistent visual representation of the formal definition and by

reflecting on their evolving conceptions as they compared their interactions with the visual representation to the written formal definition. Cory and Garofalo (2011) put forth the possibility that as result, their participants demonstrated a more coherent, enduring understanding of limit ideas eight months later. Their findings are consistent with the *principle of manipulation* (Plass et al., 2009) which suggests that learning from visualizations is improved when learners manipulate the content of the visualization and with Mayer's (2009) Theory of Multimedia Learning which holds that a crucial step in learning involves integrating one's pictorial model of a concept with one's verbal model. For the present study, we adapted Cory and Garofalo's (2011) dynamic sketches to coincide with the language and symbols the students used to create their definition and to include several graphs the students generated during Day 1. Six months later, the students will repeat the reinvention process so that the dynamic visualization's impact, if any, on their re-development of the definition can be investigated.

**Emerging Results**

Leading up to producing their final definition, Joann and David engaged in many challenges that provided opportunities for learning through the thoughtful resolution of identified problems during the creation of their sequence convergence definition. On the last day of the teaching experiment, the students used *Sketchpad*'s dynamic capabilities to continue to explore their definition and how it applied to many of the sequences they generated earlier. In many ways, the sketches appeared to reinforce the students' ideas about sequence convergence by giving them opportunities to manipulate a coherent visual representation of their definition. We describe three challenges encountered by Joann and David, how they resolved these problems, and how *Sketchpad*'s dynamic capabilities seemed to reinforce the resolutions they had made.

One challenge Joann and David faced was to understand the importance of the universal quantifier on the "barrier *b*" (corresponding to $\varepsilon$ in standard formulations). This concept appeared on Day 2 when Joann first mentioned the idea of "breaking a barrier" while investigating the graph of a monotonically increasing sequence converging to 4.9 rather than 5. She explained, "If [the sequence] was going to 5, then it would cross the 4.9 and it would cross the 4.99 and…the 4.999….You have to…break that barrier of 4.9." After some discussion, the participants wrote the definition in *Figure* 2. As the students were invited to compare various definitions they had developed and to make their ideas more concise and precise, they created the phrase, "for all decreasing decimal barriers", and ultimately settled on the words, "for any barrier, *b*." Later, as they manipulated the *b*-value on the *Sketchpad* sketches to show any value they desired, the concept of the universal quantifier was reinforced (see *Figure* 3).

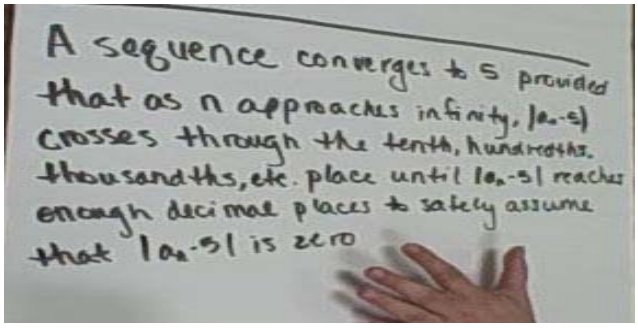


*Figure* 2. Participants' definition involving decimal "barriers"

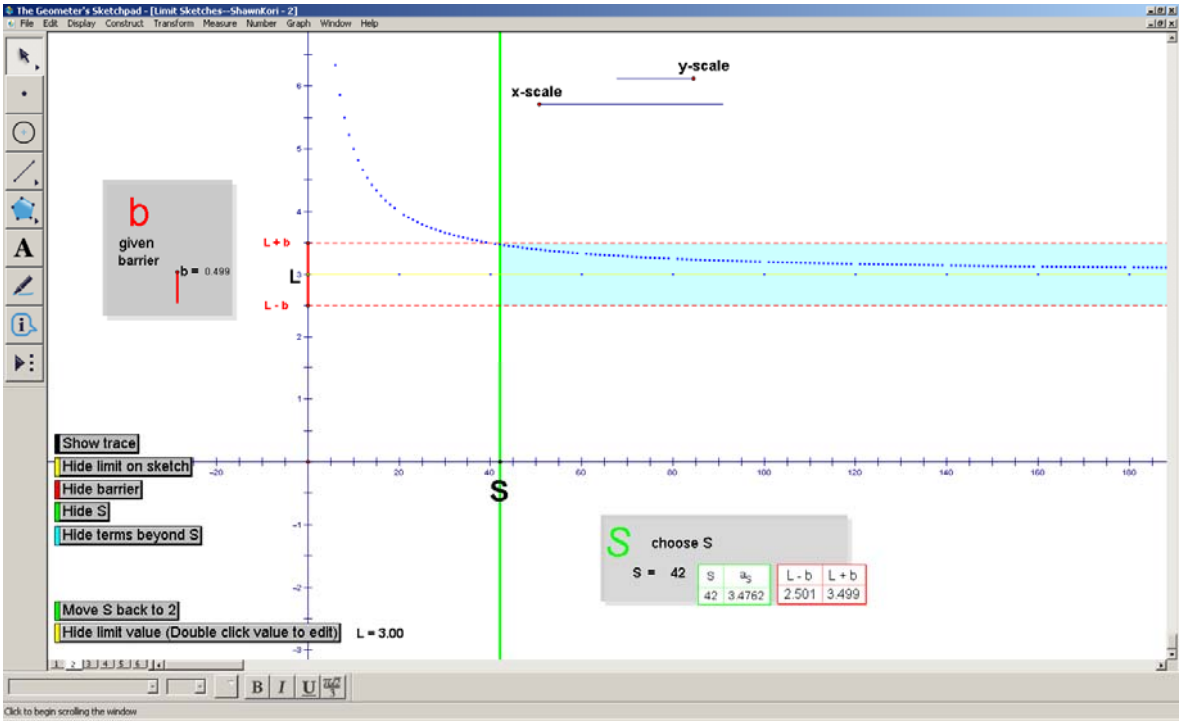


*Figure* 3. An interactive sketch of the formal definition of the limit of a sequence.

During Days 3 to 5, one of the participants' challenges was recognizing that using a "peak" on a graph was not an effective way to establish an error-bound. For them, a peak became a local extremum as seen in an oscillating sequence or could be any particular point in a monotonically decreasing sequence since all subsequent points are "below" that particular point. After much discussion, they developed the definition in *Figure* 4. Immediately the students had difficulty clearly defining a peak and struggled in applying their "peaks" definition graphically, eventually realizing that their definition did not exclude some of their non-examples. In addition, they interpreted this "peak" idea as setting the error bound for subsequent points, a conception in which an error bound is dependent upon a peak's height. Following this, the facilitator guided them to think of the error-bound as an independent variable which could be placed anywhere on the *y*-axis and to compare their "peaks" definition to their earlier definition. The removal of their "peaks" idea and eventual acceptance of error bounds were reinforced on the last day of the teaching experiment as both participants took turns manipulating values for the *b*-band on the sketches (see *Figure* 3) which no longer needed to correspond with points on the graph.

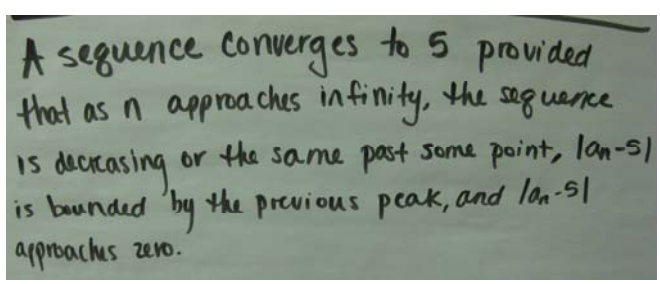
A sequence converges to S provided that as n approaches infinity, the sequence is decreasing or the same past some point, $|a_n - S|$ is bounded by the previous peak, and $|a_n - S|$ approaches zero.

*Figure* 4. Participants' definition involving peaks.

As the teaching experiment progressed, the participants also worked to resolve problems leading to their acceptance that all terms *past some point* must fall within a barrier and that this point depends on the barrier. The "past some point" idea had already appeared in their "peaks" definition (see *Figure* 4), and after returning to their "decreasing decimal barriers" definition and attending to their convergent graph with early random behavior, they immediately reincorporated the "past some point" idea in a new "decreasing decimal barriers" definition. They eventually chose "*s*" to identify the point after which all terms must fall within a specified barrier. Finally, after exploring the relationship between *s* and *b* using various graphs, they revised their definition so that *s* depended on the barrier (see *Figure* 5). These resolutions were repeatedly reinforced on the sketches, as one participant chose a *b*-value for which the other participant chose a "good" *s* by sliding the *s*-line along the graph until all dots beyond *s* fell inside the horizontal lines set by *b* (see *Figure* 3). Each time they carried out these manipulations, the participants were guided to explain why their *s* "worked," thus giving them opportunity to connect the visual representation to their verbal model and the written definition.

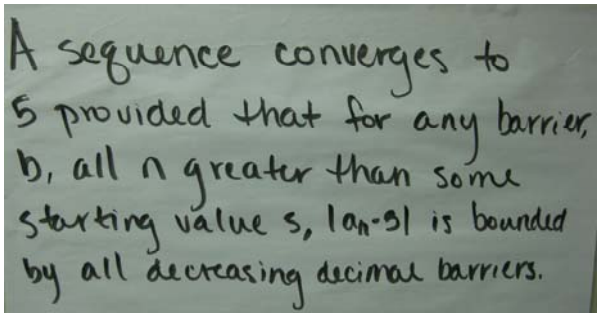


*Figure* 5. Participants' definition involving the dependence of *s* on *b*.

**Discussion and Questions**

As in Oehrtman et al. (2011), Joann and David wrestled with the problem of rigorously articulating their ideas as they focused on relevant quantities and their relationships. The universal quantification of the barriers, the move away from terms determining the value of barriers, and the cognitive shift to focus on *s* as a function of *b* were all seen as viable solutions to problems. On the teaching experiment's last day, we gave the students an occasion to strengthen connections between their definition and their visual representations by using interactive dynamic visualizations. Our remaining questions include: How might we better isolate the dynamic visualizations' effects? How might the dynamic visualizations be incorporated into the reinvention itself? How could the dynamic visualizations be modified to support students in using their definitions to address genuine mathematical problems?

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# Mathematical Modeling and Engineering Majors

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**Abstract:** A first course in differential equations for engineers and scientists is intended to introduce the students to key principles and techniques involved in using mathematics as a modeling tool. However, a great many students emerge with only a limited number of analytic techniques that are applicable only to a narrow selection of equations, despite the inclusion of word problems in the curriculum. Previous research into mathematical modeling competencies indicates that the students' difficulties can be traced to coordinating mathematical with physical reasoning. The purpose of this research is to develop tasks for a data collection instrument that will allow for the development of a cognition based model of how such skills grow.
**Keywords:** differential equations, mathematical modeling, design experiment

**Background** Differential equations, as a mathematical domain, arose from the study of change in physical systems over time. Many mathematics faculty and even some engineering faculty (Standler, 1990) insist that linking a differential equation with the particular applied problem it embodies should receive little attention in mathematics courses because such is the purview of engineering courses. In contrast, other educators argue that modeling is an interdisciplinary enterprise (English, 2010) and so coordinating the mathematical model and the situation it represents is critical (Shternberg & Yerushalmy, 2003).

According to Blum (2011), modeling is cognitively difficult for students because of the dialectic nature of modeling tasks: they require *Grundvorstellungen* (appropriate fundamental mathematical ideas), real-world knowledge, and the ability to translate back and forth between the two. Niss, Blum, and Galbraith (2007) identify two reasons to teach with applications in mathematics: (1) to use mathematical modeling (MM) and applications for the learning of mathematics and (2) to learn mathematics in order to develop competency in applying mathematics and building mathematical models. The latter is the primary motivation for including differential equations in science and engineering programs. Engineering majors struggle in applying mathematics to build mathematical models or to manipulate them.

The two goals of the broader research project are (1) to study the development of students' "ability to construct and to use mathematical models by carrying out those various modeling steps appropriately as well as to analyse or to compare given models" (Blum, 2011, p. 18), and its components, from a cognitive perspective and (2) to create tasks that could be adopted by mathematics faculty as instructional aids. The focus of the present research is instrument design. The intention is to present a set of modeling tasks appropriate to engineering students and differential equations and to also provide some preliminary data and analyses arising from field testing the tasks with engineering students.

**Mathematical Modeling** In the mathematics, mathematics education, and engineering education literature bases, MM is presented as a process that bridges two worlds: the real and the mathematical. As in problem-solving, there are four overarching stages: identify a real world problem, idealize and express the phenomenon mathematically, analyze the mathematical model, and interpret the solution in real world terms. Kehle and Lester (2003)

presented four processes that link these stages together: the modeler begins with a realistic problem which he simplifies into a realistic (idealized) model. The idealized model is then abstracted into a mathematical model and calculations lead to mathematical results which must be interpreted. The process students have the least success with is mathematiziation.

The term "mathematizing" is used to encompass activities like symbolizing, algorithmatizing, and defining (Rasmussen, Zandieh, King, & Teppo, 2005) or the the application of mathematical tools such as creating standard representations (Kwon, Allen, & Rasmussen, 2005; Rasmussen & Blumenfeld, 2007). In the MM literature (see, for example, Blum, 2011; Niss et al., 2007; Lesh, Doerr, Carmona, & Hjalmarson, 2003), the term refers to the arc of cognitive activities that lead from the description of a life-like problem to rendering that problem in mathematical terms so that well-known tools (e.g., equations) can be identified or expressed. Lesh and Yoon (2007) distinguish between "mathematizing reality" and "realizing mathematics," where the latter refers to dressing up mathematical problems with language of lifelike situations. Mathematizing reality involves simplification and abstraction (Kehle & Lester, 2003), specifying assumptions and making mathematical observations (Zbiek & Conner, 2006, see Figure 1), and distilling life-like problems into an idealized "situation model" (Haines & Crouch, 2007). A series of processes inverse to mathematizing, but less well-theorized, is carried out after the mathematical analyses take place. In this phase, the modeler examines the results of the mathematical analysis in light of the purpose for building the model. Finally, the model must be validated and refined.

There may be a great deal of oscillation among portions of the modeling process before a stable idea or representation is reached (let alone a viable model). Most diagrams represent the modeling process iteratively. Zbiek and Conner's (2006) schematic details critical sites within the overarching cycle where deliberations may occur as well as which and how they recruit cognitive processes. Kehle and Lester (2003) explained these cognitive transitions between the two worlds as different modes of inference. Abduction bridges experience to a sign system, deduction is the drawing of conclusions based on the manipulation of those signs according to rules, and induction applies a sign system to an experience that is thought to correspond to the structure of that system. Induction and abduction work together to help interpret experience. Students are best prepared in their mathematics classes for the analysis portion of the cycle (Gainsburg, 2006) and they need experience in connecting real world to mathematical world connections in order to develop modeling competency (Crouch & Haines, 2004).

The questions guiding this project are: How do engineering students carry out mathematization? How do they validate their models? Are their techniques stable or do they change over time? How do the students "keep track" (Gainsburg, 2006) of the transitions among realistic situations, idealized situations, and mathematical models? What features of life-like situations to students attend to? What criteria do they use to analyze and evaluate models? What mathematical, and in particular differential equations, competencies are modeling tasks most suited to enhance (Niss et al., 2007)? What elements, behaviors, or cognitive activities of the modeling process might be unique to differential equations?

**Methodology** Given the two objectives for the larger research context, a design experiment methodology was selected (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003; Kelly, Baek, Lesh, & Bannan-Ritland, 2008). Task development has focused on modeling competencies

and has proceded iteratively. Modeling competencies include "the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model" (Niss et al., 2007, p. 12). Selection of appropriate tools, whether mathematical or cognitive, depends on recognizing the underlying structure of a problem (English, 2010). Since many application and modeling problems emphasize the analysis of an already mathematized situation, Lesh, Hoover, Hole, Kelly, and Post (2000) developed model eliciting activities (MEAs) to serve the dual role of revealing students thought processes as they solved significant mathematics problems while simultaneously providing learning experiences for the students. However, one of the primary challenges in using MEAs in undergraduate engineering courses is to discover ways to blend them with other pedagogies (Hamilton, Lesh, Lester, & Brilleslyper, 2008), particularly those often used in undergraduate mathematics classrooms.

These ideas guided the initial creation of the modeling tasks, which highlight different stages of the modeling process and a variety of modeling competencies. Thus, both whole modeling tasks and competency-specific tasks were developed drawing on multiple mathematical domains. A series of one-on-one task-based clinical interviews will be conducted with engineering students enrolled in a differential equations course, in accordance with the design experiment methodology, in order to assess and modify the tasks relative to students' knowledge and development.

**Results** At the time of this submission, instrument construction has proceeded iteratively with content-validity checks. Concurrent validity will also be assessed. Relevant literature has indicated various phases of the modeling process that students should encounter as they solve the problems and these phases will be used to frame the students' activities while addressing the tasks. Through analysis of the protocols and students' written work, I expect to assess the feasibility of using the instrument to identify and map cognitive activities crucial to the development of modeling competencies. My goal for this presentation is to generate feedback from other researchers about how to best improve the data collection instrument.

**Questions** Based on the preliminary data collected, I would request feedback to improve this instrument:

- Are these tasks representative of the different stages of the modeling process? Are there aspects of the modeling cycle that are being neglected? Aspects that could be better assessed?
- How can tasks be modified, extended, or added to include more student reflection?
- What additional paradigms that could be used to explore students' development of modeling skills?
- What other literature, frameworks, theories, or considerations might be essential to this work?
- Does authenticity of the tasks matter? To what extent?

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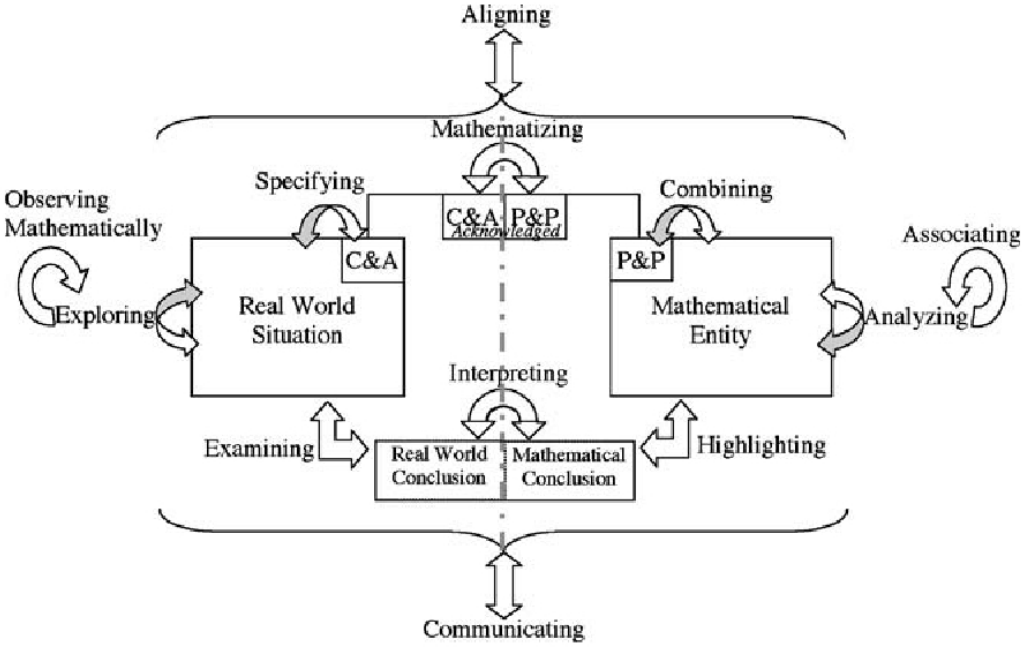
*Figure 1.* : Schematic of the modeling process (Zbiek & Conner, 2006)



Title:

# Future Teachers' Views of Mathematics and Intentions for Gender Equity: Are These Carried Forward into Their Own Classrooms?

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Abstract (Preliminary Research Report):

Previous work indicated that an interdisciplinary mathematics and gender course about women mathematicians and their contributions to the field shifts students' views away from seeing mathematics as the study of numbers and toward a more expert view of what the subject entails. In addition, at the end of the course in the reflective writing portion of a portfolio, future teachers frequently volunteer their intentions to foster gender equity in their own classrooms. This preliminary research report will explore whether, and how, this enriched view of mathematics and the resolve for equity persist and influence the classroom teaching of four former students. It will also seek to determine the particular learning experiences that most contributed to any positive findings. Ethnographic methods, including interviews and classroom observations, will be employed.

Keywords: K-12 teacher preparation, gender equity, views of mathematics, case study

Students throughout K-12, as well as many in college, even those majoring in STEM (Science, Technology, Engineering and Mathematics) fields consider mathematics to be the study of numbers (Dewar, 2008; Forringer, 2010). In contrast, experts in the field, namely mathematics faculty, describe mathematics as being concerned with patterns, proof, abstraction and generalization (Devlin, 1994; Dewar, 2008). Concern about students' understanding of their discipline is longstanding. Schwab (1964) argued the importance of undergraduates, and especially future teachers, learning the underlying structures and principles of their majors. Teaching disciplinary specific practice continues to be a matter of concern to this day (Leamnson, 1999; Riordan & Roth, 2005). In mathematics, the phenomenon known as stereotype-threat (Steele, Spencer, & Aronson, 2002), wherein the performance of members of a group about which there is a negative stereotype suffers due to anxiety that that their performance will conform to the stereotype, makes excelling in mathematics even more challenging for female and minority students.

A small study (n = 7) of future teachers enrolled in 2004 in an interdisciplinary mathematics and gender course titled, Women and Mathematics, indicated that this course was successful in moving students toward a more expert view of mathematics, based primarily on a content analysis of their descriptions of mathematics as a field of study at the beginning and the end of the course, whereas traditional courses in the mathematics major curriculum did not. In addition, reflective writing in their end-of-course portfolios revealed that the students were very determined to present mathematics in their future classrooms as a desirable activity for *all* students. Eighteen months later, three of these seven students were interviewed, two of whom were teaching. These interviews

suggested that the two who were teaching had maintained their richer views of mathematics. The former student who was not teaching (she was working in student affairs on a local college campus) had shifted back to a description of mathematics being mostly about numbers. These results were intriguing and begged to be explored in greater depth with additional students, especially with those who had actual K-12 classroom teaching experience.

This new study explores whether, and how, the enriched views of mathematics and the resolve for equity persist and influence the classroom teaching of a new cohort of former students. For any positive findings, it seeks to determine the particular learning experiences that most contributed to those. Specifically,

- Do the former students' enriched views of mathematics persist?
- Is there evidence that the more expert views espoused at the end of the Women and Mathematics course are influencing the instruction of these former students who are now teaching in their own classrooms?
- In what ways have the former students carried out their stated commitment to equitable mathematics instruction?
- What courses, learning experiences or other factors influenced the teachers' views of mathematics or their approaches to equitable instruction?
- What role, if any, did participation in several pre-professional opportunities (presenting workshops at conferences for future teachers or at a math/science career day for junior high girls) associated with the second cohort, but not experienced by students in the first study, play in developing students' resolve to provide equitable mathematics instruction and helping them to achieve this goal once they were teaching?

The subjects of the current study, four former students of the Women and Mathematics course in 2008, are now in their third year of teaching. Data similar to that gathered in the first study was collected during the course to determine their views and intentions for gender equity. Ethnographic methods including classroom observations and interviews are being employed to determine whether their views of mathematics persist and are influencing instruction, whether resolve to create an equitable classroom is carried out, and what courses, learning experiences or other factors contributed to any positive findings. The observations and interviews are being conducted in October and November of 2011. The data being gathered relative to gender equity in the observations and interviews includes seating assignments, grouping assignments, classroom displays, differences in classroom discourse, how teachers describe an equitable classroom, how their classroom fits that description, their views of the similarities and differences between the girls and boys in their class relative to cognition, behavior, motivation, beliefs about their ability to do mathematics, and how those views influence the way they design their instruction. Relative to the teachers' views of mathematics, the evidence being collected is how they currently describe mathematics, what they want their students to think mathematics is all about, how that was reflected in the lesson observed, how it might appear in other lessons, whether student work samples reflect those aspects of mathematics. They are also being asked to identify which courses, learning experiences or other factors influenced their views of mathematics and equity.

The findings of this study have the potential to be useful to undergraduate mathematics major programs as well as mathematics teacher preparation programs. Presumably, college faculty have an intrinsic interest in what views of the discipline their students hold. Program review and assessment

requirements certainly invite and encourage departments to investigate student understanding of their discipline. Further, the importance of this question for future K-12 teachers can hardly be exaggerated, since what views they hold will influence their choices about what content they teach and how they approach it, given that precollege-level mathematics teaching is so constrained by the realities of State standards and "No Child Left Behind."

Enlightening future teachers about the facts and fallacies that underlie the widely held idea that boys are better at math than girls is one way to empower them to confront these stereotypes personally and then, in turn, with their students. Providing information about role models and awareness that women have contributed to the development of mathematics is another important strategy. Convincing students that mathematics is as important for girls to learn as for boys is yet another challenge faced by K-12 teachers (Gilbert & Gilbert, 2002). The Women and Mathematics course addresses all of these topics in addition to displaying mathematics as a study of patterns, emphasizing and contrasting the use of inductive and deductive reasoning, and providing multiple representations for many mathematical concepts. Which of these aspects of the course, if any, has a positive and enduring influence on future teachers is something this study seeks to answer.

For this Preliminary Research Report suggested Discussion Questions are:

- Would undertaking similar observations and interviews with teachers who have not taken the Women and Mathematics courses, as points of comparison, be a worthwhile undertaking? If so, how should this comparison group of teachers be chosen?
- How does one determine how a view of mathematics influences instruction?
- How does one accurately determine what factors influenced a person's view of mathematics?
- How does one accurately determine what informs a future teacher about the need for equitable instruction?
- How might one determine what is effective in helping them develop the resolve to achieve that and give them useful tools toward that end?

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*Preliminary Research Report*

**Title**: Authority dynamics in mathematics discussions

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**Abstract:**

We employed grounded theory techniques to examine the evolution and influences of authority relationships in an undergraduate mathematics education research study. Our analysis focused on video data from a five day teaching experiment with two faculty researchers engaging two second-semester calculus students in a guided reinvention of formal limit definitions. We will discuss our model for authority in a mathematical discussion and characterize the patterns, influence and evolution of authority that we identified in the guided reinvention. Finally, we illustrate the need for researchers to be cognizant of authority patterns in group data collection settings, since such patterns can mask individual evidence of knowledge and reasoning.

**Keywords**: authority, mathematical authority, mathematical group settings, model of authority

## Introduction and Research Questions

The role of interviewing in qualitative data collection requires researchers to consider the strategies researchers employ to obtain interview data (Patton, 2002). Authority dynamics between interviewers and participants have been identified as one factor influencing the authenticity of interview data; Langer-Osuna and Engle (2010) and Brubaker (2009) emphasize the need to attend to authority patterns in these settings. Authority can be socially or content based; social authority is defined as charismatic authority derived from social norms, while content authority is derived from the community of practice, instructors, or textbooks (Amit & Freid, 2005). While both types of authority have strong impacts on academic discussions, social authority has been observed to overwhelm content authority, potentially leading groups in directions not based on sound reasoning (Langer-Osuna & Engle, 2010). We use a grounded theory approach to answer the research question: what role does authority play in the guided reinvention process, and what model can be developed to assist researchers in understanding authority dynamics in mathematical group settings?

Esmonde and Langer-Osuna's model for authority in science discourse has four components: socially negotiated influence, degree of perceived authority, access to the conversational floor, and access to the interactional space (Engle, Langer-Osuna, & McKinney de Royston, in press). While the work by these and other authors have focused on K-12 settings, here we address mathematics educators' need to understand how authority, both mathematical and social, develops in guided reinvention with undergraduate mathematics students. Relevant work in undergraduate mathematics settings is limited (Langer & Engle, in press), but the work of Szydlik (2000) and Frid, (1994) indicates an aspect of mathematical authority called source of conviction, which dictates how mathematical statements are justified internally (sense-making) or externally (from outside authoritative sources). While SoC can be used as a measure of

mathematical authority, and Freid and Amit's (2006) framework can serve as a basis for social authority, a comprehensive model of authority in undergraduate mathematics group settings has not been formulated or used to analyze group dynamics in these settings.

**Theoretical Perspective and Methods**

Examining roles of participation and authority in group settings are two of the basic constructs of situated cognition (Brown, Collins, & Duguid, 1989; Lave & Wagner, 1991; Salomon & Perkins, 1997). In this perspective, researchers focus on how constructs such as roles and authority contribute to the progressive discourse within the group, which is what the theory defines as learning. We employed this lens to investigate participation and authority and their effects on group dynamics and learning (Bereiter, 1994; Jordan & Henderson, 1995; Sfard's 1998).

Given the limited literature on authority in guided reinventions, we used grounded theory (Patton, 2002) to develop a model of authority dynamics. We initially open coded the first two days of the guided reinvention of limit concepts (Martin, Oehrtman, Roh, Swinyard, & Hart-Weber, 2011; Oehrtman, Swinyard, Martin, Roh, & Hart-Weber, 2011; Swinyard, 2011) using a constant comparative method, and then developed our initial categories (Corbin & Strauss, 2008). After we conducted a literature search on social and mathematical authority, we adapted the model and standards of evidence proposed by Engle, Langer-Osuna, and McKinney de Royston (2008) to fit the group size and content discussed by our participants (Figure 1), and then coded the first five days of the guided reinvention using this new framework. The goal of our model was to categorize the types of interaction between the participants and interviewers to model authority dynamics.

| Social Authority | Math Authority | Evidence of Authority |
|---|---|---|
| Socially negotiated influence (SNI) | | declared a new position; influenced someone else to change positions; strengthened or weakened someone's position |
| Access to the conversation/space (Access) | | participant is granted/required permission to speak; access to space; participant interrupts others; body orientation/attention |
| | Degree of authority (DoA) | evaluation of another persons' mathematical credibility, acting as a credible source of information |
| | Sources of conviction (SoC) | statements made appealing to empirical evidence, intuition, logic, or consistency, indicating the concept is logically structured, sensible, and connected to reality, or that they could figure out for themselves, statements/appeals to authority, statements made to appeal to the mathematical structure of the argument |

Figure 1. Authority framework.

**Results**

We found the two categories of authority (social and math) were distinguishable in the five day teaching experiment. Influence statements decreased after the second day, but access

stayed relatively constant throughout the guided reinvention. As for mathematical authority, one participant, Belinda, exhibited a higher degree of mathematical than her peer, Megan. Over the teaching experiment, Belinda made statements indicating a stronger sense of internal authority, while Megan had a stronger sense of external authority (figures 2 and 3).

As influence exchanges lasted approximately a minute, we axial coded the code with a simple count. To axial code access, we employed micro-ethnography to weight the strength of each code, since the clips were not of the same duration. Craig and Megan had the greatest influence on the discussion, while Belinda and Jason, who has less social authority, could not influence the discussion without providing mathematical evidence (Figure 2).

| Participant | Declare position after being neutral | Strengthen Position of others | Weakening another's position | Convincing others to switch positions | Total |
|---|---|---|---|---|---|
| Craig | 4 | 3 | 3 | 3 | 13 |
| Megan | 2 | 1 | 5 | 1 | 9 |
| Belinda | 2 | 0 | 1 | 0 | 3 |
| Jason | 2 | 0 | 0 | 0 | 2 |

Figure 2 Socially negotiated influence

For access, we used micro-ethnography to generate a list of all behaviors participants displayed when obtaining or being denied access, such as being pointed at to speak or taking control of writing implements. Belinda's co-opting of all writing material allowed her more access than any other participant, but Belinda and Jason had significantly less body orientation towards them than Craig and Megan (Figure 3).

| | Access | Orientation | Overtalk | Permission | Total |
|---|---|---|---|---|---|
| Megan | 49 | 21 | 1 | 12 | 83 |
| Craig | 37 | 25 | 3 | 2 | 67 |
| Belinda | 59 | 15 | 0 | 9 | 83 |
| Jason | 21 | 7 | 0 | 0 | 28 |
| Total | 166 | 68 | 4 | 23 | |

Figure 3 Access

| | Megan evaluates Group | Belinda evaluates Group | Megan Treated as Authority | Belinda Treated as Authority |
|---|---|---|---|---|
| Day 2 | 9 | 15 | 1 | 3 |
| Day 3 | 11 | 15 | 3 | 4 |
| Day 4 | 7 | 19 | 2 | 3 |
| Day 5 | 5 | 10 | 3 | 4 |
| Total | 32 | 59 | 9 | 14 |

Figure 4 Socially negotiated authority

Using an idea as the unit of analysis, we totaled the counts for each day made by the participants. From the table we saw a relatively stable pattern in the number of evaluations made by each participant and the number of times the participant was treated with authority. Belinda tended to have higher evaluations of the group each day of the interview.

| | Megan | Belinda | Megan | Belinda | Megan Math | Belinda |
|---|---|---|---|---|---|---|

| | Internal SoC | Internal SoC | External SoC | External SoC | SoC | Math SoC |
|---|---|---|---|---|---|---|
| Day 2 | 19 | 29 | 1 | 0 | 1 | 0 |
| Day 3 | 19 | 37 | 8 | 0 | 2 | 4 |
| Day 4 | 7 | 27 | 2 | 1 | 0 | 3 |
| Day 5 | 6 | 28 | 3 | 2 | 0 | 6 |
| Total | 51 | 121 | 14 | 3 | 3 | 13 |

Figure 5 Sources of conviction

Again using an idea as the unit of analysis, we totaled the counts of the source of conviction made by each participant. Belinda exhibited more statements coded as internal authority and mathematical authority, while Megan had more external source of authority statements.

**Conclusions**

Understanding the relation between authority and group dynamics is important for mathematical settings such as interviews, focus groups, and teaching experiments like the guided reinvention. We observed diminishing social authority while mathematical authority patterns remain fairly constant in our teaching experiment employing the guided reinvention heuristic. Ongoing research is aimed at understanding the causes of such patterns. Being aware that interviewers often have authority over the learners is important, especially in our case, where we saw how a multi-day teaching experiment created an environment where authority dynamics initially established persisted over later days. We suggest this could be because the participants perceived the researchers as authority figures (an aspect of social authority), causing the participants to attempt to foster appeals to mathematical reasoning (an aspect of mathematical authority). Overall, our current model describes the authority relationships in this interview. In our preliminary report, we are interested in obtaining feedback, particularly the following questions: (1) what factors may foster shifts in authority dynamics?, (2) what are typical considerations interviewers take regarding authority?, and (3) what additional information would need to be incorporated into a model of authority dynamics to usefully inform data collection methods?

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**Learning trajectories and formative assessment in first semester calculus: A case study**

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*While formative assessment, assignments given for feedback rather than grades, raise student achievement, the literature lacks an explanation for how these assessments affect student learning. The purpose of this case study of an introductory calculus class using the approximation framework was to investigate how adding formative assessments to an introductory class using the approximation framework changed the learning trajectory for the class. The preliminary analysis of the formative assessments suggested that the assessments appeared to scaffold metacognition, self-reflection, and transfer of the approximation framework between units.*

**Keywords**: approximation framework, formative assessment, learning trajectory, transfer

## Introduction and Research Questions

Formative assessments, low stakes assignments given to assess students' current level of understanding, increase student achievement (Black & Wiliam, 2009; Clark, 2011), but little is known about how implementing formative assessments facilitates this achievement gain. The purpose of this case study is to study the impact of formative assessment on students' learning trajectories in a calculus course with Oehrtman's (2008) approximation framework; the main research question that guided my investigation was: How does formative assessment impact students' Zone of Proximal Development of the Approximation Framework (Oehrtman, 2008) between contexts in introductory calculus?

Understanding how formative assessment affects how undergraduates learn and transfer the approximation framework helps to advance the theory of formative assessment, which has been primarily developed on European students in primary and secondary school (Black & Wiliam, 1998, 2001, 2003, 2006, 2009). Furthermore, a better understanding of how formative assessments scaffolds student achievement allows us to improve our calculus pedagogy.

Black & Wiliam's (2009) framework of formative assessments suggests that there are five major benefits of formative assessment: (1) to communicate clearly what the learning goals are, (2) allowing instruction to be based on students' current level of understanding, (3) providing learners with feedback that scaffolds learning, (4) giving peers a common experience to talk to each other about, and (5) raising students ownership of learning. Researchers have found that transferring concepts from the initial contexts in which the concept is learned is difficult for students (Barnett, 2002; Lobato & Siebert, 2002), but since formative assessment can increase student self-monitoring (Clark, 2010), which can facilitate transfer (Ning & Sun, 2011), we hypothesized formative assessments could also help facilitate transfer, which could also impact students' learning trajectories.

## Theoretical Perspective and Methods

Examining the role of peripheral participation in group settings, such as a formative assessment, is a basic constructs of situation cognition (Brown, Collins, & Duguid, 1989; Lave & Wagner, 1991; Salomon & Perkins, 1997). A situated cognition perspective allows researchers to focus on how these constructs contribute to the progressive discourse within the group, which is what the theory defines as learning. Using the established frameworks of situated cognition, we chose to use this lens to investigate participation and its effect on group dynamics and learning (Bereiter, 1994; Jordan & Henderson, 1995; Sfard's 1998). These frameworks guided my standards of evidence and were effective tools for investigating our research question about how authority dynamics can be modeled for mathematical group settings.

In the figure below (Figure 1), we have included a typical formative assessment. The first questions of our formative assessments were conceptual questions related to the current content. The two open questions always appear as the last two questions of every formative assessment. When we analyzed students' documents, we looked for student errors in the content questions, and checked the homework assignments and test following the formative assessment how persistent the error was. For the open questions, we coded all student questions and comments about what they did not understand that were directed to the instructor as peripheral participation, and considered the written statement in the penultimate question to identify concepts students claimed to be transferring from other areas. After coding the open response questions, we looked at students' summative homework assignments and exams for further evidence of improvement and transfer.

Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question. Please type your answers into this word document and email it back to [Me] at [Your.instructor@email.edu] by [9 pm tonight].

1. Fill in blanks with the letter(s) from the definition of the derivative to label the quantities marked on the graph of $y = f(x)$ as illustrated below.

   Error Bound = _____

   Average Rate of Change = _____

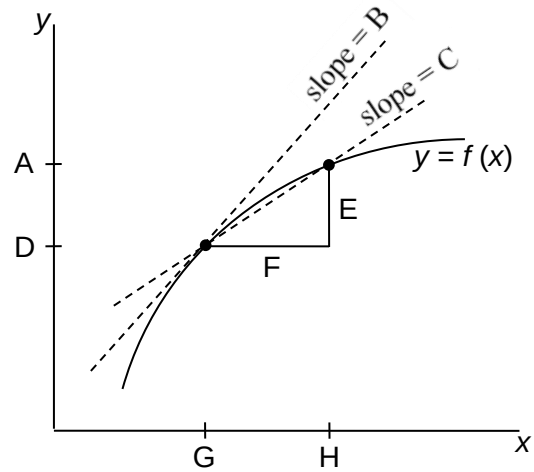
   Instantaneous Rate of Change= _____

   Δy = _____

   Δx = _____

   $x$ = _____

   $x + h$ = _____




2. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
3. What questions do you have about the material we have covered so far in class?

*Figure 1* Formative Assessment Two

Given the lack of qualitative literature on formative assessment, particularly with American undergraduates, we chose to conduct an exploratory study. The first level of our analysis was the classroom , where we conducted a macro-level analysis of the learning trajectory of the classroom for two introductory calculus classes. At the second level, we analyzed formative assessments, homework assignments, and exams as artifacts of the learning trajectory (Patton, 1990) from four students in each class. The first author also observed the classrooms the day before and the day after the weekly formative assessment was distributed to the students and debriefed the instructors on a weekly basis to obtain their observations of student and classroom learning trajectories. We analyzed the data using an open coding thematic analysis, which was peer checked (Patton, 2002).

The classes we recruited participants from utilized Oehrtman's (2008) approximation framework as a coherent instructional approach which uses limits to develop the concepts in introductory calculus. This framework is built upon an approximation metaphor for limits Oehrtman (2008, 2009) based on approximating an unknown quantity. For each approximation there is an associated error which one needs to bound in order to have some sense of the accuracy of the approximation. While the actual student usage of approximation metaphors can be highly idiosyncratic (Martin & Oehrtman, 2010), systematic structuring of the elements and relationships among approximations, errors, error bounds reinforce common limit structures within and across different limit contexts. The goal of the instructional framework is for students' use of the metaphor to become more systematized in ways that reflect the structure of

formal limit definitions but are intuitively accessible to the students (Oehrtman, 2008); the goal of the formative assessments was to facilitate this systemization. The systematic metaphor can encourage the abstraction of a common structure while engaging in multiple activities within a limit context and the results of such abstractions further support abstractions of common structures across different limit contexts that can provide a more coherent understanding of the role of limit throughout all of calculus and beyond. As a student's approximation schema becomes well organized these ideas become a cognitive tool that can guide students' informal investigation into concepts formally defined in terms of limits.

**Results**

The classroom-level learning trajectory outlined by Oehrtman's (2008) approximation framework was unchanged with the addition of formative assessment; the class still needed to engage in the same cognitive challenges to master the framework. However, since the instructor was able to use formative assessments to provide feedback that immediately addressed misconceptions, students who completed the formative assessments appeared to make fewer mistakes on their unit tests than students who did not do the assignments. While the early questions on each formative assessment allowed instructors to communicate with students what material was important, the final two questions of each formative assessment contained some evidence of student self-monitoring and actor-orientated transfer.

The thematic analysis of the data suggested three factors helped individual students develop more systematic and less idiosyncratic conceptual structures related to the approximation framework. First, the formative assessment provided students a legitimate peripheral participatory role; the open response question allowed students to ask questions of their instructor without any loss of face, and gave students some say over what happened in class. As Max explained after class one day,

> Everyone at my table is so much smarter than me, and I know they really get it, but when we do the formative assessments, it's over email, so no one has to see me not get it. I know they [my table] get bored the next day, but it makes all the difference for me to have my questions answered.

Second, by asking students to reflect on what concepts they did and did not understand, the formative assessment scaffolded student self-monitoring. As Robin explained:

Before the first one [formative assessment] I thought I understood most everything. But them when I had to sit down and write a paragraph about what I didn't understand I started to realize I really didn't know how the pieces fit together. Then, when we talked about it [the formative assessment] the next day, I knew I had to pay extra close attention.

Third, by asking students to reflect each week on what concepts they had seen before, together with the improvements in self-monitoring, students improved their incidence of actor-orientated transfer. The responses on the next to last question on each formative assessments that asked students to make connections not only increased in length, but students began to correctly claim that approximation ideas were applicable from week to week. In the figure below (Figure 2), we have provided the responses of a typical student's responses from the first two formative assessments in the derivative chapter, in the third and fourth weeks of class. While the student is mostly noticing common vocabulary words at this stage, this is a necessary first step to further transfer of concept (Barnett, 2002).

| Formative assessment #1 | Formative assessment #2 |
|---|---|
| This week I recognize the phrase average speed. That is when you take the change in height and divide it by the change in speed. I recognize instantaneous speed from physics. I'm not exactly sure how you find it, but I do recognize the word. I recognize the slope of a line and relate it to when I learned about it back in algebra. The approximation value, remind me of last week when we worked on limits. | I recognize slope from past math classes and the instantaneous rate of change from physics classes as well as calc. I also recognize error bound because we have been discussing it over the last few classes. |

*Figure 2.* Sample Formative Assessment Responses.

The analysis of individual students' artifacts suggested that the opportunity to ask questions and gain specific feedback was crucial in addressing individual misconceptions.

**Conclusions**

While the formative assessments are graded for completion and only worth a few token percent of the students' final grades, the act of completing the formative assessment help students understand what concepts the instructor values, reflect on what they understand, ask questions without losing face, and ponder connections between topics on weekly, unit and semester scales. This suggests that, for undergraduate mathematics students using asynchronous formative assessment, the peripheral participatory role can be included in Black & Wiliam's (2009) theoretical framework. Since their framework is based on verbal and whole class formative assessments, students who have questions about the material must feel safe admitting this in front of their peers; participation is not peripheral. Since formative assessment improved students' self-monitoring, formative assessments could be designed and implemented for any introductory mathematics course. As we move forward on data collection, we are interested in obtaining feedback from peers, particularly the following questions: (1) how else might formative assessment influence the learning process? (2) How can we improve our coding scheme for evidence of transfer?

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**Operational and Structural Reasoning with Linear Transformations**
(Preliminary Research Report)

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**Abstract**
*In this report we detail linear algebra students' interpretations of linear transformations. Data for this analysis comes from mid semester, semi-structured problem solving interviews with 13 undergraduate students in linear algebra. We identified two main categories for student reasoning students in completing three tasks: 1) students who used structural reasoning with entries of the matrix, columns of the matrix, and orientation of the shape and 2) students who used operational reasoning through matrix and vector multiplication. We examine the patterns that emerged from student strategies, and discuss possible explanations for these patterns.*

**Key words:** linear algebra, linear transformations, operational and structural reasoning, concept development

## Introduction

A longstanding concern in mathematics education is the balance and relationship between knowing how to do something and knowing why something is the case. The research community has addressed this issue by developing a number of explanatory frames, including procedural versus conceptual understanding (Hiebert, 1986), process versus object conceptions (Breidenbach et al., 1992), concrete versus abstract modes of reasoning (Wilensky, 1991), instrumental versus relational understanding (Skemp, 1976), synthetic versus analytic thinking (Sierpinska, 2000), and operational versus structural reasoning (Sfard, 1991). Although there are differences in these constructs (both nominal and theoretical), there is a general consensus that both modes of reasoning are necessary to develop mathematical proficiency. Indeed, each of these types of understanding is represented in NCTM's five strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001), which highlights the need for students to develop both forms of reasoning.

In the domain of linear algebra, researchers have expanded on these dual modes of reasoning. For example, Sierpinska (2000) describes different modes of student reasoning as *synthetic-geometric, analytic-arithmetic,* and *analytic-structural.* Related to these modes of reasoning, Hillel (2000) describes three modes of representations: *geometric* (using the language of $R^2$ and $R^3$, such as line segments and planes)*, algebraic* (using language specific to $R^n$, such as matrices and rank)*,* and *abstract* (using the language of the general formalized theory such as vector spaces and dimension). A number of student difficulties in linear algebra have also been documented (see Carlson, 1993; Hillel, 2000; Dorier, Robert, Robinet, & Rogalski, 2000; Sierpinska, 2000; Stewart & Thomas, 2009), with many of these difficulties attributed to the disconnect between various representations and students' modes of reasoning. For example, some researchers have been interested in how a geometric introduction to linear algebra may (or may not) help students make connections to algebraic and abstract modes of reasoning.

In this study we examine students' conceptions of linear transformations by analyzing their solutions to a series of tasks involving geometric representations of linear transformations. These tasks differed in their level of complexity. In increasing order of complexity, the first task was a matching problem, the second was a prediction problem, and the third was a creation problem. The research questions related to these tasks are: (1) What are students' strategies on these three types of problems? (2) What patterns exist in students' strategies across the three types of problems? In answering these questions we also sought to account for any patterns that we identified in student reasoning.

## Methods

Data for this analysis were collected from one extensive, semi-structured problem-solving interview (Bernard, 1988) with 13 undergraduate students. The interview questions were used to gather information related to participants' understanding of linear transformations, with an emphasis on geometric representations on linear transformations. For this study, the last three questions of the interview were analyzed: a matching question consisting of five parts, a prediction task, and a creation task. These tasks will be discussed in detail below. The students were primarily engineering majors at a large southwestern university. Four of these students received a final grade of a 'C' in the linear algebra course, six students received a 'B', and three received an 'A', and pseudonyms were developed that reflect these grades. The interview was the second of a series of three interviews that was part of a semester-long classroom teaching experiment (Cobb, 2000). The interview was conducted after students had discussed geometric

and algebraic interpretations of linear transformations, but before they had begun a unit on eigen-theory. Each interview was videotaped, transcribed, and thick descriptions were developed for students' solutions to each of the tasks that included students' written work (Geertz, 1994). The videos, transcriptions, and thick descriptions were analyzed through grounded analysis (Corbin & Strauss, 2008).

We analyzed student responses to three tasks from the interview: a matching task, a prediction task, and a creation task. The matching task consisted of five problems of increasing difficulty, beginning with a positive, diagonal matrix and ending with a matrix with no zero entries. The prediction task was created to be slightly more difficult than the matching tasks, and the creation task was thought to be the hardest. This task design was modeled after Artigue's (1992) interview task design involving student understanding of differential equations.

**Interview Tasks.** The prompt for each matching task was follows: "In each of the following questions, you are given a matrix transformation and a corresponding set of images. Identify any images that correspond to the image of the unit square (as shown below on the left) under the given transformation." There were five parts, each part involving a different matrix and a different set of possible images under the given transformation. The five matrices that were provided were: $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -1/3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 1/3 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. See Figure 1 for the first of the five matching tasks.

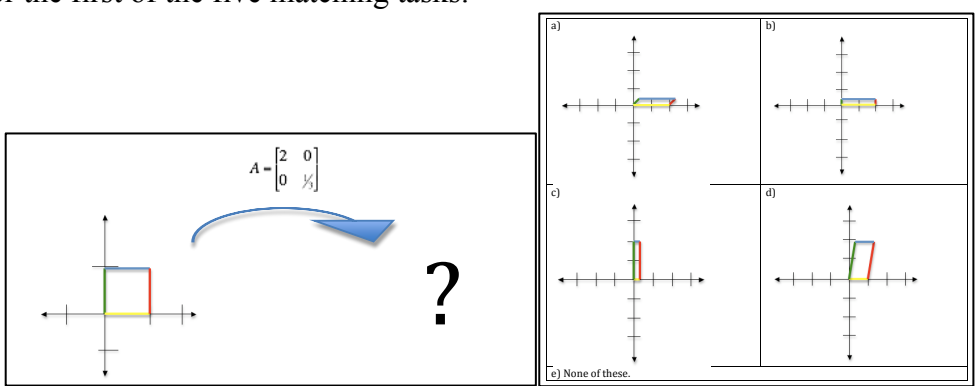


*Figure 1 a and b.* Task A of five matching tasks.

The prediction task asked students to "Please find the image of the picture below under the matrix transformation $\begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$" and provided an image of a 'T', as in Figure 2.

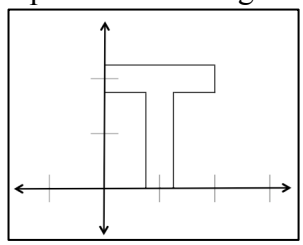

*Figure 2.* Prediction task.

The creation task required students to find a matrix that that fit a given transormation, as represented by an initial and final figure. The exact prompt was, "Please find a matrix that transforms the image on the left into the image on the right. Note that the rectangle on the left is

3 units by 2 units." Students were shown two images of a 3 x 2 rectangle, one 'untransformed' and one transformed under a to-be-determined matrix, as shown in Figure 3.

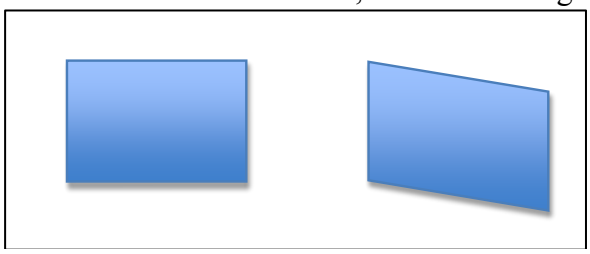

*Figure 3.* Creation task.

## Results

In this section we present analysis of students' strategies while solving the matching, prediction, and creation tasks. Students approached these tasks with a wide variety of strategies, and appeared to either view the matrix as a tool that performs the actions of the transformation (for example, by inputting vectors into the matrix to compute the resultant vector), or as an entity that provides information about how the transformation acts (for example, what do the individual entries in the matrix tell you, or what do the columns of the matrix tell you). We interpreted these different conceptions as viewing the matrix as a process or viewing it as an object, and made use of Sfard's (1991) distinctions between *operational* and *structural* conceptions to differentiate students' solutions.

Student reasoning on these tasks were further classified into six strategies, three of which related to a structural conception of linear transformation and three to an operational conception. We refer to these six strategies as *Structural entries (Se)*, *Structural vector (Sv)*, *Structural orientation (So)*, *Operational identify (Oi)*, *Operational unit-vector (Ou)*, and *Operational vector (Ov)*. We operationally defined each of these categories in Table 1.

*Table 1.* Student Strategies

| | |
|---|---|
| *Structural entries (Se)* | A student categorized as using an Se strategy reasoned by treating the two by two matrix as being composed of four pieces, the entries of the matrix. |
| *Structural vector (Sv)* | A student categorized as using an Sv strategy reasoned by treating the two by two matrix as being composed of two pieces: the two column vectors of the matrix. |
| *Structural orientation (So)* | A student categorized as using an So strategy attended to the visual and/or geometric properties of the original shape/graph as opposed to properties of the matrix. *So* often appeared when the students discussed the orientation of the box as well as how the colors of the sides should be oriented. |
| *Operational identify (Oi)* | A student categorized as using an Oi strategy reasoned by performing multiplication with the identity matrix. |
| *Operational unit-vector (Ou)* | A student categorized as using an Ou strategy reasoned by performing multiplication dealing with the unit vectors. In the matching tasks, the unit vector (1,0) was colored green, and the unit vector (0,1) was colored yellow, and thus students who performed operations on the 'green' and 'yellow' vectors were considered to be employing this strategy. |
| *Operational vector (Ov)* | A student categorized as using an Ov strategy reasoned by performing multiplication dealing with a non-unit vector, such as (1,1). |

Frequently students' overall strategies for solving these tasks involved many sub-strategies; for example a student may solve a task by using an overall strategy of SeOuOv (first using the entries of the matrix, then performing computations on both unit vectors and non unit vectors). In Table 2, we report students' overall strategies for each task. Sub-strategies were coded in order of use, and a green sub-strategy indicates that this strategy was used correctly, and a red sub-strategy indicated that it was used incorrectly. For example, on matching task d, Alex used an overall strategy of SeOvSo, indicating that he first used the entries of the matrix to inform his solution (correctly), then performed a computation using a non-unit vector correctly, and last reasoned about the orientation or colors of the matrix incorrectly. Entries that are highlighted in blue indicate that these strategies relied only on structural strategies, and those highlighted green indicate that a purely operational strategy was employed. The times under each entry represent the amount of time the student spent on the task.

This table was the main data source used for the analysis of these tasks. These tasks were grouped as follows: the matching tasks into three groups (the diagonal matrices (a and b), the non-diagonal matrices with at least one zero entry (c and d), and the matrix with no zero entries (e). The analysis of the data was conducted in two ways: first we looked for patterns *within* each of the individual tasks, and then we looked at the individual student strategies *across* the tasks.

*Table 2. Student Reasoning by Student*

| | **Match. a** | **Match. b** | **Match. c** | **Match. d** | **Match. e** | **Prediction** | **Creation** |
|---|---|---|---|---|---|---|---|
| **Alex** | Se | SeSo | SeOvOu | SeOvSo | SeSoOv | OvSoOv | OvSe |
| **Andrew** | SeOu | Se | SeSvSo | OuOvSo | OuOvSo | SeOvSo | Se |
| **Anthony** | Se | SeSoOv | SeSoOv | OvOuSo | OvSoSv | OvSoSe | SeOv |
| **Bailey** | OuOvSe | SeSo | Ou | Ou | OuOv | SoOvSe | OvSv |
| **Bart** | SvSeOv | SeSoSv | OvOuSo | SeOvOu | SeOvOu | OvSo | OvSo |
| **Becca** | OuSe | SeOuSo | SeSoOuSv | OuSvSoOv | OuOv | OvSeOvSo | OvSe |
| **Ben** | Ou | OuSo | OuSo | OuSo | OuSo | OvSeSo | Ov |
| **Bill** | SeOu | OuSo | SeOuSoOv | Ou | Ou | SoOvSe | Ov |
| **Brad** | SvOiSe | Se | OiSvSo | OiSe | So | SeSo | SeOv |
| **Caden** | OiSo | SoOv | Se | SvSo | SvSo | SeSo | SvSo |
| **Chad** | Se | SeSo | SeSo | SeOi | SoOiSv | OvSeSo | SeOv |
| **Charles** | SeOi | SeSo | Se | SeSoOiSv | SoSv | SeSoOv | SeSvOv |
| **Chris** | Se | SeSo | SeOvOuSo | SeSo | SeSo | SeSo | SeSo |

## Discussion

One of the clearest patterns that we saw in the data was transition from predominantly structural to a combination of structural and operational reasoning. Sfard (1991) described concept development as a shift from an operational conception to a structural conception. Thus, we may explain this shift in student strategies as indicative of students' stronger understanding of the geometric implications of linear transformations represented by diagonal matrices versus transformations with matrix representations that contain non-zero entries on the non-diagonals, prediction tasks or creation tasks. This is not surprising, especially considering the geometric results of diagonal matrices versus non-diagonal matrices, and the visual ease of understanding stretching compared to skewing.

What is surprising is that C-students overall exhibited a much higher frequency of purely structural strategies. Do C-students have fuller concept development of the geometric implications of linear transformations than A and B-students? Or do C-students have a weaker operational understanding of matrices and thus instead rely on their structural conceptions? In these tasks we were not specifically interested in how strong students' procedural competency was, and thus have no way to assess if this explains C-students' preference for structural strategies. However, a weak understanding of matrix multiplication certainly would result in a low grade in any linear algebra course. These differences suggest that further investigation into the differences between A, B, and C-students' operational and structural conceptions is needed.

**Discussion Questions**

1. How do you think these results could be best leveraged in a classroom environment?
2. Are the differences between the codes well understood and effectively differentiated?
3. A main result is the difference in student reasoning between grade categories. Other than differences in concept development, what else may explain these differences?

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**Title:** The Role of Technology in Constructing Collaborative Learning Spaces
Preliminary Research Report

**Authors:** Brian Fisher and Timothy Lucas, Pepperdine University

**Abstract:** Traditionally, research on technology in mathematics education focuses on interactions between the user and the technology, but little is known is about how technology can facilitate interaction among students. In this preliminary report we will explore the role that iPads versus traditional laptops play in shaping the learning spaces in which students explore concepts in business calculus. We will report on classroom observations and a series of small-group interviews in which students explore the concepts of local and global extrema. Our preliminary results are that the introducing the iPad, a portable device with intuitive applications, enhances collaboration by allowing students to transition back and forth from private to public learning spaces.

**Keywords:** learning spaces, classroom technology, iPad, social constructivism, business calculus

**Proposal:** For the past half-century mathematics educators have been contemplating the role of technology in mathematics education. Recent decades have seen significant growth in student access to technology in the classroom. Among the key strands of research are:

- Handheld devices and calculators, e.g. (Burrill et al., 2002).
- Technology designed to accumulate real data for student exploration, e.g. (Konold & Pollatsek, 2002).
- Dynamic geometry software and other microworlds, e.g. (Jones, 2000).

Like the strands mentioned above, the bulk of research on technology in mathematics education focuses on interactions between the user and the technology. Little is known about how individuals use technology to interact with one another. However, the current generation of undergraduates is likely to incorporate technology throughout their social interactions with each other. In this preliminary report we will explore how students use iPads while negotiating mathematical meaning in a community of learners.

There are many ways that technology can facilitate learning, but our goal is to understand the role of technology in facilitating joint explorations of mathematical concepts. We view a students understanding of mathematics to be directly impacted by both the medium in which the student encounters the concept and the interactions of the student with others in his/her learning community. Our study of interaction leads us to draw, primarily, from the perspective of social constructivism, which views learning as an inherently social process, e.g.(Vygotsky, 1978; Cobb & Yackel, 1996; Stephan & Rasmussen, 2002). However, we view technology as one of many ways in which a student may physically interact with a mathematical concept, and we view these interactions via technology as a significant element of our students understanding of mathematics. This viewpoint leads us to take the perspective of embodied cognition (Lakoff & Núñez, 2000) in the sense that we cannot divorce the ways a student may physically interact with concept from their perception of the concept. By taking this perspective we are emphasizing the physical role of technology within

student interactions and, in particular, students abilities to convey to their peers their embodied understanding of a concept developed using technology.

The motivation for this study originated with a university wide study of the effectiveness of the iPad as a classroom tool. In the fall of 2010, Pepperdine University distributed iPads to one section of Business Calculus along with two applications, Numbers (spreadsheet) and Graphing Calculator HD. Students used the iPads both inside and outside the classroom for the entire semester. In contrast, a second section of the course used laptops throughout the course with Excel and a java graphing applet. Much of the course is designed around activities that allow students to reconstruct mathematical principles within a small group setting. The university study focused on the effect the iPad had on student performance on specific learning outcomes, but during that fall study we became aware of how the iPads were changing the social dynamic in the classroom. This prompted a revised study that focused on recording student interaction in two sections of Business Calculus in the fall of 2011.

In order to analyze the role technology plays in collaborations we adapted Granott's framework for student interaction (Granott, 1993). Granott's two dimensional model is contructed from the relative expertise of the students in a group and the degree of interactions among the group. Our framework incorporates a third dimension which measures the depth of conversation amongst the students. We also chose to borrow the notion of public and private spaces from a study that contrasts a class that uses private handheld devices with one that incorporates public handheld devices that connect to shared LCD displays (Liu et al., 2009). This language of private versus public spaces allows us to describe the role that iPads and laptops play in constructing learning spaces.

In Figure 1 we present some diagrams of student behavior that depict the three dimensions of student interaction. The first group of diagrams depicts students working in parallel, either in isolation from one another or with some discussion that is limited to simply verifying answers. Here the students use technology entirely as a private space to interact with the mathematics. The second group of diagrams demonstrates how students may choose to use the technology as a public learning space. Within this public space, a strong student may use the technology as a teaching tool or two or more students at similar levels may use the technology to collaborate. In those cases the conversations about the mathematics may be richer and more meaningful.

We are currently using the following qualitative methods to conduct this study:

1. Classroom Observations: We will record student behavior during in-class activities using the three-dimensional framework outlined above.
2. Group interviews: We will conduct a series of small-group interviews focusing on the concepts of local and global extrema. Students often approach these concepts from a purely computational perspective, but would benefit from the use of technology to visualize the problem. We will observe how students incorporate technology while negotiating the problem with their classmates.

From our study in the fall of 2010 we have already seen evidence of how students can transform the private space on their iPad into a public space. For example, we observed a lesson on limits that requires the use of spreadsheet and graphing calculator. During that lesson we witnessed that the size and portability of the iPad allowed students to share their screens as part of their dialogue. The fact that the class is using a uniform device also facilitated students assisting each other in the learning process. Throughout the class activities the students were fully engaged and did not stray

to online distractions. In contrast, students with personal laptops had trouble working as a team due to the physical barriers that their screens presented. Students using laptops often chose not to share their screens with others unless there was a specific request from another group member. The private spaces created by laptops also tempted several students strayed to Facebook. Our task-based interviews revealed that students working with the iPad immediately incorporated graphs into their calculations of maximums and minimums. The students with laptops were reluctant to turn them on and only did so when the problems became too complex to solve by hand.

Based on our experiences this semester, we would like to ask for feedback on future analysis of our data. We ask the audience to consider the following questions:

- Is there relevant literature that we have not considered?
- Are there other means of interpreting the data that we have not considered?
- As we re-examine the videos, are there other types of interactions that we might observe?
- The university conducted a survey of general technology use for the students involved in the study. Should we use these surveys to classify students by technological comfort and track how that influences student interaction with the technology and each other?
- The criteria for the university-wide study included having one section taught with iPads and one section taught without. Is the comparison between the iPad section and the section where students use personal laptops of interest to the mathematical education community?

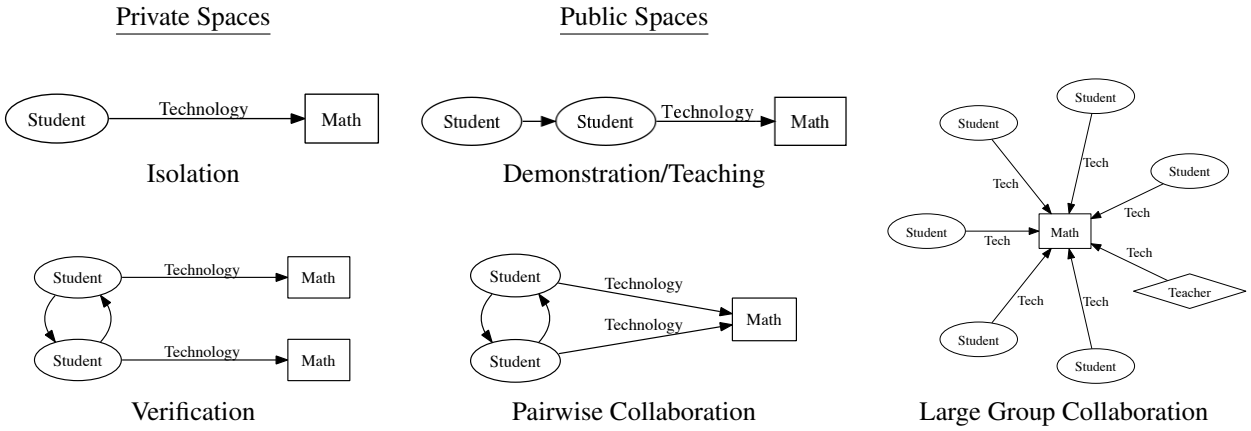


Figure 1: Examples of Student Interactions

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# Student Note Taking Behavior in Proof-based Mathematics Classes

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**Abstract:**

There is a need to explain the relationship between teaching (classroom activities) and learning. This study is one attempt to explore student note-taking as a form of mediation between teaching and learning outcomes. We will adapt the theoretical framework described by Weinberg and Wiesner (2010), who applied ideas of literary criticism to describe factors that impact the ways students read and understand mathematics textbooks. The two concepts we will use are those of the implied reader and reading models.

We are investigating student note-taking in the context of a proof-based abstract algebra class that is taught, primarily, via a lecture. We are recording the lectures and creating a set of expert-notes that are then compared with the student notes. We then interview the students to better understand the decisions that they make vis-à-vis note taking and how they "read" the text of a lecture.

**Keywords:** codes, behaviors, competencies, students reading of lecture, proof-based mathematics

## 0. Introduction and Background

The focus of most upper-level mathematics courses is on presentations of definitions, theorems and proofs of key results. Although some "inquiry-based" curricula have been designed as alternatives to the standard curriculum (e.g. Davison & Gulick, 1976; Dubinsky & Leron, 1994; Larsen, 2004), most of these courses are still lecture-focused. In order to understand how students learn the material from these lectures, it is important to understand how various aspects of the lecture relate to what the students "take away."

While there are studies relating the taking of notes with later scores (Bligh, 2000; Johnstone & Su, 1994; Kiewra et al., 1991), these studies are focused on recall and subsequent exam performance, primarily in lower-level undergraduate courses. There are studies of how students make sense of presented proofs (Mejia-Ramos, et al., 2010) and studies of how students read textbooks (e.g. Weinberg et al., 2011), but there is no corresponding study of how students "read" the text of a lecture. In this vein, research on student learning, even of topics directly related to undergraduate coursework, is often done without reference to the teaching that the students experienced and how the students made sense of their classroom experience (*e.g.,* Mejia-Ramos, et al., 2010. As a result, there is a need to explain the relationship between teaching (classroom activities) and learning. This study is one attempt to explore student note-taking as a form of mediation between teaching and learning outcomes.

## 1. Research Goals

We will investigate the following questions:

1) What is the relationship between the written, spoken and gestural text of a lecture and the students' written text of their notes? What do students include in their notes and how do they decide what to write?
2) Viewing the lecture as a text, who is the implied reader of this text, the actual readers and what is the relationship between them?

## 2. Theoretical Perspective

We will adapt the theoretical framework described by Weinberg and Wiesner (2011), who applied ideas of literary criticism to describe factors that impact the ways students read and understand mathematics textbooks. The two concepts we will use are those of the implied reader and reading models.

The *implied reader* of a mathematics text is the "embodiment of the behaviors, codes, and competencies that are required for an empirical reader to respond to the text in a way that is both meaningful and accurate" (Weinberg & Wiesner, 2011 p. 52). The *behaviors* of the implied reader are "sequences of actions (physical or mental) enacted by the implied reader" (Weinberg & Wiesner, 2011 p. 52). For example, the implied reader of the lecture might actively think about previous examples and theorems and trying to make connections with what they are currently observing. The *codes* of the implied reader are the ways that the implied reader interprets the language, symbols, words, gestures (etc.) that are part of the lecture. For example, a lecturer might say (and write): "Let *G* be a group…." ; the implied reader might interpret the word "group" as an algebraic object and recognize G as the standard symbol to represent it. Finally, the *competencies* are the "mathematical knowledge, skills, and understandings" to understand the text (Weinberg & Wiesner, 2011 p. 55). For example, the implied reader might know what a group is and be familiar with the axioms that are related to the mathematical context in which the group is being discussed.

The *empirical reader* of a mathematics text is the person who is attempting to interpret the text—in this case, the students in the class. The students' *reading models*—their strategies for reading and and beliefs about their role in a classroom shape the transaction between the students and the lecture. Weinberg and Wiesner (2011) describe two key types of beliefs that affect students' strategies. Students who have a *text-centered* model "believe that they are receivers of meaning" (Weinberg & Wiesner, 2011 p. 56); they may be likely to try to transcribe aspects of the lecture as literally and "accurately" as possible for later memorization and replication. In contrast, students who have a *reader-centered* model if they think of their participation in the lecture—even if it is passive—as a meaning-making process; these students may be likely to be selective about the aspects of the lecture that they record and construct their own interpretations of important aspects of the lecture.

## 3. Methods

Data is being collected on an on-going basis in an introductory abstract algebra class during the Fall 2011 semester. The instructor self-identifies as a traditional teacher who uses lecture as his principal in-class pedagogical technique and maintains nearly complete control over the content.

*Classroom observations.* Approximately seven class meetings will be observed and video recorded throughout the Fall 2011 semester. Field notes will focus on the relationship between gestures, speech, and the text written on the board. Thus far **two** class meetings have been observed and recorded. The observations will be selected to capture a variety of typical

lecture content (including the introduction of a new concept, a proof-writing episode, and working of a homework-type problem) and a variety of presentation formats (including episodes where the instructor discusses an idea without writing on the board and an episode where the instructor writes ideas on the board non-linearly.

All of the instructor's talk and board work during the relevant portion of the lesson will be transcribed—including the order in which the board work was developed—and all gestures will be described. We will construct tables with 3 columns to describe the written text, spoken text and described gestures.

**Analysis of classroom data**. Data will be coded in two distinct manners that will then be synthesized during analysis. First, each piece of text will be analyzed to explicitly describe the mathematical meaning that it conveys to an expert reader, including a description of any explicit links to mathematical ideas from outside the lecture. Each of these pieces will be marked for the expert observers' perception of the instructor's emphasis of importance. In order to describe the implied reader, we will create a set of notes that capture a possible "expert" observer's explanation of content, mental habits, and required competencies for learning advanced mathematics that incorporates all aspects of the text. Some specific aspects of analysis include describing the requirements in terms of symbols, proof-skills, knowledge of examples and properties, and the various verbal, symbolic, and gestural codes. Finally, we will compare and contrast the aspects of the implied reader across the different aspects of the text (written, spoken and gestural) to describe the barriers and supports to understanding the mathematics that these different aspects may provide.

**Data from students**. Seven students will participate in this study. We will collect their classroom notes from the observed course meetings and assess their understanding of the relevant content with a written instrument. We will conduct semi-structured interviews with each student them about their note-taking habits and beliefs about their role in a lecture-based classroom, and ask them to give a short summary of how they use their notes as part of doing homework and preparing for exams.

Prior to conducting each interview, we will identify excerpts where the student's notes differed from what the instructor wrote on the board, what the instructor said, or the gestures that the instructor used; these excerpts will be used during the interview to prompt discussion. The interview will include the following questions:

1. How do you take notes in this class? For you personally, what is the purpose of taking notes in this class?
2. How do you plan to use your class notes?
3. [Using a video clip where the instructor attempted to convey a difficult idea or example:] Was there something in [this video clip] that you felt was difficult for you to take notes on?
4. [Using a video clip where there was (or wasn't) something in the lecture—verbal, written, or gestural—that wasn't recorded in the notes:] How come you didn't (or did) record this aspect of the lecture in your notes?
5. [Using an example where the board work isn't developed linearly:] What aspects of this part of the class/lecture do you think are significant? What aspects did you decide to capture in your notes?

**Analysis of student data.** The analysis of the students' notes will focus on differences and similarities between the text of the lecture (as described by the "expert" observers) and the students' notes. We will identify the implied reader of the lecture and use this construct to try to

understand some of these differences. For example, the course instructor may rely on various proof heuristics—such as an "onto proof"—and we will describe whether these heuristics are part of the implied reader, whether the students' notes are guided by this heuristic, and whether the students' notes and their interview responses indicate that the underlying codes are meaningful to them. In addition, we will compare the students' notes across types of episodes (definitions, examples, etc.) and during instances where the board work is not developed linearly to understand and characterize the implied reader.

We will also characterize the students' reading models and use these to interpret patterns in their note-taking habits and the extent to which their notes match the written part of the lecture-text.

**4. Proposed discussion questions**:

1) What would you most want to know about how students take notes?
2) What are the benefits and drawbacks of framing this study using the ideas of the implied reader and reading models? What critiques would you offer?
3) Is this a fruitful line of inquiry for mathematics education? Given our interest in non-lecture-based classes, is the RUME community interested in this focus of research?
4) The implied reader of a mathematics lecture may very well be different from the empirical readers. As a result, using this framework for analysis will very likely portray lecturers as "out of touch" with their students. What kinds of things should we be thinking about and doing in order to help ensure the continued engagement of our colleagues with research that they might see as adversarial to their teaching practices?

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**Title:** Student Troubles with Simple Harmonic Motion Models

**Category:** Preliminary Research Report

**Abstract:** Many studies exist on student difficulty transferring mathematical knowledge to physics, on student understanding of trigonometry, and student ability to create graphical representations of functions. However, there are no studies that exist in the intersection of these issues. This study sought to explore student understanding of simple harmonic motion by examining how their approach to graphing the sine and cosine functions impacted their ability to graph sine and cosine based models of simple harmonic motion. The findings of this study conclude that neither an object perspective or process perspective of the graphical representations of sine and cosine is sufficient for the ability to graph simple harmonic motion modeled based on cosine. There seems to be an element missing, a connection students must make between the changes in input type, that needs to be addressed in order for students to create a graphical representation of a cosine-based simple harmonic motion model.

**Keywords:** student understanding, trigonometric functions, simple harmonic motion, graphical representations

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## 1 Introduction and research questions

Many students find physics, both algebra and calculus-based, to be a challenging subject. This may be in part due to some of the difficulties that students experience when they are expected transfer their mathematical knowledge to models of various physical concepts, such as simple harmonic motion. Several studies have sought to describe these difficulties by studying students' abilities to transfer knowledge of algebra and calculus concepts to applications within physics (Cui, et. al., 2006; Ozimek, et. al., 2004). Students that are enrolled in algebra-based physics may face additional difficulties with this transfer the equivalent of a high school algebra course is often the only prerequisite to taking the course.

The work that is currently available on student transfer of mathematics to physics has focused primarily on algebraic or calculus skills. While much work has been done in the field of mathematics education to uncover student understanding of trigonometry concepts, how these understandings impact student performance in physics has not yet been explored in depth. In particular, there seems to be some room for clarification on how student understanding of the multiple representations of trigonometric functions may impact their performance on graphing modified forms of the trigonometric functions such as those that model simple harmonic motion.

This study was inspired by a project performed with a physics educator that was interested in assisting students graph position versus time functions modeling simple harmonic motion. Using the framework provided by Moschkovich, Schoenfeld, and Arcavi (1993) to identify which graphical representation perspective a student possesses, this study sought to uncover which perspective was necessary or sufficient for students to be able to analyze the trigonometric equations that model simple harmonic motion. Students were also given instructional activities designed to help them understand the change from angular input of traditional cosine and sine functions to the expected input of time in the physical situation. Thus we sought answers to the following questions:

1) Which perspective of the graphical representations of sine and cosine, if either, is sufficient for the ability to correctly graph $x(t) = 2m\cos\left(\frac{2\pi}{5s}t\right)$?
2) What connections does the student make between graphing $x(t) = 2m\cos\left(\frac{2\pi}{5s}t\right)$ and the motivation for the switch from angular measure as input to time as input?

Here we take "correctly graphing" to mean that the student is able to create the correct shape, show appropriate scaling, and accurately label intercepts and maxima and minima of the function.

## 2 Literature and Framework

The work of Ozimek, et. al. (2004) suggests that students can successfully transfer their knowledge of trigonometry to applications in physics. However, these findings only confirmed transfer in the cases where the physics problems the students were given mirrored specific instances of right triangles.

Hence a review of the literature on student understanding of trigonometry was in order to address what aspects of student understanding of trigonometry impact student ability to work with equations modeling simple harmonic motion and oscillating behavior. Unfortunately, though much work has been done on student understanding of trigonometry none seems to focus specifically on students' abilities to switch between function, tabular, and graphical representations. Luckily, looking at the available literature on students' abilities to switch between representations at a more general level yields a useful framework for analyzing student work and understanding of functions.

The framework developed by Moschkovich, Schoenfeld, and Arcavi (1993) has two dimensions. The first dimension encompasses the means available for representing the functions, namely algebraic, graphical, and tabular. The second dimension addresses the perspective, object or process, from which a function is viewed. In the object perspective a function or its graph are thought of as "entities" that can be picked up and rotated or translated whereas in the process perspective a function is thought of as linking *x* and *y* values (Moschkovich, et. al., 1993). These distinct perspectives are well applied in the case of sine and cosine where students are often encouraged to memorize a table of values, a process perspective, and a portion of the graph that can be replicated due to the periodic nature of the functions, an object perspective.

**3 Data collection and methodology for analysis**

Data was collected in two phases. The first phase was conducted during the Spring 2011 semester with students enrolled in the second course of a two course algebra-based physics sequence. At the beginning of the semester the students were presented with a physics lab designed to review several trigonometric topics that would arise throughout physics that semester. One activity on the lab was a treatment for helping students connect angular measure with time. These labs were collected and scans were made so that the labs could be returned to the students. Based on answers to these trigonometry labs, a round of task-based interviews was conducted with 3 participants. The second phase was conducted during the Summer 2011 semester with students enrolled in the first course of the two course algebra-based physics sequence. During this phase only task-based interviews were conducted. The treatment for connecting angular measure to time and emphasizing their linear relationship was given in the form of a task during the interview.

During both phases each interview contained the following four tasks:

1) Sketch the graph of a basic function. This was to establish the participant's ability to connect an algebraic representation of a function such as a parabola or line with its graphical representation. Here participants were allowed to proceed in whichever manner they chose, though it was anticipated that they demonstrate a process perspective. We wanted to be sure that they recognized the process perspective as a valid method for producing a graphical representation.
2) Sketch the graphs of $y = \sin(x)$ and $y = \cos(x)$. This was to capture their natural perspective regarding the graphical representations of sine and cosine, to determine whether they first approached using an object perspective or process perspective. Participants were then prompted to attempt to use the perspective not chosen in order to establish whether they were capable of both.
3) Sketch a position versus time graph to model a given physical situation, namely a glider on a track attached to a spring. This was to establish their inherent comfort with the situation being modeled and to determine the level to which they were comfortable with their intuition.
4) Sketch the graphical representation of $x(t) = 2m\cos\left(\frac{2\pi}{5s}t\right)$. It was during this task that we hoped to see how the student's graphical representation perspective worked in combination with the angular measure treatment to enable the student to sketch this graph with greater facility.

All interviews were transcribed and open-coded using the framework developed by Moschkovich, Schoenfeld, and Arcavi (1993) in order to determine which perspective was used by the student during a task.

## 4 Significance and directions for further research

All of the students interviewed were able to sketch and correctly label a position versus time graph to model the oscillations of a glider attached to a spring. Thus it does not seem that the difficulties they encountered in the final task, graphing $x(t) = 2m\cos\left(\frac{2\pi}{5s}t\right)$, are due to a lack of understanding the physical situation. However, the students may not have associated the equation as a representation of such a physical situation.

The students seemed to be primarily relying on their understanding of sine and cosine as functions in order to produce the graph of $x(t) = 2m\cos\left(\frac{2\pi}{5s}t\right)$. Based on their responses to the first task, sketching the graph of a linear or quadratic function, all students were capable of using a process perspective in order to produce the graph. When it came to graphing sine and cosine as functions, the participants clearly split on their perspectives. Three of the participants were able to use both a process perspective and object perspective in discussing the graphs. Two participants only possessed an object perspective and were unable to identify intercepts, maxima, and minima. The remaining two participants had no object perspective of the graphs of sine and cosine and were only able to demonstrate a process perspective using integer inputs.

Based on student responses and preliminary analysis of the final task, it seems neither the object nor process perspective is sufficient on its own for students to be successful. Those individuals that showed only a process perspective, continued to use a process perspective using integer inputs rather than more informed inputs. The participants that showed a preference for an object perspective easily recognized what shape the graph should have and identified the new amplitude, but froze in identifying the new period and often wouldn't even sketch the shape. Even the ability to switch between object and process perspectives wasn't enough to guarantee success. Those individuals started by identifying shape and amplitude, but didn't initially sketch the cosine shape and resorted to a process perspective of inputting integers in order to try to determine how the period of the function changed.

The main implication of these findings is that there seems to be some element lacking. The students rarely referred to the instructional motivations meant to help them identify the new period. Either a new motivation technique, some sort of "informed" process perspective, or an improved object perspective where the student feels more confident in his or her ability to correctly scale the base function appears to be needed.

One way this could be addressed is that mathematics instructors could spend more time emphasizing the validity of multiple representations of functions and how to translate between them. Students that froze on an object perspective were often reluctant to use a process perspective. This result is confirmed by Leinhardt, Zaslavsky, and Stein (1990) who found that students seem to steer clear of process perspective as the focus in their classroom instruction is primarily on using an object perspective. Another issue that has arisen as a result of these findings is that another method is needed for guiding students to understand the change in input from angular measure to time. The two methods investigated during his study seemed to have no lasting impact.

## 5 Questions for discussion

1) As an alternate motivation, what about introducing the translation from angular measure in radians as input to time in seconds or minutes as input as the conversion of units?
2) What other aspects of student knowledge, besides graphical representation perspective, should be taken into account when observing students translating to a graphical representation of a function?

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**Title:** What do Students do in Self-formed Mathematics Study Groups?

**Category:** Preliminary Research Report

**Abstract:** While it is widely taken as understood that students should be spending additional time outside of the typical undergraduate mathematics classroom studying, little is known about how students spend that study time. Currently available research has investigated how much time students spend outside of the classroom studying, whether they work alone or with others, and what materials students keep on hand while studying. However all of these studies rely on self-reported data in the form of interviews or anonymous surveys. This ethnographic study undertakes to expand our understanding of what activities students are engaged in when they say that they are "studying" through direct observation, journal entries, and interviews. Particular attention is given to how students study together in groups and how students make use of the materials they bring with them for studying purposes.

**Keywords:** group work, study habits, discourse analysis

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## 1 Introduction and research questions

The majority of the university system is set up with the didactic contract that students are expected to spend up to 3 hours per hour spent in class studying outside of the classroom context (Wu, 1999). In fact, the predominant advice both from instructors and study guides (Greenman, 1993; Swain, 1970) to a student that is struggling in a mathematics course is to spend more time outside of the classroom working through the material to improve his or her understanding. This advice is often supplemented with suggestions to work with peers from the class. However, when this advice is given little instruction or guidance is given regarding how to effectively work on the material with peers or what resources to have on hand while studying.

In order to provide more accurate and valuable advice to improve student study habits, we need a better idea of what goes on when students are working together outside of the classroom beyond the reach of the instructor. Though there is some literature that addresses students working in groups, these groups are often situated within classrooms where a goal has been established by the instructor and the instructor is available as a resource to answer questions. There is nothing to inform us on how students set their own goals as a study group or how they proceed without an instructor nearby to keep them on task.

This study seeks to address this gap in information. Students participating in this study have the ability to drop in to a study lounge to work together on their schedule where they are video recorded. Through these recordings, supplemented by the collection of journal entries at the end of each study session and interviews, this study seeks to answer the following research questions:

1) What roles do students assume while working together in self-formed groups?
2) How are these roles impacted by the goals of the study group and the course content the group is working on?
3) What material resources, such as textbooks, class notes, or websites, are the groups utilizing? How are these resources being utilized?

## 2 Literature and framework

The primary inspiration for this study is drawn from the work of Uri Treisman (1985). During his dissertation work he observed that students working together in groups were more successful in learning calculus concepts. Although his work focuses on the creation of the workshop program at Berkeley and its subsequent impact on student learning, his references to how students created communities of study for themselves resonated with student behaviors I have observed as both a teaching assistant and an instructor.

Efforts to find further details of what transpired in student study groups or additional studies of student study behavior yielded mostly studies based on student self-reported data. Some of this information was collected through anonymous surveys and tracked time each student spent working on the subject outside of the classroom or how confident the student felt about the material (Cerrito & Levi, 1999; Rohrer & Pashler, 2007). Other studies conducted interviews with students to uncover how they spend their time outside of the classroom preparing for tests (Danish Institute for Educational Research, 1970; Hong, Sas, & Sas, 2006). However, there is little in the way of observational data to back any of these findings up. Thus it is still unclear what it actually looks like when a student studies.

In order to observe students in their studying situations, an ethnographic approach is needed. Adopting the perspective that students in these situations socially construct their knowledge means the framework used to justify what data to collect and how to analyze it must be able to account for symbolic gestures in addition to dialogue. Using symbolic interactionism

as a lens for observing these interactions provides one way to combine a student's utterances, gestures, and other performed microtasks in order to interpret a student's intentions or the role the student has taken in the group (Blumer, 1969; Charon, 2010). The discourse analysis work of Goos, Galbraith, and Renshaw (2002) and Blanton, Stylianou, and David (2009) provide a compatible framework for coding student utterances in order to more carefully analyze a student's contribution to the dialogue.

**3 Data collection and methodology for analysis**

Participants for this study have been drawn from a second year undergraduate mathematics course that encourages students to work together in an inquiry-based tradition. This is achieved by using a lecture room equipped with round tables for the students to work at, assigning group projects, and allowing students to work together on homework assignments.

The participants in this study have been given access to a study lounge during hours scheduled to meet their studying needs. By restricting access to this lounge to only the participants there is no struggle for them to find seating or space to work and it provides a respite from the noisy dormitories if they dislike working in their rooms. The study lounge is equipped with two computers with internet access and several suites of mathematical software, round tables with chairs, and a white board. Students are video-recorded while working in this room, whether together, near each other, or alone. Students that have opted to do so are also invited to complete journal entries recording what materials they worked with during the session, which individuals they worked with during the session, what course content they worked on, and how successful they felt their study session was. Students will also be interviewed twice throughout the semester, once roughly halfway through the semester and once at the end of the semester. These interviews will be designed to gather additional information about comments students leave on the journal entries and particular behaviors they exhibited during recorded study sessions.

All video recordings and audio recordings are being transcribed and coded. In particular, I am looking to create a catalog of microtasks that occur during the study sessions. Actions such as sharing a print out of the homework assignment, consulting class notes, and writing on the white board are considered microtasks. At this level each student utterance is also considered a microtask. The use of the coding scheme developed by Goos, Galbraith, and Renshaw (2002) and Blanton, Stylianou, and David (2009) will help identify types of student utterances and will serve as microtasks as well. Once this list of microtasks is compiled, this study intends to search for patterns in the microtasks performed by each participant in order to determine what sort of role that participant is playing in the group.

**4 Results and significance**

Data collection is still underway at this time. Thus far however, some interesting phenomena have been observed. Discrepancies are arising between how I, as the researcher, would describe some of the events that have transpired and how the students appear to perceive these same events. For instance, in one event it happened that two individuals came to the study lounge around the same time to work on a homework assignment. Their arrival times were staggered and although they sat at the same table, they left an empty chair in between them so that they were not sitting immediately adjacent to each other. Although they engaged in some discussion over one problem the majority of their time was spent in silence as they worked on their individual tasks. From my position as observer, I would not have considered these two individuals to be working "as a group." On the journal entry for that day however, one of the individuals reported that he "worked in a group" with the other individual that was present.

Thus, "the reason why observation is so important is that it is not unusual for persons to say they are doing one thing but in reality they are doing something else" (Corbin & Strauss, 2008, p. 29). So one major implication of this study is its potential to confirm or contradict some of the earlier findings that have been published based on student self-reported data.

On another occasion, during one of the more lively study sessions, a group of 5 individuals came in to work together on a homework assignment. They proceeded to outline what homework problems still needed to be worked on and then split into 2 or 3 subgroups of individuals clarifying their understanding of a problem or checking their answers. They would suddenly converge as one group again, re-assess what problems individuals still needed to work on, then again split into 2 or 3 subgroups, comprised of different individuals than before. This process continued for the duration of their 2 hour study session. What is interesting about this is that there were very few times when the group focused on one homework question all at the same time. There was instead a very natural ebb and flow as students took turns being an authority on a question and aiding peers depending on which question a subgroup was working on. Yet their occasional convergence to assess everyone's completion status indicates that they were organizing their efforts as an overall group.

Studying the way the group breaks out into subgroups and then reconvenes in addition to understanding the roles that arose in those subgroups and in the overall group, provide a way to describe different study groups of students based on their dynamics and the roles they are composed of. Hence another implication of this study is that it lays the groundwork for comparing groups to assess efficacy by providing a means of describing the group based on dynamics and role composition.

Finally, this study also contributes information regarding what resources students are using to find answers to their questions when the instructor is not around. In addition to simply generating a list of textbooks referenced and websites visited, this study provides a means for assessing how these materials are being used. For instance, from video-recordings and interviews, it can be determined whether a website was used to generate a correct answer to a homework question or whether it was used to gather further information about the concept in order to develop an improved solution strategy. With such knowledge instructors can create assignments that take these material utilizations into account.

**5 Questions for discussion**

There are many questions that could be raised for discussion regarding the methodology, the chosen framework, or even the implications of the findings. I am choosing to focus on the following questions for discussion:

1) What other perspectives or frameworks may provide an insightful analysis of the data being collected?
2) What information is there to be learned from observing students working alone or silently near each other?

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**Undergraduate Proof in the Context of Inquiry-Based Learning**
Preliminary Research Report

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This research project explores students' proof abilities in the context of an inquiry-based learning (IBL) approach to teaching an introductory proofs course. IBL is a teaching method that focuses on student discussion and exploration in contrast to lecture based instruction. Data was collected from three sections of an introductory proofs course, which included 70 students total. Data collection included a portfolio from each student, consisting of their work on every proof assigned throughout the course, as well as each student's final exam. Contrary to previously published research relating to courses taught in a more traditional lecture based setting, this data analysis suggests that students developed a strong grasp on how to correctly use definitions and assumptions within the context of their proofs. Results also suggest that within the IBL setting, students generally organized their proofs in an efficient, thoughtful, and logical manner.

**Key-Words**: Proof, Inquiry-Based Learning, Undergraduates, Definitions, Assumptions

Current methods for teaching mathematics often consist of lecture-based lessons followed by students completing homework on their own. This classroom structure does little to encourage the development of deep problem solving techniques that will stay with students after they have moved on to higher-level classes. An emerging method to combat these potential problems is Inquiry Based Learning (IBL). Stemming from the Modified Moore Method, IBL focuses on student discussion and exploration in contrast to lecture-based instruction. Instructors typically place a high responsibility on students for their own learning and use leading questions to prompt students' problem solving. "As mathematics education researchers turn their attention to IBL, evidence mounts that this approach to the teaching of mathematics is ideal for the teaching of proof" (Schinck 2011). Studies conducted by Boaler (1998) and Rasmussen and Kwon (2007), summarized in Schinck's (2011) article, deduce that IBL students experience mathematics in a way that deepens their comprehension of abstract ideas essential to proofs.

This report focuses on three sections of an introductory mathematical proofs course taught using IBL. The structure of the course required students to present various assigned problems, which the class would then discuss together to encourage further student collaboration. The 70 students also each completed a portfolio consisting of all assigned problems, some of which were also turned in as homework.

For the purposes of this research, we chose to evaluate ten problems from each student using a coding scheme developed using previous work related to mathematical proof. We chose two similar problems from each content area covered in class – one presented and one not. Our coding scheme consisted of two parts to evaluate the selected student work. The first level of the coding scheme is adapted from work by Harel and Sowder (1998) and the second level is adapted from Andrew (2009).

The first level of coding focused on categorizing student proof attempts as *analytical* or *empirical* (Harel & Sowder, 1998). Proofs that did not belong in either category were coded as *other*. Coded problems were deemed analytical more than 95% of the time. The second level of coding was used on these analytical student proof attempts. Using codes developed by Andrew

(2009) to address the results of Moore (1994), the second level coding focused on structure of student proof and identified errors with implications and steps in the proof. We were also interested in student use of definitions and assumptions since Moore (1994) suggested these were significant issues in a lecture-based introductory proofs course. Therefore, the second level of coding also recorded the number of assumptions and definitions used in each proof and kept a tally of those that were incorrect.

In direct contrast to Moore's (1994) observation regarding his research, we found that students consistently used definitions and assumptions appropriately. The table below shows the percentages of incorrect definitions and assumptions for all problems coded.

| Total Problems | % Incorrect Definitions | % Incorrect Assumptions |
|---|---|---|
| 473 | 2.574 | 3.525 |

The low percentages may suggest that the students' understanding of definitions and assumptions will be an asset to their future work in mathematical proof. This begs the question of whether the drastic variation in teaching method made a difference in the conceptual understanding and use of definitions and assumptions.

The second level of coding also utilized codes developed by Andrew (2009) related to the structure (S) and understanding (U) of proof. The table below describes the codes.

| Codes for Structure | | Codes for Understanding | |
|---|---|---|---|
| S3 | Ideas not in logical order | U4 | Crucial step missing or not justified |
| S4 | Extra details or hard to follow | U5 | False statement; incorrect implication or equivalence |
| S5 | Illegible or difficult to read | U6 | One aspect of proof not addressed |
| S8 | Nonstandard or confusing notation | U7 | Forgot conclusion |

The table below shows the per problem average, as well as averages on presented (P) versus not presented (NP) problems, for structure (S) and understanding (U) codes.

| | Total Problems | S3 | S4 | S5 | S8 | Avg S | U4 | U5 | U6 | U7 | Avg U |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **All** | 473 | .125 | .180 | .002 | .412 | .719 | .687 | .326 | .448 | .214 | 1.675 |
| **P** | 259 | .193 | .208 | .004 | .39 | .795 | .734 | .398 | .734 | .263 | 2.127 |
| **NP** | 214 | .042 | .145 | 0 | .439 | .626 | .631 | .238 | .103 | .154 | 1.126 |

U4 and U6 have the highest averages, implying that students had some difficulty addressing all components necessary to prove a statement. Though S8 was the most common Structure code, the low occurrence of S3 codes suggests that students are relatively competent in organizing their thoughts, even though they may struggle with expressing them using standard notation. Overall, there is less than half the number of recorded codes in the S category than in the U category, meaning that the bulk of student error did not lie with proof structure but with understanding proof techniques.

Each problem presented in class received on average approximately one more U code than those that were not presented. In fact, every code in the *Presented* category, excluding S8, has a higher average than the same code in the *Not Presented* category. This may seem unexpected since one might assume that students would commit fewer errors on problems that

were discussed in class. However, due to the teaching method used in the class, there is a possibility that students merely copied down the problems they saw presented while not fully understanding what they were writing. Another explanation could be that since students took part in correcting the presented problems, they understood the common errors and learned how to avoid them when attempting similar problems on their own, resulting in a lower frequency of error. Also for the presented problems, students received on average almost three times as many U codes as S codes. Although this difference is lower for the non-presented problems, there are still almost twice as many U codes as S codes. This again suggests that students struggled more with understanding of implications within proofs than with language and notation.

Using the previously described coding scheme, we extended our research to code the part of the final exam that required proving theorems related to previously unseen definitions. Students were asked to use new definitions and hence make assumptions related to ideas that they had not previously been exposed to.

The table below shows the percents of incorrect definitions and assumptions for all coded final exam problems.

| Total Problems | % Incorrect Definitions | % Incorrect Assumptions |
|---|---|---|
| 95 | 13.3 | 2.96 |

When comparing the final exam problems to the coursework, students had more than four times as many incorrect definitions in the final. Though this seems like an extreme difference, the reality is that 86% of definitions used on the final were used appropriately. In consideration of the unique circumstances that make up those found within the confines of a final exam (including time constraints and stress), one may consider this 86% rate commendable. Moreover, students consistently used assumptions correctly, shown by only a slight variation (.565%) between the statistics of assumption use from the coursework and final exam. Thus, it is logical to conclude that the class successfully prepared students to properly use assumptions and definitions.

The table below shows the average structure (S) and understanding (U) codes for all final exam problems.

| Total Problems | S3 | S4 | S5 | S8 | Avg S | U4 | U5 | U6 | U7 | Avg U |
|---|---|---|---|---|---|---|---|---|---|---|
| 95 | .083 | .307 | 0 | .561 | .951 | .702 | .547 | .311 | .063 | 1.623 |

Students had limited issues with language and notation and regularly made conclusions for their proofs. Similar to the course statistics, the highest average codes in the final exam problems came from S8 and U4. U5 also had a high average in the final exam problems. Thus, once again it is clear that students struggled with fully understanding what was needed to prove all aspects of the problem. Overall, there is little difference between the final exam statistics and those of the course problems.

The high percentages of analytical proofs imply that this course provided students with a foundational understanding of formal proof development. Students used definitions and assumptions correctly over 95% of the time, which suggests that this particular IBL classroom environment gave students a firm foundation of how to correctly use definitions and assumptions. Fewer errors in the non-presented problems than in the comparable problems discussed in class further supports the claim that class collaboration prepared students to competently complete proofs on their own. Almost two implication errors (U codes) per problem suggest that at this level in their mathematical career, the observed students still struggle

somewhat with understanding how one step leads to the next. Less than one language and notation error (S code) per problem on average is evidence that this IBL class taught students how to convey their thoughts in an efficient and logical manner.

**Questions**

1. What categorizations of proof are most interesting to investigate in this context?
2. What analysis of the remaining final exam problems would be most beneficial?
3. Which of the U and S codes is most meaningful to focus on?
4. Is there another analysis of definitions and assumptions that would be meaningful in relation to this data?

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**Contributed Report:**

# Title: Genetic Decomposition of Integration

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## Abstract
In this paper I present a theoretical analysis (genetic decomposition) in the sense of APOS theory, of the cognitive constructions for the concept of infinite Riemann sums and the Fundamental Theorems of Calculus as a linking tool between the derivative and the integral, following Piaget's model of epistemology. This genetic decomposition is primarily based on my own mathematical knowledge as well as on my personal continual observations of students in the process of studying integration. I also present empirical data in the form of informal interviews with students at different stages of learning. The analysis of those interviews will later suggest a review of the initial genetic decomposition. Based on this analysis I also suggest instructional procedures that motivate the mental activities described in the proposed genetic decomposition. This study will shed new lights on the concept and make the connections more obvious between two key concepts in calculus.
***Keywords:*** genetic decomposition, APOS theory, Calculus, integration, interviews, observations, Piaget

## Introduction: Motives for the research:
Riemann sums and areas are generally taught in isolation from the antiderivative per se. This study was triggered by my dissatisfaction, as a teacher, with textbooks' general tendency to overlook the role of Riemann sums as a bridge between derivatives and integrals; the reason being that Riemann sums are hard to teach as such: a fine understanding of the Riemann sums justifies that the area under the curve of a positive function can be interiorized into a continuous function; and applying the Mean Value Theorem to this function yields to the Fundamental Theorem of Calculus, and hence connects the definite integral to the indefinite integral. Connecting the two faces of the integral is a lot of work, no wonder the two types of integrals (definite and indefinite) are usually presented in isolation, as if one is a geometric meaning and the other is a detached analytical meaning. In this paper I discuss ways of promoting this connection and a genetic decomposition of the Riemann sums and the process as a whole. Calculus instructors tend to avoid this connection because it involves heavy work, such as the transformation of the definite integral into a function $G(x)=\int_a^x f(t)dt$ that requires the cognitive operation of interiorizing the action of evaluating the area under a curve into a process, encapsulating the process into a function, and later differentiating that function and executing other actions on it. In general, this segment of the course is covered just about as lightly as the notorious delta-epsilon definition of a limit.

## Framework for research
In my study I adopt as framework for research an interpretation of constructivism and Piaget's ideas on reflective abstraction (Dubinsky, 1991). This paradigm has been applied to diverse topics including functions, mathematical induction, calculus, quantification, and

abstract algebra, and equivalence classes and partitions (Hamdan, 2006) and has lead to major curriculum changes.

## Preliminary Genetic decomposition

**Required constructions**

The schemas that one needs to have prior to the introduction of the Riemann sums are: A schema for functions of one variable and for real numbers; a schema for the Cartesian planes including points as objects, and distances between points as processes and actions; a schema for limits in general including limits at infinity; actions related to the sigma notation for the finite case; a schema for basic geometry that includes areas of rectangles; a schema for (finite) sequences including observing a pattern of objects and labeling them using appropriate indices; a schema for the derivative including its properties and rules and theorems around it, including a deep understanding of the MVT; and finally a schema for the concept of average in general.

**Mental Activities and Constructions needing Analysis:**

1. *Construction 1:* Finding the area under a curve and over a certain interval $[a,b]$ (will take the special case of a function and specific interval $[a,b]$
2. *Construction 2*: Generalizing the previous activity to a generic function over a general interval $[a,b]$ and naming it the indefinite integral $\int_a^b f(x)dx$ of *f* over the interval $[a,b]$.
3. Will skip this analysis: Deducing properties about the indefinite integral inspired by geometrical rules
4. *Construction 3:* Converting this area into a function $G(x)=\int_a^x f(t)dt.$
5. *Construction 4:* Applying MVT on $G(x)$ to deduce the average value of $f(x)$ over $[a,b]$: $f(c)=\frac{1}{b-a}\int_a^b f(x)dx$
6. *Construction 5:* Applying differentiation to $G(x)$ and deducing the Fundamental Theorem of Calculus (part I): $\frac{dG(x)}{dx}=\frac{d\left(\int_a^x f(t)dt\right)}{dx}=f(x)$
7. Define the indefinite integral simply as $F'(x)=f(x)$ and deduce rules for it by running derivative rules backward. We shall skip the analysis of these rules.
8. *Construction 6:* Deduce the Fundamental Theorem of Calculus (part II) $\int_a^b f(x)dx=F(b)-F(a)$

**Analysis of the required constructions:**

I. *Analysis of Construction 1:*

1. First *coordinate* between the schemas for the real numbers with the schema for the Cartesian plane (including intervals and distances) through the *action* of subdividing the interval $[a,b]$ into *n* subintervals, for a fixed positive integer *n*. Then *interiorize* these actions into a *process* that gives all the equidistant *n* points $x_i$ on that interval $[a,b]$, or just as well that gives the *n*

subintervals $[x_i, x_{i+1}]$ of equal length, $d_n = \frac{b-a}{n}$. It will be agreed that *a* and *b* will be assigned the points $x_0$ and $x_n$, respectively.

2. Then *coordinate* between the schemas of functions of one variable and the schema of Cartesian plane through the action of evaluating the function *f* at those stops/spots $x_i$ and then "lifting" those verticals of length $f(x_i)$ from $[x_i, 0]$ to $[x_i, f(x_i)]$ on the graph of *f*. These *actions* then get *interiorized* into the *process* that results in an arrangement of adjacent evenly spaced vertical segments. This is followed by the *action* of connecting the tops of those verticals using horizontal segments from the point $[x_i, f(x_i)]$ say, to the point $[x_{i-1}, f(x_i)]$ starting at $x_1$. This last action will be *interiorized* into the process that results in the construction of those adjacent rectangles of equal width but different lengths. These *n* rectangles are labeled $R_1, R_2, ... R_n$ using the previously mentioned indices. Now the geometric set up is prepared, and the areas of the resulting rectangles can now be evaluated.
3. Next, *coordinate* between the schema for basic geometry and that of sequences together with the schema for the sigma notation through the action of evaluating the area $f(x_i) * d_n$ of one "typical" rectangle (the $i'th$) from the finite sequence obtained in the previous step. This is followed by the *coordination* with the schema for sigma notation through the action of summing over all $i = 1, 2.., n$ to obtain the finite sum $\sum f(x_i) * d_n$. This action is *interiorized* into a process that results into viewing the sum $\sum f(x_i) * d_n$ as a function $S(n)$ of *n*. Note that it is quite difficult for students at this stage to foresee that neither *x* nor *i* would figure in the last result.
4. Next *coordinate* between the schema for limits and the schema for sigma notation through the action of evaluating the limit of $S(n)$ as *n* tends to infinity.

II. <u>*Analysis of Construction 2:*</u>

At this stage, the students could have an action conception for the definite integral $\int_a^b f(x)dx$ as the area over a FIXED interval $[a,b]$: Following the discussion on the process-object duality, it seems that students would need to interiorize the action of forming $\int_a^b f(x)dx$ for various intervals $[a,b]$ into a process with *b* as a parameter. It makes more sense to refer to *b* as *x* and refer to the above expression as $\int_a^x f(t)dt$ over the interval $[a,x]$.

III. <u>*Analysis of Construction 3:*</u>

Then one needs to *encapsulate* the resulting process into an *object* which they may now denote as the function $G(x)$.

IV. <u>*Analysis of Construction 4:*</u>

Through geometric guesswork, students are lead to deduce the average value $f(c)$ of $f(x)$ over the interval $[a,b]$: this is a simple construct that is inspired

by geometric speculation: the simulation of a rectangle of width $b-a$ and of height the magical $f(c)$: thus they reach the conclusion:

$f(c)=\frac{1}{b-a}\int_a^b f(x)dx.$ So the mental activity witnessed here is the interiorizing into a process of the action of finding the average value $f(c)$, given a function $f$ and an interval $[a,b]$. Finally, this process is encapsulated into the object: the average value $f(c)$ of $f$ over the interval $[a,b]$.

V. <u>*Analysis of Construction 5:*</u>

1. In the meantime, and on the back burner, the constructed function $G(x)$ has been encapsulated into an object; it becomes reasonable to execute on it an action, namely that of differentiation through using the formal limit definition of the derivative itself: $\frac{dG(x)}{dx}=\lim_{h\to 0}\left(\frac{\int_a^{x+h} f(t)dt-\int_a^x f(t)dt}{h}\right)=\lim_{h\to 0}\left(\frac{\int_x^{x+h} f(t)dt}{h}\right).$
2. Now, one needs to *reverse* the existing internalized process of finding the average value and view the last limit as the average value $f(c)$ of $f$ over the interval $[x,x+h]$. This will yield us we obtain $\frac{dG(x)}{dx}=\lim_{h\to 0} f(c)=f(x).$
3. Note that one can describe this step as "the undo of the operator $G(x)$ is the operator derivative.)

VI. <u>*Analysis of Construction 6:*</u> Deducing FTC II: $\int_a^b f(x)dx=F(b)-F(a)$ now that the concept of antiderivative or indefinite integral $F(x)$ is defined such that $F'(x)=f(x)$ and denoted (surprisingly) by $F(x)=\int f(x)dx.$

1. <u>*Method 1*</u>:
   Note that since both $F(x)$ and $G(x)$ have the same derivative, then, according to a previous theorem from the derivative, the two functions differ by a constant and consequently, since $F(a)=0$ and $F(b)=G(b),$ the result simply follows.
2. <u>*Method 2*</u>:
   Alternatively, one can decompose the difference $F(b)-F(a)$ into $\sum\left(F(x_i)-F(x_{i-1})\right)$ and then by *reversing* the process that produces (the definition) of the derivative, one can deduce that each difference is some $F'(x_k)\Delta x_k$. Hence $F(b)-F(a)=\sum F'(x_k)\Delta x_k=\sum f(x_k)\Delta x_k$ which is the area under the curve, namely, $\int_a^b f(x)dx.$

## Instructional Procedures:

I assume that any successful instruction of mathematical constructions would take into consideration the cognitive structures, as well as the mechanism (reflective abstraction) on which these constructions are built. The preceding epistemological analysis serves as a guideline for planning

instruction. In the following I present a selection of activities that were designed to help students along the cognitive steps in the genetic decomposition.

**Activities promoting the different mental constructions:**

*I.* *Activities promoting evaluating area*

1. Exercises on Sigma notation and its properties
2. Activities on estimating area under a curve using different numbers and heights of rectangles.
3. Make up theorems and rules about areas in case of linear functions or in case of increasing/decreasing functions.
4. Find areas in case of a linear function.
5. Given an infinite sum, try to express it as a definite integral over a certain interval; alternatively, describe the area that it represents.
6. Explain how it is that if $b < c$ then we have $\int_a^b f(x)dx < \int_a^c f(x)dx.$
7. Shift of emphasis: Compare what the expressions $\int_a^{0.1} f(t)dt$ to $\int_a^{0.2} f(t)dt$ and $\int_a^{0.3} f(t)dt$ represent geometrically. Try to find the generic expression, a way to refer to all these expressions in terms of a generic *x* once you observe the distinguishing factor. (this exercise is a recall for the exercise where a pattern is solicited in the introduction to the section on functions: e.g. express $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$as a function of 2, 3, 4,5, etc.

*II.* *Activities promoting understanding the function* $G(x)$:

1. Exercises on comparing areas under two different curves: if $f(x) < g(x)$ over the interval $[a,b]$ then compare $\int_a^b f(x)dx$ to $\int_a^b g(x)dx.$
2. Characterize properties of $G(x)$: when is it an increasing function? Is it continuous? Does $f(x)$ need to be increasing for $G(x)$ to be increasing?
3. Compare what $G(x)$ measures to what $f(x)$ measures for a particular point *x*.
4. Express the function $G(x)$ as a composition of two functions.
5. Explain the presence of two letters in the expression of $G(x)$ and what the difference between the roles of *x* and that of *t* really is.

*III.* *Activities promoting the concept of the average value:*

1. Have students guess the average value through experimenting with linear functions at first.
2. How to decide, in the case of a line, whether the point $c$ is to the right or left of the midpoint of the interval $[a,b]$? What characteristic of a line makes you decide which side it is on?
3. Connection between MVT and average value: note the similarity of the terms: Mean/Average

*IV.* *(Challenging) Activities promoting construction of the definition of an integral by reversing that of the derivative:*

Students are challenged to look back at both the formal definitions of the derivative and that of the definite integral (as an infinite sum) and deduce how the two definitions are in retrospect) inverses of one another.

Note: I have already conducted informal interviews with the students in various stages of learning these topics. And I am in the process of analyzing them.

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**Interculturally Rich Mathematics Pedagogical Content Knowledge for Teacher Leaders**
*Preliminary Report*

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***Abstract.*** *We report on our work to build a theory about teacher leader development of interculturally aware mathematics pedagogical content knowledge (PCK). The effort is based on existing and continuing work on developing pre- and in-service teacher classroom PCK and intercultural competence. This preliminary report seeks feedback from RUME-goers on two discussion questions: Discussion Item 1: How do we identify and capture evidence of what might be called "teacher leader pedagogical content knowledge" in interculturally aware ways? Discussion Item 2: What question formats might be productive for eliciting information from teacher leaders about their awareness of/attention to the intercultural aspects of mathematics instruction? ... of mathematics itself?...of teacher leadership? This includes questions for written instruments as well as interview prompts and possible survey items.*

## Relation of the Work to the Research Literature

Teacher leaders are experienced teachers who take on responsibilities and risks to improve students' educational opportunities while working collaboratively with fellow teachers, administrators, and others (Yow, 2007). Many teacher leaders are mentors to colleagues (e.g., as math coaches or facilitators of teacher professional development, Borko, 2004), conduits of communication with administrators, and collaborators on educational policy, research, and product development – from curriculum to school budget and school law (Dozier, 2007; York-Barr & Duke, 2004). Many who identify themselves as teacher leaders report entering leadership positions without any formal training (Dozier, 2007; Lieberman & Miller, 2007; York-Barr & Duke, 2004). And, few have preparation in the teaching and learning of adults. Much of the work of a teacher leader involves negotiating meaning across professional and personal cultural differences. While the significance of diversity as a factor in the education of American children has been widely discussed for many years, the nature of "diversity" continues to evolve in U.S. schools (Aud, Fox, & KewalRamani, 2010). Several frameworks currently exist for professional contexts that involve understanding, interacting, and communicating with people from various "cultures" (see Figure 1 for working definition). In particular, healthcare and international relations groups have generated tools for personal and professional growth based on the theory of intercultural development and communication (Bennett, 1993, 2004; Hammer, 2009). "Culture" can include professional and classroom environments as well as personal or home experience. In this sense, several cultures – sets of values and ways of communicating about them – are developing for teacher leadership in the United States. A university partnership, the Mathematics Teacher Leadership Center (MathTLC), is exploring this area of collegiate mathematics education, and the potential for university-based methods in teacher leadership development. Members of the program include teachers whose current or near-future job roles are leadership positions, university mathematics and mathematics education professors as instructors in the program, and graduate student and faculty mathematics education researchers. Our goal includes building a theory about teacher leader development of interculturally aware mathematics *pedagogical content knowledge* (PCK) that is based on existing and continuing work on classroom PCK (Hill, Ball, & Schilling, 2008; Jackson, Rice, & Noblet, 2011) and intercultural competence development among teachers (DeJaeghere & Cao, 2009).

**Research Questions**
What is teacher leader PCK (TL-PCK)? How can attention to intercultural competence play a role in the development and refinement of responsive TL-PCK? In what ways do self-awareness and awareness of others as cultural beings support mathematics teacher leadership development?

**Conceptual Framework**
Our efforts rely on two theories: one theory for intercultural competence development for mathematics teaching and learning in post-secondary settings and one for PCK. The first framework is based on the *Developmental Model of Intercultural Sensitivity* (Bennett & Bennett, 2004). As a developmental model, it includes lower and upper anchor orientations, intermediate orientations, and descriptions of the transitions among the orientations. Associated with this framework is an explicit attention to aspects of discourse based on effective intercultural conflict resolution (Hammer, 2005). The continuum of orientations runs from a monocultural or ethnocentric "denial" of difference based in the assumption "Everybody is like me" to an intercultural and ethnorelative "adaptation" to difference. The development from denial to the "polarization" orientation comes with the recognition of difference, of light and dark in viewing a situation (e.g., Figure 2a). The polarization orientation is driven by the assimilative assumption "Everybody should be like me/my group" and is an orientation that views cultural differences in terms of "us" and "them." A developing tendency to deal with difference by minimizing it and focusing on similarities, commonality, and presumed universals (e.g., biological similarities – we all have to eat and sleep; and values – we all know the difference between good and evil) leads to the minimization orientation. A person in minimization will, however, be blind to deeper recognition and appreciation of difference (e.g., Figure 2b, a "colorblind" view). Transition from a minimization orientation to the "acceptance" of difference involves attention to nuance and a growing awareness of oneself as having a culture and belonging to cultures (plural) that differ in both obvious and subtle ways. While aware of difference and the importance of relative context, how to respond and what to respond *in the moment of interaction* is still elusive. The transition to "adaptation" involves developing frameworks for perception, and behavior shifting skills, that are responsive to a full spectrum of detail in an intercultural interaction (e.g., the detailed and contextualized view in Figure 2c). Adaptation is an orientation wherein one may shift cultural perspective, without loosing or violating one's authentic self, and adjust communication and behavior in culturally and contextually appropriate ways. There are several ways that knowing one's orientation, or the normative orientation of a group, can inform teacher leader work.

In thinking about TL-PCK we have relied on the layered model shown in Figure 3, where the yellow region (classroom) is the "C" of "content" in TL-PCK. In our presentation we will talk about how intercultural aspects of TL-PCK and PCK live in the model as we frame the research questions and forms of their answers and engage in RUME Session Discussion Item 1 (next page). We note that we have not yet tackled the other kinds of socio-cultural knowledge needed for teacher leaders to work with administrators, policy makers, and others.

**Goals for RUME 2012.** The work on the research questions is shaped by the program goals (see Figure 4). For example, intercultural theory gives a language for thinking and talking about how we come to communication – including communication across orientations – and how we each respond to the variety of orientations in a room (e.g., meet people where they are). The theory also gives a language to develop awareness, as someone who has perspectives about difference and similarity in educational contexts, and for calibrating self-efficacy (e.g., adjust judgments of ability to successfully complete task X to take into account how others involved in task X define "success"). In particular, at the conference we will focus on:

***RUME Session Discussion Item 1:*** How do we identify and capture evidence of what might be called "teacher leader pedagogical content knowledge" in interculturally aware ways?
***RUME Session Discussion Item 2*:** What question formats might be productive for eliciting information from teacher leaders about their awareness of/attention to the intercultural aspects of mathematics instruction? ... of mathematics itself?...of teacher leadership? This includes questions for written instruments as well as interview prompts and possible survey items.

**Research Methods**

The exploration of the culture of teacher leadership being developed by the members of the project and the nature of pedagogical content knowledge for teacher leaders is mixed-methods. All members completed a 50-item validated and reliable *Intercultural Development Inventory* (see idiinventory.com) that provided intercultural orientation profiles of stakeholder groups. These profiles were shared with all groups. To date we also have completed thematic and categorical coding of teacher leader application essays (coding of subsequent reflective essays by teacher leaders and university staff is ongoing), and initial cognitive interviews and piloting of written assessments of teacher leader pedagogical content knowledge. Further interviews with teacher leader experts developing and facilitating the program are being collected and will be analyzed, preliminary results may be shared at RUME2012 (not reported on here).

**Preliminary Results**

To give a sense of the population and a preliminary portrait of their TL-PCK and cultural awareness, analysis of application essays for 14 teacher leaders (the first of four planned cohorts) is summarized in Figures 5 and 6. Essay prompts were about (1) ideal classroom, (2) significant experiences prompting a move to leadership, and (3) personal and professional goals. Many talked about the desire to understand another persons' perceptions: "I hope the program will help me gain a deeper understanding of how other teachers view their teaching of mathematics" and a to "translate my knowledge and skills as a classroom teacher into pedagogical knowledge about adult teachers learning math and learning to teach math to diverse populations." Reports on goals included "My hope would be that through my participation in this program I would gain the skills and knowledge to improve my own teaching, better meet the needs of the diverse population of County High School and to influence more classroom teachers to be involved in the school improvement process from the classroom to the national level."

For context, we offer Figure 7, showing the distributions of intercultural orientations of program members along with a reference set of additional stakeholders: secondary mathematics teachers (the "students" of the program's teacher leaders). As a group, the teachers' orientation was normatively in polarization while the teacher leaders were largely at the lower end of minimization and university folk were largely in minimization. As part of the research process, we have conducted group profile debriefing sessions with teachers, teacher leaders, and university staff and asked how knowledge of these orientations (for oneself and awareness that they exist for others) might play a part in their professional work. We have also created items used on a written instrument and in interviews with teacher leaders to look at the various aspects of the TL-PCK model shown in Figure 3. Below, we give an example of such an item and will share others at the conference as we explore RUME Session Discussion Item 2.

*Part 1.* Create a story problem whose solution would require 8$^{th}$ grade students to solve the following for $x$: $5x - 3 = 12$.

*Part 2.* What challenges might you expect the students to encounter in doing your story problem?

*Part 3.* Now think about helping teachers in a PD workshop to build skills in writing story problems. What challenges might you expect 6th to 8th grade teachers to encounter in creating such a story problem?

*Part 4.* [Give examples of two different teachers problem posing efforts] How would you respond to each of the teachers?

**Conclusion**

Intercultural orientation is embedded in each component of the TL-PCK model in Figure 3. How and what a teacher leader notices, how and what a teacher notices, and what a teacher leader does with the noticed things in working with teachers are all connected to self-awareness and other-awareness, (i.e., to the intercultural orientations of all in the professional development classroom – teacher leaders and teachers). Though beyond the scope of this proposal, we are also aware of yet another layer that can be added to Figure 3, of university teacher-leader educators, whose students are teacher leaders and for whom the "content" is the entirety of Figure 3.

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Figure 1. Working definition of "culture."

| |
|---|
| *Short definition of culture*: A dynamic social system of values, beliefs, behaviors, and norms for a specific group, organization, or other collectivity; the shared values, beliefs, behaviors, and norms are learned, internalized, and changeable by members of the society (Hammer, 2009). |

Figure 2. The intercultural competence developmental continuum.


Figure 3. Layered model for intercultural teacher leader pedagogical content knowledge.

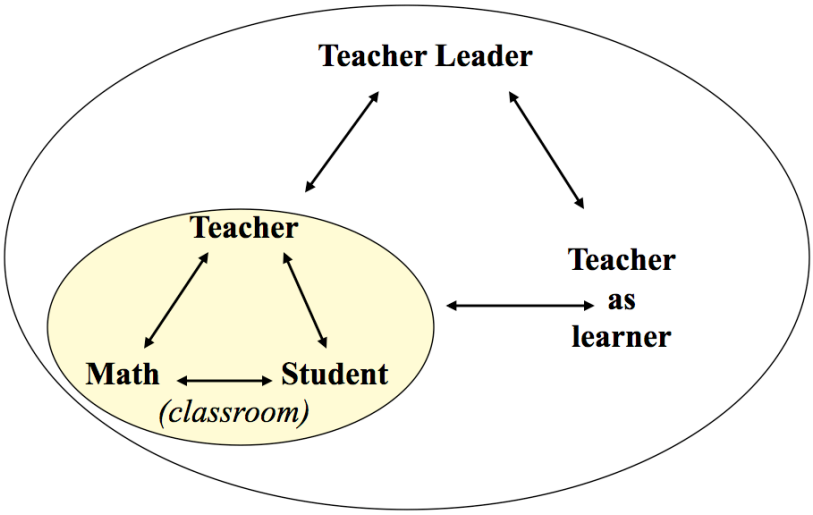



Figure 4. Goals of the Teacher Leadership Program

- Develop a shared vision of mathematics teacher leadership
- Enhance mathematics content knowledge
- Expand understanding of how teachers build knowledge for teaching mathematics
- Increase pedagogical content knowledge for teaching teachers
- Develop understanding of equity and culture in mathematics in schools and districts
- Build self-efficacy as teacher-leaders of mathematics

Figure 5. Teacher professional learning goals

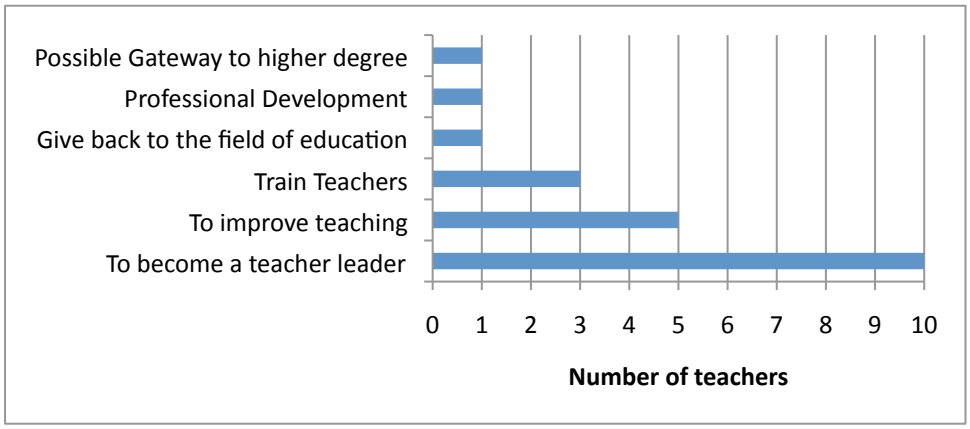



Figure 6. Teacher reports of significant experiences prompting a focus on leadership.

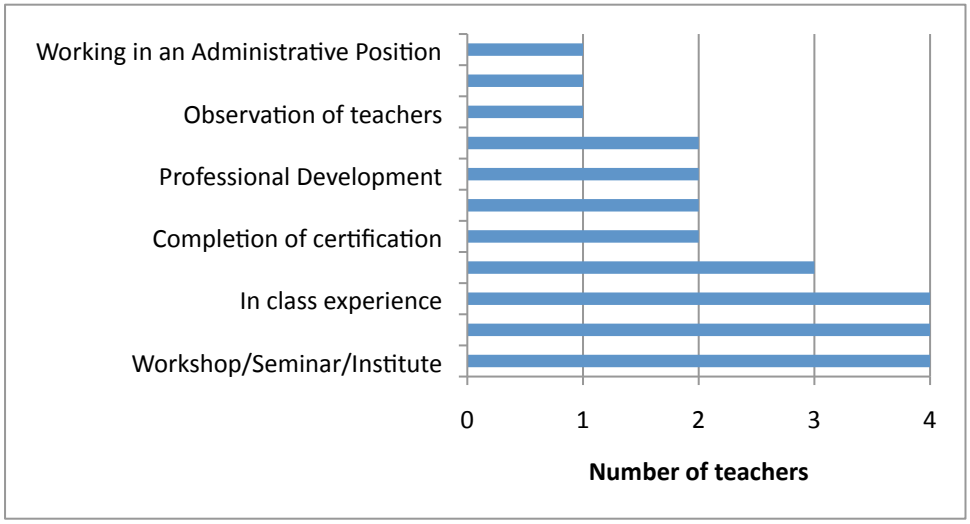
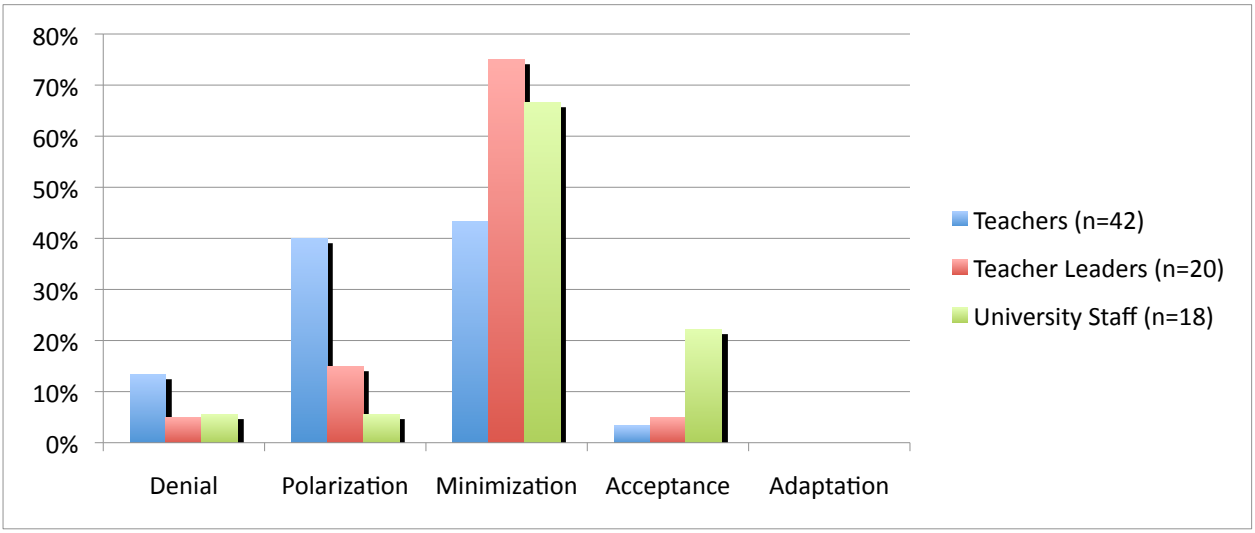



Figure 7. Distribution of intercultural orientations for stakeholder groups.



# Exploring success of underrepresented groups in university mathematics courses

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## Abstract

*Although some research indicates that the number of women in science, technology, engineering, and mathematics disciplines have been growing (Astin et al., 1983), women and other minorities in mathematics classrooms that serve these disciplines are still largely absent (Pattatucci, 1998). Given the lack of women and minorities in the classroom, how can instructors develop equity and quality in mathematics programs and fields where mathematics acts as a gatekeeper? By utilizing data collected in a differential equations course, we engage in a discussion that explores what leads to students' success in mathematics. We were interested in the interrelationship between students' demographic backgrounds and classroom dynamics to see how we can better serve women, minorities, and those from rural and first generation university backgrounds*

*Keywords: Gender, student success, equity, differential equations*

## Purpose

Although some research indicates that the number of women in science, technology, engineering, and mathematics (STEM) disciplines have been growing (Astin et al., 1983; Eisenhart & Holland, 2001), women and other minorities are still largely absent in mathematics classrooms that serve STEM disciplines (Pattatucci, 1998; Wyer et al., 2001). This is particularly the case at the Midwestern land grant institution that was the focal point of our research, where the ratio of men to women enrolled in differential equations at the time of the study was 9:1. Since this course is required of mathematics majors and many other STEM fields, it prompted us to investigate this issue.

Given the lack of women and minorities in the classroom, how can instructors at a rural land grant university develop equity and quality in mathematics programs and fields where mathematics acts as a gatekeeper? By utilizing data collected during a spring semester differential equations course, we engage in a discussion that explores what leads to students' success in mathematics. In particular, we were interested in the interrelationship between students' demographic backgrounds and classroom dynamics to see how we can better serve women, minorities, and those from rural and first generation university backgrounds.

## Perspectives

Previous research provides insight into understanding the lack of women's presence and success in the STEM fields (Correll, 2001; Eisenhart & Holland 2001; Keller, 1985 & 2001; Zuckerman, 2001). Much of the research emphasizes the ways in which young women are discouraged, through gender socialization, to take seriously a career in the sciences (Correll, 2001; Eisenhart & Holland, 2001; Keller, 1985, 2001; Muller & Pavone, 1998; Zuckerman, 2001). Correll discusses the importance of gender in men and women's choices to step into careers in STEM. In particular, cultural beliefs about men and women's ability to do

mathematics impact women's self perception of competency in this field, which ultimately impacts the career paths that women take. In addition, Muller and Pavone discuss how young women are more likely to "internalize failure, and are thus less apt to persist in an area in which they have not been particularly encouraged" (p. 250). Research also emphasizes that the first year of college is essential to tracking majors in the sciences, and if young women are lacking self-confidence and encouragement from their experiences in middle- and high school, they will be less likely to move into those major areas of study.

Although there are studies that provide personal narratives about women's success in STEM fields (Keller, 2001; Pattatucci, 1998; Sands, 2001), there is little research that examines women and other underrepresented groups' success in mathematics classrooms. From the personal narratives and interviews of successful women and minority students in the sciences, their success is largely a product of a variety of factors, ranging from parental support to personal tenacity. It is our goal to broaden this understanding and provide a working model that can encourage systemic support for women and other underrepresented students in the mathematics classroom. In turn, we explore the following research questions: (a) Who is succeeding in mathematics courses?, (b) When are students choosing their mathematics-based majors?, and (c) What do students feel has contributed most to their success in mathematics coursework?

## Methods

To examine the success of women and other underrepresented groups, students who had almost completed a differential equations in mathematics were purposively selected as participants in the study. By reaching this level of mathematics, they have proven to be successful in mathematics. We define success by the fact that students in a differential equations class have passed the calculus series in mathematics, which are often used at universities as "weed out" courses for the STEM fields. At this point, they are in their last required mathematics course for engineering, and the students who are not engineers are likely to be continuing on to another STEM field. At the end of the spring semester, five sections of students (n = 150) enrolled in a differential equations course were surveyed. One hundred and one surveys were completed for a response rate of 70%.

### *Survey Instrument*

The survey instrument, which was created by the authors, was used to examine students' perceptions of their success in mathematics courses, both in high school and at the university level. A mixed-method (Johnson & Christensen, 2004) approach was used when creating the survey. Specifically, the survey contained 10 open-response items and 23 closed-response items. The formation of these questions was informed by the background literature as well as by questions we had about student success in mathematics. In the surveys, we collected information on each student's academic background, university classroom experiences, demographic background, and parent's educational and economic background. We were particularly interested in what students attributed to their success in their major and how they dealt with challenges

### *Data Analysis*

The researchers worked as a team to enter all of the closed-ended survey responses into EXCEL spreadsheets and the open-ended responses were entered into the HyperRESEARCH software program. From here, the researchers compiled the data both section-by-section and as a whole. Demographic information was compiled first to allow data from females, first generation college students, and non-traditional students to be recorded separately, as well as with each larger group. Thus far, the researchers have only examined the quantitative data by using simple

descriptive statistics. In the final paper, the researchers will include results that compare responses to closed-response items that have been analyzed using statistics that allow for group comparisons. Open-ended response items will be used for triangluation purposes to help ensure reliability of the results. These results and the coding scheme will be discussed further in the final paper.

## Results

In our analysis of the data thus far, we have focused on who is succeeding in mathematics courses, as well as to what they attribute that success. We also wanted to look at when students are choosing their mathematics-based majors and how students "feel" about being in these classes. Results pertaining to students' feeling about their coursework will be included in the final paper. Here we provide a brief overview of the preliminary findings.

*Who is succeeding in mathematics courses?*

We began our data analysis by determining the demographic nature of the students who had made it to the level of differential equations. It was important to know who was succeeding in the mathematics before we were to examine why they thought they were successful in mathematics. The gender gap was quite pronounced, as only 15.84% (n = 15) of the respondents were female and 84.16% (n = 85) participants were male. It should be noted that the university is made up of 57% males and 43% females. In addition, only 1.9% (n = 2) of the respondents were non-white. There were no African-American respondents. These descriptive statistics, in and of themselves, tell us that some groups are underrepresented in the final mathematics course required from most STEM majors (in particular for the field of engineering) at this institution. With an expectation that these discrepancies may arise after obtaining information from the registrar on the demographics of the students enrolled in differential equations in the previous year, we were interested in finding out when these students selected their majors.

*When are students choosing their mathematics-based majors?*

Interestingly enough, nearly 75% (75 out of 101) of students surveyed are deciding that they are going into a mathematics-based STEM field before or during their freshman year at the university. This follows other research findings about the significance of the first year of college to students' selection of mathematics-based majors (Muller & Pavone, 1998). This tells us that the window of opportunity to recruit majors into mathematics-based STEM fields begins even before they enter the university system. This also means that institutions need to develop ways to understand how women and other underrepresented groups experience the mathematics classroom in order to develop a receptive climate that encourages their success in the classroom.

*What do you feel has contributed most to your success?*

Overall, the majority of students—both men and women—attributed their success to their personal drive and ability (males (42): 49.4%; females (6): 37.5%) and to their classmates (males (20): 23.5%; females (4): 25%). Parental support (males (1): .03%; females (0): 0%) and enjoyment of material (males (0): 0%; females (1): .06%) were the lowest contributing factors selected by the students. More details, including comments from students, will be included in the final version of this paper.

## Educational significance

Our goal in this paper was to provide insights on how professors can better serve underrepresented groups in the mathematics-based STEM disciplines to be successful and have a pleasant experience in their mathematics courses. The results from our research once again illustrate the pronounced lack of representation of particular groups (e.g., females and non-white students) in disciplines that require a strong mathematics background. However, these data tell

us that the students who are successful in mathematics are selecting their majors either very early on in their college careers or before they enter college. This should be a call to educators to communicate with high school teachers and people who are teaching algebra courses at the university level.

Full analysis of the data will also help professors be cognizant of how they can help to develop a community and culture that supports women and underrepresented groups. This study provides a starting point for discussion and a call for additional research on building an institutional environment that fosters women's success and values their presence in STEM disciplines. We will pose the following questions to the audience to push the research beyond the preliminary stage.

1. What ways do you know of that institutions understand how women and other underrepresented groups experience the mathematics classroom?
2. What ways do you know of that institutions develop a receptive climate that encourages their success in the classroom?
3. What would be interesting to come out of the analysis of the open-ended questions?

With this discussion, we hope to gain thoughtful insights to strengthen our research, data analysis, and further data collection so that we can disseminate quality research to the mathematics/mathematics education community.

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# ASSESSING PROOF SCHEMES: AN INTERESTING "PROOF" BY MATHEMATICAL INDUCTION

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**Abstract**

Students face an array of difficulties when they learn to understand and write proofs by mathematical induction (MI). This paper describes the responses of students in an inquiry-based (IBL) number theory course when presented with a false proof by MI asserting that all humans are the same height. The *proof schemes* of Harel, Sowder and others provided a lens through which to analyze student responses. Some were consistent with the misconceptions already in the literature on MI, while others may be especially revealing of IBL students' ways of understanding mathematical proof in general and MI in particular.

**Keywords:** Inquiry-Based Learning, Mathematical Induction, Proof Schemes

## 1 Introduction

Undergraduate mathematics students experience a variety of challenges learning to understand and construct proofs by Mathematical Induction (MI). Ernest (1984) reported that many view the basis step as unnecessary. Avital and Libeskind (1978) and other authors have noted students' serious difficulties with the logical complexity of the inductive step due to the universally-quantified implication

$$\forall k \geq 1 \left[ P(k) \rightarrow P(k+1) \right],$$

where one wishes to prove the statement $P(n)$ for all natural numbers $n$. Baker (1996) and others have described how many students focus on form over substance when writing and reading proofs by MI, and Harel (2002) has described how students accept the MI procedure as a rule handed down by an authority – a textbook author or an instructor – without developing an understanding of why MI constitutes valid reasoning and when it should be used.

An excellent way to assess a student's beliefs about proof is to ask him or her to critique a proof attempt. Some textbooks offer students an opportunity to critique a purported proof by MI of a silly and clearly false statement. Pólya (1954) offered an early example in *Mathematics and Plausible Reasoning*, suggesting a "proof" that all ladies have the same color eyes. Brumfiel (1974) reported that in a university honors calculus class, none of the students readily identified the error in a similar argument asserting that all billiard balls are the same color.

What can be learned from undergraduate students' written reactions to a Pólya-style argument? We propose to analyze the work of students in an Inquiry-Based (IBL) Number Theory course critiquing a false proof that all humans are the same height. Please see Figure 1.

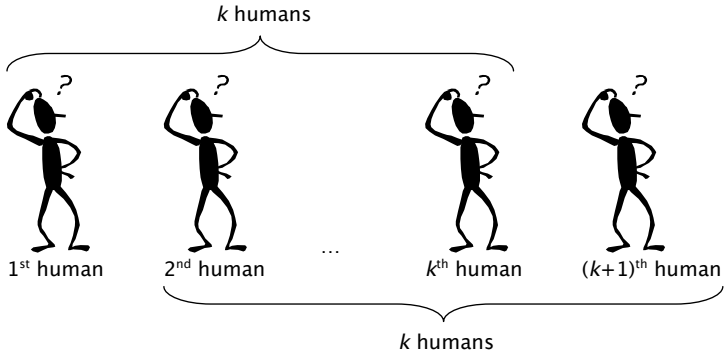
## 2 Theoretical Framework and Methodology

The proof analysis task in Figure 1 was given to 27 students in a Number Theory class at a commuter university in a working class, urban/suburban area in California. The majority of undergraduates at this university are first-generation college students, and the main ethnic subgroups in 2009 (when the task was assigned) were Hispanic 40%, White 28% and African American 11%. At this university, the Number Theory course functions as a transition to the upper division mathematics curriculum.

The instructor used IBL (also known as the Modified Moore Method) along with the text by Marshall et al. (2007). Students had spent the first class meeting exploring MI problems from their text. At subsequent class meetings students

You receive a free trial subscription to *Amazing Induction!!!* Magazine. The cover of the first issue you receive reads "Breakthrough! Proof that all humans are the same height!" Here is the argument printed inside:

*We will show that in any group of $n$ humans, all are the same height. Since there is a finite number of humans in the world, this will show that all humans in the world are the same height. As a base case for the induction, consider a group containing just one human. Of course that person is the same height as him or herself, so the statement is certainly true when $n = 1$. Next we will show that if the statement is true for a positive integer $k$, it must also be true for $k + 1$. Suppose we know that in any group of $k$ humans, all must be the same height. Now consider a group of $k + 1$ humans. We have the situation shown below.*




*The group on the left contains $k$ humans, therefore all in that group have the same height. Similarly, the $k$ humans in the right group also must have the same height. All of them have the same height as the $k - 1$ humans who belong to both groups. Therefore, in any group of $k + 1$ humans, all must have the same height. By induction, this proves that all humans have the same height.*

You know that something must be wrong here. After all, the conclusion is false! Write a brief letter to the editor of *Amazing Induction!!!* explaining what is wrong with this argument. Be as specific as possible. [Hint: it may help to look at examples to see how this argument works for specific numbers of people.]

Figure 1: Proof Analysis Task

presented their solutions and engaged in whole-class discussion of MI facilitated by the instructor. All students had Junior or Senior standing and just over half were mathematics majors.

Students were required to write their own critique of the Pólya-style argument. This assignment was graded for completion (as were similar short writing assignments given throughout the course). Written responses were collected and coded for common responses.

The work of Harel and Sowder (1998) on proof schemes provided a theoretical framework for understanding student responses. A student's proof scheme describes what that student tends to find convincing in mathematical argumentation. Students with faulty conceptions of proof may have an external proof scheme – taking the instructor's authority or surface features of an argument (such as the two-column format or the use of algebraic symbolism) as sources of validity. Studies by Harel with Sowder and others describe students moving through empirical schemes – in which a general statement is "proved" if it is seen to be true for a particular figure or for several example cases – towards deductive schemes as students gain mathematical maturity. Within the category of deductive schemes, students may possess a transformational scheme – where one freely manipulates a generic expression or figure to explain why a statement holds in all cases – or an axiomatic scheme approaching a modern mathematician's concept of proof.

Student responses shed light on their ways of understanding mathematical proof in general and MI in particular. A few responses may be especially revealing of IBL students' ways of understanding.

## 3 Preliminary Results and Discussion

Fifteen students consented to have their work considered for this study, and this sample appeared to give a faithful representation of the entire class. Their responses are summarized in Table 1. Some students incorporated several coded responses in their work, so the total in the Frequency column exceeds 15.

The flaw in Figure 1 (and all similar arguments) lies in the implicit assumption that $k$ is a generic *large* number (at least 2) in the inductive step, whereas the basis step can only establish the trivial case $k = 1$. The picture shows $k$ greater than or equal to 3, with some unspecified number of people to the right of the 2nd human and to the left of the

| Code | Response | Frequency |
|---|---|---|
| $A$ | Correct: Pointed out failure of argument when $n = 2$ | 1 |
| $B$ | Correct: This basis step does not support this inductive step | 2 |
| $C$ | Incorrect: Base case is trivial | 4 |
| $D$ | Incorrect: We can't assume the inductive hypothesis | 4 |
| $E$ | Incorrect: Misunderstanding of set theory or other points of logic | 4 |
| $F$ | Incorrect: Counterexample only | 1 |
| $G$ | Incorrect: Critique based upon context | 2 |
| $H$ | Incorrect: Critique based upon symbol manipulation | 3 |

Table 1: Student Responses ($N = 15$)

$k^{\text{th}}$. But when $k = 1$, we have $k - 1 = 0$ and the two sets of equal-height humans have an empty intersection. This argument cannot show that any two humans are the same height. We discuss some particularly interesting responses below.

### 3.1 Response $B$: Basis Step and Inductive Step Don't Work Together

A remarkably sophisticated critique that showed a mature understanding of the relationship between the two steps in a proof by MI was offered by two students who appeared to possess a modern axiomatic proof scheme. The student response excerpted below is notable for its reliance on the properties of equivalence relations:

> *As I was trying to understand how the proof worked, I realized that the base case did not give enough ground to work with. Proving the base case works for a group of one only shows a reflexive property: that a person is identical to his or herself. What is needed and what is being used in the rest of the proof is a relation between two or more people: that they have the same height. If we were to show that any two people have the same height, we could see how a larger group of people would have to have the same height. [ ...] We could follow the pattern forever and prove that it would work for groups of $k + 1$. Of course, proving that any two people have the same height would not be possible because plenty of counterexamples could be given.*

### 3.2 Response $C$: The Basis Step is Trivial

$C$ responses suggested that because the case $n = 1$ is vacuous it is illegitimate, or even that a comparison can *only* be made between two distinct objects.

> *The problem with this proof is the base case. You can't say for $k = 1$ compare your height to your own. $k = 1$ should have been a comparison of at least two humans because this is a comparison proof.*

The frequency of $C$-coded responses was striking, considering the literature on student attitudes towards MI. A fairly common misconception uncovered by Ernest (1984), Harel (2002) and others, holds that the basis step is needed only to satisfy the instructor. In a study by Baker (1996), large numbers of secondary and university students failed to recognize a missing base case in a proof analysis task. A student who views the basis step in MI as an unnecessary formality should be less likely to find fault with one that is vacuously true. But the students offering $C$-coded critiques held the opposite view. What explains these students' level of unease with the basis step in this argument?

Some $C$ responses suggested an empirical proof scheme. Smith (2006) suggests that the use of IBL with this group of students may have been significant as well. In IBL environments, students grew more likely to expect mathematical proofs to explain *why* a given statement is true. A vacuously true statement, while perfectly valid, may raise more suspicion among IBL students than ones who place less emphasis on the explanatory role of a proof. IBL students were also observed frequently using examples to help them make sense of mathematical statements whereas their non-IBL peers appeared to view examples as unhelpful because they were not proofs. Smith's findings suggest that the use of examples by IBL students can a highly effective way of working and may lend itself to a deductive (transformational) proof scheme, but it may raise a question for IBL practitioners. Does the role of examples as a sense-making tool have implications for the way IBL students understand the basis step of MI?If so, how should IBL users support students in the transition towards axiomatic reasoning?

### 3.3 $G$ and $H$ Responses: Contextual and External Symbolic Proof Schemes

A response was coded $G$ when the writer showed suspicion related to the non-mathematical context – that a mathematical argument, even if correct (!), simply cannot apply to people:

> *Humans are not numbers.*

Nearing the opposite extreme were some $H$-coded responses which focused on the performance of algebraic procedures and suggested an external symbolic proof scheme:

> *When the substitution step is done ...*
>
> *Also there's a problem when they're evaluating $k+1$ ...*
>
> *If the $1^{\text{st}}$ human through the $k^{\text{th}}$ human was set to $k$ [ ...] we could substitute in for $1^{\text{st}}$ human through the $k^{\text{th}}$ human with $k-1$ but even then, $k-1$ does not equal $k+1$ and there is no way to prove that.*

Although zero $H$-coded responses would surely be preferable to three, the literature on MI describes a significant reliance on procedural thinking. Perhaps the low frequency of $H$-coded responses should be encouraging.

## 4 Questions

1. How might these student responses differ from the responses of students in a non-IBL environment?
2. Do IBL students differ from non-IBL students in their ways of understanding (or misunderstanding) the basis step in MI?
3. how should IBL users support students in the transition towards axiomatic reasoning?

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# MATHEMAITCAL ACTIVITY FOR TEACHING

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*This work aims to establish a new theoretical construct, mathematical activity for teaching, and to investigate relationships between students' mathematical activity, mathematical activity for teaching, and teacher's instructional moves. Mathematical activity for teaching refers to the mathematical activity teachers engage in as they work to support students' mathematical activity. This construct represents one component of the Mathematical Activity for Teaching Cycle, a conceptual framework that guided my analysis of classroom interactions between mathematicians and their abstract algebra students. Through this analysis I was able to exemplify each component of the Mathematical Activity for Teaching framework and begin to identify relationships between teachers' mathematical activity and those of their students'.*

*Key Words*: Teaching, Mathematical activity, Mathematical knowledge for teaching

As a way to address the challenges mathematicians face while implementing inquiry-based curriculum, researchers have looked to link mathematical knowledge to certain teaching demands (Johnson & Larsen, 2011; Speer &Wagner, 2009). While these studies begin to identify the process by which teachers' knowledge influences their teaching, there remain questions about how teachers' mathematical knowledge directly relates to the mathematical activity of their students. Presumably, it is not enough for teachers to simply *have* the mathematical knowledge that underlies their curriculum. Teachers also need to be able to *use* their mathematical knowledge in a way that supports their students' mathematical activity. With this distinction in mind, one question that could be asked is: *What types of mathematical activity for teaching do teachers engage in to support their students' mathematical activity?* This paper addresses this question in the context of an inquiry-oriented, abstract algebra course.

## Conceptual Framework

Guiding my work is a framework, the Mathematical Activity for Teaching Cycle, developed to investigate the relationships between students' mathematical activity, mathematical activity for teaching, and teacher's instructional moves (see figure 1). Because the abstract algebra curriculum for the course was heavily influenced by the Realistic Mathematics Education heuristic of *guided reinvention* (Freudenthal, 1991), as students work to reinvent group theory concepts, I expect to see instances in which students make conjectures, pose questions, and generalize ideas. Additional activities of interest include symbolizing, algorithmatizing, and defining. Such activities serve to exemplify the *students' mathematical activities* component of the Mathematical Activity for Teaching Cycle.

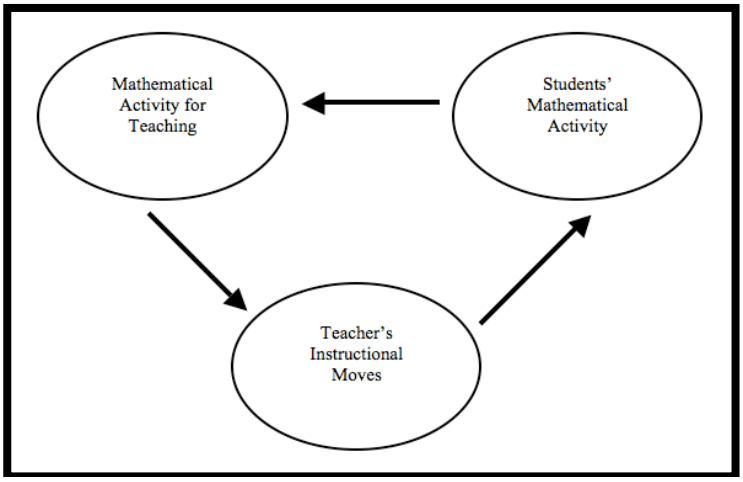


Figure 1. The Mathematical Activity for Teaching Cycle

As students engage in such mathematical activity, one would expect that teachers would also need to engage in mathematical activity. For instance, faced with a novel proof, the teacher may need to evaluate a student's proof to determine the validity of the argument and possible (dis)advantages of this new approach, both in terms of the current task and in terms of their students' mathematical development. Such evaluation may include proof analysis (Lakatos, 1976; Larsen & Zandieh, 2007) and identifying connections between the student's proof technique and other mathematical justifications the students would be likely to encounter during the course of the curriculum.

Additionally, within the last few years there has been research done to investigate mathematicians' abilities to engage in specific skills related to the implementation of inquiry-oriented curriculum. For instance, Speer and Wagner (2009) investigated a mathematician's ability to provide *analytic scaffolding* during whole class discussions, where "analytic scaffolding is used to support progress toward the mathematical goals for the discussion" (p. 493); and Johnson and Larsen (2011) investigated a mathematician's ability to *interpretively and/or generatively listening* to their students' contributions, where interpretive listening involves a teacher's intent of making sense of student contributions and generative listening reflects a readiness for using student contributions to generate new mathematical understanding or instructional activities (Davis, 1997; Yackel, Stephan, Rasmussen, & Underwood, 2003). While such skills may not necessarily be mathematical in nature, I hypothesize that they may rely on a teacher's ability to engage in certain mathematical activities. For instance, in order to engage in interpretive listening, a mathematician may need to interpret a student's imprecise language, generalize the student's statement into a testable conjecture, and then identify a counterexample (see Johnson & Larsen, 2011)

Indeed, while both Speer and Wagner (2009) and Johnson & Larsen (2011) connected the mathematicians' ability to successfully engage in these activities to the mathematicians' mathematical knowledge for teaching, Ball et al. (2008) warn against a purely static view of mathematical knowledge for teaching. Instead stating that, their interest was not limited to the knowledge that teachers hold, but also in "how teachers reason about and deploy mathematical ideas in their work" including "skills, habits, sensibilities, and judgments as well as knowledge" (p. 403). The *mathematical activity for teaching* component of my framework consists of the mathematical activity that supports such skills and reasoning.

The last component of the Mathematical Activity for Teaching Cycle is *instructional moves*. This component represents the mechanism through which the teacher's mathematical activities influence the students' mathematical activities. Such instructional moves could include providing counterexamples, restating student concerns for class discussion, exhibiting a proof for the class, or types of pedagogical content tools (Rasmussen & Marrongelle, 2006).

## Research Method

To understand ways that instructors engage with the inquiry-oriented, abstract algebra curriculum we have collected data from the classrooms of three mathematicians over the course of two years. During these two years, there have been four implementations of the curriculum. For each implementation, every class session was videotaped and members of the larger research team took field notes. Additionally, mathematicians participated in interviews related to their experiences both in class and in using the curriculum materials.

The Mathematical Activity for Teaching Cycle guided the data analysis process. Initially, instances in which students would likely engaged in mathematical activity were hypothesized based on an analysis of the curriculum materials. For instance, during the group unit students are asked to prove some basic theorems related to the order of group elements. Given such a task, I would expect students' mathematical activity to include proving. Such analysis of the instructor materials served to inform my first round of classroom videotape data analysis, in which I identified instances in which students' mathematical activity of interest appeared. These episodes were reanalyzed to see if and how teachers were engaging in mathematical activity, using cases of listening and analytic scaffolding as a signal that teachers may be engaged in such activity. In the third round of data analysis instructional moves that bridged the teacher's mathematical activity and that of the students' were identified. Finally I looked for changes in students' mathematical activity following the teacher's instructional moves.

## Results

Here I will provide an example to illustrate my analytic process, the components of the Mathematical Activity Cycle for Teaching, and the relationships I am trying to investigate. During the deductive phase of the group unit the students were asked to prove that, if the order of $b$ is 4 and $ab = b^3a$, then $ab^2 = b^2a$. After a chance to work alone, a student presented a proof by contradiction to the class. In this proof the student assumed that $ab^2 \neq b^2a$ and was able to deduce that $ab \neq b^3a$. However, this student's steps relied on the fact that if you start with two things that are not equal ($b^3ab \neq b^2a$) and multiply both expressions on the left by the same element, then your resulting expressions are still not equal ($bb^3ab \neq bb^2a$). The creation of this proof represents an example of *student mathematical activity.*

Following this proof, some students questioned if was valid to assume $bb^3ab \neq bb^2a$ based on the fact that $b^3ab \neq b^2a$. The teacher, Dr. Bond, generatively listened to these students' concerns and used them as a way to guide the trajectory of the course, asking the students, "if we take two things that we know aren't equal and we multiply, do we know that they are still not equal"? Initially Dr. Bond stated that this question did not need to be resolved, instead she just wanted to make sure that the students were aware that "this is an important question to ask". Indeed, during the debriefing meeting following this class, Dr. Bond admitted that, " I hadn't decided if it was valid or not … I really hadn't thought it through yet".

However, in the process of raising this question to the class, Dr. Bond gained insight into the justification of the step in question by connecting the student's proof to previously established

result, if $ab = ac$ then $b = c$. She then shared this realization with the class, stating, "*my gut at the moment is that … what is, our cancelation property says that if ab equals ac then b equals c, right. And what was the contrapositive to this"?* Having made this connection for herself, Dr. Bond was then able to verify the steps of the student's proof with the class.

Given Dr. Bond's debriefing statement, it is clear that this result was not knowledge that she carried with her into class. Instead, Dr. Bond drew on her mathematical knowledge in order to carry out mathematical activity in the moment. As such, this example of proof analysis was categorized as an instance of *mathematical activity for teaching*. The *instructional move* implemented by Dr. Bond to connect her mathematical activity to the students' was that of justification exhibition, and this instructional move resulted in a resolution of the proof.

## Implications for Future Research

I see this work as a first step in establishing the mathematical activity for teaching construct and the Mathematical Activity for Teaching framework. Both this construct and the framework can serve as analytic tools for better understanding the relationships between teacher activity and student activity in the classroom. Further, by investigating how mathematical knowledge for teaching supports mathematical activity for teaching, it may be possible to identify specific processes by which teacher knowledge can impact student learning.

## Questions for the Audience

1. What other questions might this framework suggest?
2. One motivation for the *mathematical activity for teaching* construct was dissatisfaction with an acquisition interpretation of mathematical knowledge for teaching. Is this interpretation of mathematical knowledge for teaching consistent with its use in the literature?

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# A CASE OF BRIDGING COMMUNITIES: MATHEMATICIANS AND PRESERVICE ELEMENTARY TEACHERS

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*This report focuses on an ongoing project that is developing a calculus course required for all preservice elementary teachers at a large southeastern university. In the process of designing and implementing the new materials, several research-based tasks have been developed, tested and refined. We discuss the results of the implementation and the refined tasks. We specifically focus on the task developed for the introduction and development of students' limit understanding. Preliminary results indicate that students in our classes have difficulty thinking about the big ideas of the calculus, including limit, and that participation in these tasks, although difficult, is providing a venue for preservice elementary teachers to think more like mathematicians and come to view mathematics as more than a set of procedures to be followed. We hypothesize this experience will provide students with a stronger foundation as they begin their careers as elementary educators.*
Keywords: Calculus, Limits, Preservice Teachers

## *Introduction*

We present preliminary results from an ongoing project developing a calculus course for preservice elementary teachers at a large southeastern university. In the process of designing and implementing the new materials, several research-based tasks have been developed, tested and refined. We discuss the refined tasks and the results of the implementation. We specifically focus on the task developed to introduce the limit concept. Preliminary results indicate that students in our classes have difficulty thinking about the big ideas of the calculus, including limit, but that participation in these tasks, although difficult, is providing a venue for preservice elementary teachers to reason more like mathematicians and view mathematics as more than a set of procedures to be followed. We hypothesize this experience will provide students with a stronger foundation as they begin their careers as elementary educators.

## *Literature Review*

*Mathematical knowledge for teaching.* Studies abound that show prospective or practicing elementary teachers' lack of: knowledge of mathematics (e.g., Ball, 1990; Fennema & Franke, 1992; Ma, 1999; Mewborn; 2001), productive beliefs about the discipline (Thompson, 1992; Phillip, 2007), and a sense of self-efficacy for teaching mathematics (Enochs, Smith, & Huinkee, 2000; Utley, Bryant, & Moseley, 2005; Utley & Moseley, 2006) and these studies have sparked great concern in education. More recently, the response to the question of teachers' needed mathematical knowledge has moved toward the notion of teachers' mathematical knowledge for teaching (Ball, Hill & Bass, 2005). As defined, this knowledge includes not only what is considered common content knowledge, but also specialized content knowledge, i.e., knowledge of mathematics that is specific to the needs of teachers (Ball, Thames & Phelps,

2008). Additionally, and important to support our work, the MAA standards established in the Committee on the Undergraduate Program's Curriculum Guide (2004) state we need to go further than just the basics in our education of elementary mathematics teachers.

Within these areas, recent research has begun to show that elementary teachers who demonstrate specialized content knowledge do positively impact student achievement (Hill, Rowan & Ball, 2005). In fact, the National Mathematics Advisory Panel (NMP) noted "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (National Mathematics Advisory Panel, 2008, p. xx). Our research addresses both the more advanced perspective and the connections to other important mathematics mentioned by the NMP.

*Students as mathematicians.* Some educators posit that mathematics students should approach school mathematics in a manner similar to how mathematicians do mathematics (e.g. Papert, 1971; Seaman & Szydlik, 2007). In their study of the mathematical behavior of preservice elementary teachers, Seaman & Szydlik (2007) found, "teachers display a set of values and avenues for learning mathematics that is so different from that of the mathematical community and so impoverished, that their attempts to create fundamental mathematical understandings often meet with little success" (p. 179). However, as important rigorous mathematical practice is for students to participate in, there are necessary modifications. Wu (2006) calls mathematics education, "mathematical engineering, in the sense that it is the customization of basic mathematical principles to meet the needs of teachers and students" (p. 3) and stresses the importance of mathematicians partnering with educators in order to build appropriate mathematics for K-12 classrooms.

*Student understanding of limit and designing a limit activity.* Research on student understanding of limits has identified both common misconceptions students hold, as well as a number of features instructional activities for limit should include. For instance, students are likely to believe that a sequence cannot reach its limit and may confuse the limit with a bound (Davis & Vinner, 1983). Furthermore, students tend to hold intuitive, dynamic images of limit as evidenced by their language of a sequence "getting closer and closer" or "approaching" its limit (Mamona-Downs, 2002; Roh, 2008). Researchers have illustrated a number of components of limit activities in order to best avoid such misconceptions including beginning by helping students develop an intuitive sense of limit and structuring activities to coordinate with formal conceptions of limit (Mamona-Downs, 2002; Oehrtman, 2008; Roh, 2008).

## *Setting and Description of Research*

*Setting.* The project is a collaboration between individuals from three fields: mathematics, elementary education, and mathematics education. Each brings valuable background and perspective to the project. The setting is an elementary education preservice program that is "STEM-focused" and students are required to take 9 hours of undergraduate level mathematics and 3 hours of statistics in their course of study. Instructors in the pilot calculus class are emphasizing the big ideas of calculus as well as modeling the teaching strategies that they hope will be implemented by the future teachers. These strategies include: inquiry, collaboration, justification of ideas, and provision for diverse learners.

*Research questions.* We investigate the following two questions: What instructional sequence may provide preservice elementary teachers with an informal understanding as well as a basis for more formal understanding of limit? How do preservice elementary teachers understand limit of a sequence both informally and formally?

*Description of Research.* Our work is primarily design-based research (Collins, Joseph, & Bielaczyc, 2004), in which we "carry out formative research to test and refine educational designs based on principles derived from prior research (p. 15). Specifically, we used research from mathematics education partnered with personal experience in calculus instruction to design the curriculum, sequence the instruction and design the specific tasks, teacher presentations, and assessments.

The task discussed introduces the concept of the limit of a sequence. We researched, developed the task, and tested it on several focus groups in spring 2011. We video-recorded these sessions, one of the researchers was the facilitator, and the other researchers took field notes. After each implementation, the research team met and revised the task. In fall 2011, we used the revised activity in the pilot class of 29 students. The class consists of 27 females and 2 male undergraduates, all freshmen or sophomores. The course is being team taught by two of the researchers (one from mathematics department, one from mathematics education). Another researcher attends, takes field notes, and video-records selected episodes. The implementation of the limit task was recorded during whole group instruction, and one small group discussion. Data are also presented from supplemental course material including field-notes and student work.

*Preliminary Results*

Results lie in two areas: 1) new instructional sequences that are research-based, tested and refined, and 2) new evidence about student learning of advanced mathematics ideas. We have identified the primary notions we will emphasize in this course as function, limit, and derivative. Thus, our research is focused on how students might learn these ideas.

*Research-based instructional tasks and sequences.* One of the tasks we have developed is called the "Sesame Street Activity." The primary goal is to provide students with an experience where they are introduced to and begin thinking about limits informally. Space does not allow the inclusion of the full task, but the introduction and two of the questions students are asked to answer in groups follow:

*Big Bird and Count von Count are traveling back to Sesame Street when they come to a bridge. Just before the bridge there is a sign: Each step on my bridge must be special: Every step you take must be exactly half of the remaining distance you have left to cross.*

5. *Without computing, do you know if Big Bird will ever have a step size less than .000000001 meters? How about $10^{-100}$ meters? How could you find the number of steps?*
6. *Big Bird makes a shocking revelation: He claims that if you call out any number, as small as you like, if he follows Lord Zeno's directions, after a certain step, the size of all his following steps will be smaller than your number. Test out Big Bird's theory.*

Task construction aligns with suggestions from research (Oehrtman, 2008; Roh, 2008), particularly structuring activities to support an informal sense of limit of a sequence that can be connected to a more formal definition of limit. While aspects of this task were successful, (e.g. students gaining an informal sense of the limit of a sequence) other aspects proved problematic. For instance, one question involved using logarithms to simplify an equation and students tended to focus heavily on procedural components of the question as opposed to the limiting idea.

The issue of using an elementary school context appears to be useful in some ways but not in others. Students are engaged early on with the ideas and willing to participate with little encouragement. However, there is some drawback, as the students seem to expect the task to focus on elementary mathematics and may not stay involved throughout. We continue to struggle with this idea that making the context elementary does not accomplish our goal of deep understanding of calculus concepts.

*Student learning of advanced mathematical ideas.* The limit task was not successful in helping students avoid some common misconceptions of limit of a sequence. Specifically, students were apt to describe limits using imprecise language (e.g. "getting closer and closer") and a common conception held by students in ensuing lessons/activities was that a sequence could never reach its limit. Students were successful in attending to the difference between the physical act of walking across a bridge and the specific mathematical task this activity presents. This activity successfully helped transition students from an informal understanding of the context and the mathematics to a more abstract, formal setting.

## *Conclusion*

Research continues all over the United States about teacher knowledge and its relationship to good teaching. The research reported here contributes to that research base in that we are developing instruction that will allow future teachers to develop deep understanding of complex mathematical ideas and connect them to the mathematics that they will teach in elementary school. The work will be disseminated, as the idea of STEM-focused elementary school teachers is growing, and the use of calculus as a base course has great potential. Further work is necessary in this project to follow these teachers who are learning calculus and evaluate how it affects their teaching and students.

## *Questions*

1. How could we implement more dynamic ways of looking at limit?
2. How can we move students into more formal thinking about function, limit, and derivative- or we do need to?
3. What ways does calculus tie to earlier mathematics?

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Title: Instructional Influence on Student Understanding of Infinite Series

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Abstract:
Many studies have documented the nature of student conceptions for various topics in college calculus. Of interest in this study are the sources of these understandings. In particular, the instructor's discourse seems to significantly impact students' views and conceptualizations of a topic. The nature and scope of this influence, on a particularly troublesome topic, infinite series, is the object of study in this research. Research questions:

1. What are the sources of students' misconceptions about infinite series?
2. Are there student misconceptions about infinite series which arise from the classroom discourse in college calculus?

The data collected consisted of survey responses, transcripts from interviews, and videotapes of instruction. Student conceptions regarding the convergence of a series agreed in key areas, and evidence indicated that this understanding was fostered during classroom discourse. Further study will reveal the extent to which the instruction influenced other aspects of students' conceptions of infinite series.

Key words: calculus instruction, infinite series, sequences, conceptual understanding

## Introduction

"When I say/write/do _____, do my students follow me?" This common and eminently practical question that instructors often ask presents a challenge to researchers. On the one hand, it is to be expected that student understanding of a topic is related to the specific content, such as examples, definitions, and diagrams, presented by the instructor. On the other hand, students also develop conceptions of topics that can deviate wildly from the material presented in class, leaving the instructor to wonder "Where are they getting this?" Certainly, there is a variety of sources for students' conceptions of a topic, but instruction is a key component (Hiebert and Grouws, 2007). This study focuses on exploring the influence of instruction on student conceptions of a duly nefarious calculus topic: series and sequences.

## Relevant Literature and Research Questions

Many of the studies on calculus learning have found a lack of conceptual understanding among students regarding specific topics in calculus, including functions (Carlson, 1998; Thompson, 1994); limits (Sierpinska, 1987; Tall & Vinner, 1981), derivatives (Monk & Nemirovsky, 1994; Zandieh, 2000), integrals (Rasslan & Tall, 2002), sequences (Mamona, 1990; McDonald, Mathews, & Strobel, 2000), and infinite series (Alcock & Simpson, 2004; Lithner, 2003). While many studies have documented the nature of student conceptions, few have traced these conceptions back to instruction or curriculum. Infinite series is known to be a particularly difficult topic for calculus students to learn, rife with misconceptions and faulty understandings (Tall&Razali, 1993; McDonald, Matthews, and Strobel, 2000; Keynes, Lindaman, and Schmitz, 2009). Certainly, students' conceptions of prior topics plays a role, as it does in the learning of other calculus topics. In the case of limits, and limit processes, the sources of these conceptions have been traced back to students' prior conceptions of functions (Carlson, 1998), or knowledge of the real line (Mamona-Downs, 2001; Sierpenska, 1987).

However, other sources for their conceptions could exist, extrinsic to the students' body of knowledge. Certainly, several relationships come to mind, but the three relationships which are most likely to influence students' conceptions are: instructor-student, student-student, and

curriculum-student. In particular, the instructor's role in creating classroom discourse seems to have a significant impact on students' views and conceptualizations of a topic. The nature and scope of this influence, on a particularly troublesome topic, is the object of study in the research. That is, this study investigates the link between student conceptions about infinite series, and the instructor's presentation of the curriculum. The research questions are:

1. What are the sources of students' misconceptions about infinite series?
2. Are there student misconceptions about infinite series which arise from the classroom discourse in college calculus?

## Methodology

Data collection focused on a single section of second-semester calculus, taught by an adjunct faculty with multiple years of teaching experience. The teaching style was predominantly lecture-based, with an emphasis on the use of examples to illustrate key concepts. The classroom interactions were mostly teacher-centered with occasional group work, although students asked several (5-10) questions per class period. The instructor selected three or four students, by drawing out notecards with their names, to provide the solution to a problem, or answer a factual question. Technology played a limited role in the course; the instructor made little use of technology during instruction, and calculators were not permitted on course exams.

The data collected consisted of responses on a short in-class survey, transcripts from four student interviews, and from videos of the classroom instruction during the unit on infinite series. During the final week of class, over 27 students in a second-semester calculus class completed a written survey in class. The survey instrument contained one item asking students to describe convergence for a series, and the other asking students whether a repeating decimal equaled an integer. Other items collected demographic information as well as preference for various topics seen in the course. Students were also asked to participate in a voluntary follow-up interview. An email was sent to each of the six students who indicated their willingness to participate in an interview. Four students responded and an interview schedule was arranged according to the students' preferences.

An interview protocol was created, based on an instrument used in prior doctoral research (Lindaman, 2007). The protocol consisted of 11 questions, focused primarily on gathering information about student understanding of series. The four participants were all undergraduate students, with three females and one male. They ranged from 19 to 23 years of age. For all four, this was the first time they had taken Calculus II. For two of the students their anticipated grade was a B, and the other two anticipated earning an A.

Grounded theory, as described in Creswell (1998), was used to analyze the transcript data from the interviews. Phrases and words were coded according to frequency and similarity. Then, codes were condensed into several categories. The videotapes are being coded by time and topic. Transcripts will then be generated specific to various subtopics of series, such as convergence, various convergence tests, convergence criteria, etc., in order to match moments in the instruction with discussion in the interviews.

## Preliminary Findings from the Student Interviews

*Misconception 1: The concept image for series convergence includes addition and sequence terminology, but neglects partial sums.*

On all 27 surveys, students were asked to "Explain what it means for a series to converge." The most frequent responses were "it adds to a number/sum", followed by "it reaches/approaches a number". The first response indicates an understanding grounded in the language of addition, while the second response indicates a limit process at work. Other responses referenced graphs, oscillations, and infinity. In all cases, however, no mention was made of the sequence of partial sums, a finding which is consistent to prior work (Lindaman and Gay, in press).

During the follow-up interviews, each of the four students were given the opportunity to clarify their responses to the item. Though they used a variety of verbs in describing the convergence, e.g. "comes to", "resolves", "approaches", etc., none of the four referenced partial sums in any sense. This is of particular concern in that the notion of partial sums provides the foundational link between series and sequences upon which all other definitions and theorems reside. For A and B level students to have so completely ignored this connection between sequences and series is noteworthy. For the most part, all of the participants use similar language in describing the convergence of sequences and the convergence of series.

For one participant, she was able to recognize that the convergence was different for sequences and series, yet like the other three participants, describing that difference proved impossible for her. One student, AJ, did mention that the "sum begins to approach a point" for a series, which could be interpreted as a referent to partial sums. However, he failed to make any more mention of this type of thinking when probed.

*Misconception 2: Sequences and functions are related but series are not related to either.*

In order to tease out each student's conception of the "big picture" in second-semester calculus, each was asked to describe the relationships among sequences, series, and functions. In true design research fashion, this question was drafted after the interview with the first student, Teri, so her response is absent. The predominant finding from responses to this question was that while students acknowledged a link between sequences and functions, even indicating $a_n = f(n)$ in one case, they struggled to connect series with sequences or with functions. Though one student did take more time, he did recognize that functions can be represented as power series, especially for the purposes of integration and differentiation.

Preliminary findings from videotape analysis:

*There is evidence that misconception 1 comes from instruction.*

On the first day in which series were discussed, the instructor drew a clear distinction between the mechanics of sequences and the mechanics of series, labeling sequences as "Easy", and series as "Hard" on the board to emphasize the distinction. He then went on to define series as converging "when the partial sums converged". No mention was made of the partial sums as being terms in a sequence, indeed, no mention made of sequences whatsoever. Then he defined partial sums as $s_1 = a_1$, $s_2 = a_1 + a_2$, etc., and gave the example $\sum_{n=1}^{\infty} n$. In writing the partial sum as $S_n = \sum_{i=1}^{n} i = n(n+1)/2$, he referenced an earlier portion of the text, and noted "To find out if it converges or diverges we're going to take the limit as n goes to infinity". Again, this language distinctly references the process for determining the limit of a sequence, but the connection to sequences is not made explicit for students.

*Is there evidence that misconception 2 comes from instruction?*

This will be investigated by conducting additional analysis of the videotapes.

Conclusions

While students from a single section of second-semester calculus did hold a variety of conceptions about series in general, student conceptions regarding the convergence of a series appeared to agree in key areas, and evidence was found that this understanding was fostered during classroom discourse. Further study will reveal the extent to which the instruction influenced other aspects of students' conceptions of infinite series.

Questions to be addressed during the session:

1. What are the methods by which large volumes of videotape can be analyzed qualitatively in a manner which reduces researcher bias?
2. Would assessing the instructor's understanding via a survey be relevant? Or could concept maps be used in a way to address the research questions?

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# Reading Comprehension of Series Convergence Proofs in Calculus II

Preliminary Report

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**Abstract**

This study examines the effect of activities and assessments concerning reading comprehension of series convergence proofs in Calculus II on students' exam performance. Two sections of Calculus II taught during a summer semester were compared. Both sections primarily used traditional lecture methods, and one section was also given reading assignments with open-ended questions and in-class quizzes evaluating reading comprehension. We compare test scores and interview data from the two sections.

**Keywords**: Calculus, series convergence proofs, reading comprehension, teaching experiment

## Literature Review

Standard evaluation methods in lower division mathematics courses measure students' ability to work problems, with the result that many students focus on mimicking algorithms rather than understanding the underlying mathematics. In a Calculus II course, the level of difficulty increases significantly because students are expected to determine convergence of an infinite series, a very abstract task, and to write an argument justifying their conclusion. Traditional instruction in Calculus II does not emphasize reading in class, and so any reading of the textbook or other related materials that students might do in order to learn these techniques must be done on their own. Students often have a hard time understanding the dense and symbol-heavy style of most mathematical writing (Watkins, 1979). It has been found that in an inquiry-oriented classroom, reading can serve multiple roles, such as focusing the inquiry, carrying out the inquiry, and communicating results (Siegel, Borasi & Fonzi, 1998). The importance of writing mathematics in Calculus has also been documented (Brandau, 1990; Porter, 1996). We believe that requiring students to critically read mathematical arguments and reflect upon their reading is a promising pedagogical technique that should contribute both to better facility with determining convergence and greater fluency in writing convergence arguments.

Stickles & Stickles (2008) found that giving students assignments that directly address their assigned reading can help motivate students to read their textbooks, and can have a positive effect on their success in Calculus. Reader-oriented theory suggests that a reader's understanding of a text is shaped in part by their goals and motivation as they read (Weinberg & Wiesner, 2011). It may be that students will read mathematical texts differently when they know that they will be evaluated based on their comprehension.

In order to design assessment instruments for reading comprehension, we need a model of reading comprehension. Mejia-Ramos et al (2010) have developed a framework of proof comprehension that can be used to create assessment tools. To illustrate their model, they presented a Calculus-level proof, and several multiple-choice items to assess the different dimensions of proof comprehension. The proof that they chose intentionally highlighted all of their dimensions, but the dimensions are not always easy to assess for every proof. We have

adapted their dimensions to the types of arguments that appear during the discussion of infinite series in Calculus II.

Several prior studies show that sequences and series arguments are problematic for students at the university level. The literature shows that students think about series in a wide variety of ways, including visual, verbal and algebraic, shaped by their own view of their role as a learner (Alcock & Simpson, 2004; Alcock & Simpson, 2005). A number of different methods for presenting the idea of convergence have been proposed (Burn, 2005; Roh, 2008; Roh, 2010). We are proposing to evaluate the effectiveness of traditional instructional methods augmented by our reading comprehension tasks on exam performance and on reading comprehension tasks as evaluated in interviews of selected students.

**Research Questions**

- Do students read mathematical arguments differently after activities that emphasize and assess reading comprehension?
- Do students comprehend more of what they read after activities that emphasize and assess reading comprehension?
- Will students' facility in determining series convergence or divergence improve after activities that emphasize and assess reading comprehension of series arguments?
- Will students have more fluency in writing justifications of series convergence or divergence after exercises assessing reading comprehension of convergence arguments?

**Methods**

Two sections of Calculus II were taught during a summer semester by instructors with similar styles and similar teaching experience. Both instructors were advanced doctoral students in mathematics who had not previously taught Calculus II. Both sections were taught in a traditional way, with the majority of each class period devoted to lecture and additional time spent on class discussion and problem solving by students. Students self-selected between the two sections, which met at the same time, with 19 students enrolling in the first section (control) and 29 students enrolling in the second (test). The two sections used identical examinations given four times during the semester and identical assignments in an online homework system.

The first in-class examination, covering techniques of integration and applications, was used as the study's pre-test. All three researchers will score students' test papers both to provide numerical scores and a catalogue of student errors on each problem. These data are used to provide a cross-section comparison of students' knowledge base and frequency of various types of errors.

After the pre-test, students in both control and test sections completed the same assignments on sequences and series in the online homework system. Students in the test section participated in additional in-class activities which emphasized comprehension of mathematical passages read by the students and completed several quizzes assessing reading comprehension of series convergence arguments. These passages were adapted from or excerpts from Stewart's Calculus with Early Transcendentals (Stewart, 2008) and were able to be used as models for students' own proofs. Assessments of students' reading comprehension were designed using a model we adapted from Mejia-Ramos et al (2010).

The second in-class examination, covering convergence and divergence of series of constants, served as the post-test. The post-test required students to determine convergence of series and justify their arguments, but it did not directly test reading comprehension. All three

researchers will score and analyze error types for students in both sections.

After final grades were submitted, two students were interviewed from each section. The interview subjects were selected from the pool of volunteers as having roughly comparable scores on the pre-test. During the interview, subjects were asked to read an argument concerning the convergence of a series and were asked to explain the argument and to answer various questions about it.

**Preliminary analysis**

Preliminary analysis based on the scoring of the pre-test and post-test by the class instructors shows no clear advantage on exam 2 to students who completed the reading comprehension activities and assessments. However, interview subjects who had completed the reading comprehension activities showed a greater degree of facility with the reading tasks requested during the interview than the subjects from the control section who had not completed any reading comprehension activities.

Further analysis of the pre-test and post-test will be conducted by the researchers. We will score the test papers from both sections with a common rubric and will compare scores with each other to look for agreement. We will then analyze the relative change from exam 1 to exam 2 for students from both sections, based on the uniform scoring of exams. Additionally we will code the types of errors seen, to look for any possible improvement in particular types of errors by students in the test section as compared to the control section.

Further analysis of the interviews will attempt to determine if the students from the test section read the mathematical argument differently from students in the control section, if they comprehended what they read differently, and if they can apply the general method in a new example (Mejia-Ramos et al, 2010). The researchers will look for instances that highlight how the student is reading the mathematics, such as evidence that they are able to re-state an argument in their own words or that they understand the big picture instead of just trying to read line-by-line, as noted in previous studies (Selden and Selden, 2003). The analysis will also look for evidence that students comprehend what they read. This evidence may come from students' answers to the reading comprehension questions in the interview, or from whether or not they are able to use the argument as a resource.

**Questions**

- Can we improve the reading comprehension activities and quizzes to lead to better results?
- Are there additional ways other than the assignments given to promote reading comprehension? Are there additional tasks or interview questions that can assess reading comprehension?
- Are there other places in the calculus sequence where it would be valuable to promote better reading comprehension by our students?

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# Improving Student Success in Developmental Algebra and Its Impact on Subsequent Mathematics Courses

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Preliminary Report[1]

**Abstract.** One direction taken by course reform over the past few years has been the use of computer-assisted instruction, often applied to large-enrollment service courses in mathematics, and justified in part by cost-effectiveness. Elementary algebra is typically taken by undergraduate students who do not place into a credit course. The goal of such a developmental algebra course has been to enhance students' "algebra skills," for example, dealing procedurally with rational expressions. Higher-order thinking may be largely absent from such an approach. Our motivating question is "What approach maximizes the student's chance to succeed in subsequent courses?" In view of our theoretical perspective that an inquiry-based approach enhances learning, a subsidiary question is "Is it effective to blend a focus on skills development with a focus on problem-solving?" Results of the analysis, not yet complete, suggest that effectiveness is a matter of what student outcomes are valued, balanced against cost-effectiveness.

**Introduction.** An elementary algebra course often is taken by undergraduate students who do not place into a credit-bearing course. Traditionally, the goal of such a developmental algebra course has been to enhance students' "algebra skills," for example, dealing procedurally with rational numbers and expressions. While this is a form of active learning, higher-order thinking may be largely absent from such an approach. Our motivating question is "What pedagogical approach maximizes the student's chance to succeed in subsequent courses?" In view of our theoretical perspective that an active learning approach enhances learning in STEM courses, a subsidiary question is "Is it possible to blend a focus on skills development (through computer-assisted instruction) with a focus on problem-solving (through cooperative group learning)?"

**Research Question**. Three studies (Mayer 2009, 2010, 2011) relevant to the current research compared treatments using quasi-experimental designs. The fundamental difference between the treatments in the two studies of a developmental algebra course (2010, 2011) was (1) incorporating one or more inquiry-based class meetings, or (2) incorporating lecture class meetings, both together with a common computer-assisted learning component. In the current research, which uses additional data gathered on the algebra student cohorts, we ask the question, "Does the treatment have a statistically significant effect on student success in the next mathematics course taken?"

**Theoretical Perspective**. Our research is based on the premise that active learning (Prince 2004) promotes retention of knowledge, concept development, and problem-solving (Marrongelle and Rasmussen 2008). We take computer-assisted instruction, a form of active learning, as a *ground* – the *figure* is blending with another type of active learning: inquiry-based

[1] **Keywords:** inquiry-based learning, computer-assisted instruction, blended instruction developmental algebra, elementary algebra.

learning (IBL) in the form of collaborative small group work and whole-group sharing. We comment here only on the figure.

In their extensive report on the IBL Mathematics Project, Laursen (et al. 2011) identifies several features of IBL "typical of their project." These features correlate well with the dimensions of the RTOP instrument for classroom observation (RTOP 2010, Sawada 2002). Where Laursen identifies features of the *course*, we modify this and list features of the *class meeting*:

1. The main work of the class meeting is problem-solving (e.g., Savin-Baden and Major 2004; Prince and Felder 2007).
2. Class goals emphasize development of skills such as problem-solving, communication, and mathematical habits of mind (e.g., Duch, et al. 2001; Perkins and Tishman 2001).
3. Most of the class time is spent on student-centered instructional activities, such as collaborative group work (e.g., Gillies 2007; Johnson, et al. 1998; Gautreau and Novemsky 1997; Cohen 1994).
4. The instructor's main role is not lecturing, but guiding, asking questions, and giving feedback; student voices predominate in the classroom (Alrø and Skovsmose 2002).
5. Students and instructor share responsibility for learning, respectful listening, and constructive critique (e.g., Goodsell, et al. 1992; Lerman 2000; Prince 2004).

The inquiry-based treatments (identified as G, GG, or GL above) were designed to incorporate these features.

**Prior Research and Relation to Literature**. Three recent studies (Mayer et al. 2009, 2010, 2011), simultaneously compared different pedagogies over one semester. There are few such direct comparisons in the literature (examples: Doorn 2007, Gautreau 1997, Hoellwarth 2005; literature review: Hough 2010a, 2010b). Nearly all previous studies have focused on courses at the calculus level and above (Hough 2011a). The results of the quasi-experimental studies of a finite mathematic course (2009), and of an elementary algebra course (2010, 2011) showed in all cases that students in the inquiry-based treatment(s) did significantly better ($p<0.05$) comparing pre-test and post-test performance in the areas of problem identification, problem-solving, and explanation (see Figures 1 and 2). Moreover, students, regardless of treatment, performed statistically indistinguishably when compared on the basis of course test scores. Outcomes of the first two studies by Mayer differed in gain in accuracy, pre-test to post-test: in the finite mathematics study, there was no significant difference between treatments, but in the first elementary algebra study there was a significant difference between treatments in favor of the inquiry-based treatment. In those studies, accuracy was assessed on a small set of open-ended problems. In the second elementary algebra study, the pre/post-test had both an open-ended and an objective portion. There was no significant difference among treatments in the second elementary algebra study with regard to the objective part of the pre/post-test. Mayer (2011) reported that students were distinctly more satisfied with a pedagogical approach that included at least some lecture meetings (see Figure 3).

**Research Methodology**. The methodology in (Mayer 2010, 2011) was quasi-experimental in that it sought to remove from consideration as many confounding factors as possible, to assign treatment on as random a basis as possible (constrained only by students being able to choose the time slot in which they take the course), and then to compare results for the same cohort of students. All students involved in the courses had identical computer-assisted instruction provided in a mathematics learning laboratory.

This methodology was described completely in (Mayer 2010, 2011). For completeness herein, we briefly describe the experimental set-up. Students registered for one of three time periods in the Fall 2010 semester schedule for two 50-minute class meetings and one 50-minute required lab meeting. Students in each time slot were randomly assigned to one of the three treatments for the semester:

(1) [GG] two sessions weekly of inquiry-based collaborative group work (random, weekly changing, groups of four) without prior instruction, on problems intended to motivate the topics to be covered in computer-assisted instruction;
(2) [LL] two sessions weekly of traditional summary lecture with teacher-presented examples on the topics to be covered in computer-assisted instruction, and
(3) [GL] a blend of treatments (1) and (2), with one weekly meeting traditional lecture, and one weekly meeting inquiry-based group work.

Students registered for one of four time periods in the Fall 2009 semester schedule for one 50-minute class meeting and one 50-minute required lab meeting. Students in each time slot were randomly assigned to one of the two treatments for the semester, similar to (1) designated [G] and (2) designated [L], above, with just one class meeting per week. Each instructor involved taught all treatments, and all instructors had previous experience in both didactic and inquiry-based teaching. Each instructor also met with his/her class in the mathematics computer lab.

Data gathered during the experiments in Fall 2009 and Fall 2010, and reported by Mayer (2010, 2011) on the two cohorts of elementary algebra students, included (1) course grades and test scores, (2) pre-test and post-test of content knowledge based upon a test which incorporated three open-ended problems, (3) for the 2010 cohort only, pre-test and post-test of content knowledge based upon a test consisting of 25 objective questions, (4) student course evaluations using the online IDEA system (IDEA 2010), and (5) RTOP observations of the instructors (RTOP 2010, Sawada 2002).

For this study, in Summer 2011, (6) data on performance of students in the next mathematics course taken after the elementary algebra course was collected from the university data base. At the time of submission of this paper, student performance in subsequent courses was available for Spring 2010, Summer 2010, Fall 2010, and Spring 2011. Thus, we have more data on performance in subsequent courses for the Fall 2009 experimental cohort than for the Fall 2010 cohort. By the time of the RUME 2012 meeting, we expect to have data for Summer 2011 analyzed, and possibly also for Fall 2011.

**Results of the Research.** Analysis of student success in subsequent courses, as measured by students' final grade in the next course, was analyzed by using the comparisons of means independent t-test with an alpha of 0.05. Students' grades in subsequent courses were coded as follows: A-5, B-4, C-3, D-2, and F-1. Figure 4 depicts statistics on students' grades for the Fall 2009 cohort in their subsequent math course making no distinction between subsequent courses. There was no significant difference between student grades in the next course based on the MA098 treatment (G or L) they received. Figure 5 breaks down the Fall 2009 cohort based on the specific subsequent course taken: MA110 is finite mathematics and is taught only in an inquiry-based/computer-assisted format and MA102 is Intermediate Algebra, taught only in a lecture/computer-assisted format. There was no significant difference between MA098 treatment groups for either MA110 or MA102 as the next course, though the MA098(L)→MA102 trajectory narrowly missed significance. There were three treatments in the Fall 2010 cohort: GG, LL, and GL. Figure 6 shows data on how these treatment groups compared pair-wise based on student success in subsequent courses, making no distinction

between the next two possible courses. There were no significant differences between any of the three MA098 treatments as measured by final grades in subsequent courses. In summary, we found no differences in success in subsequent courses ascribable to treatment in MA098.

**Questions for Further Research/Analysis.** We will be analyzing data about subsequent courses for the 2010 cohort to include in our final report. We would like the audience's input on the following:

1. What additional data on students would be useful if we want to try to understand the differences between students going on to MA110 (a terminal mathematics course) versus students going on to MA102 (a pre-requisite for pre-calculus algebra)?
2. What would be a reliable and rigorous way to determine what impact the treatment has on a student's course trajectory?

**Implications for Practice.** We now teach all regular sections of elementary algebra following the blended treatment of the Fall 2010 experimental cohort: three class meetings weekly, one inquiry-based, one lecture, and one in the lab. We made our decision to change MA098 instruction prior to analyzing student success in subsequence courses based upon gains on open-ended problems and student satisfaction. In view of the inherent coherence of algebra-related topics cutting across courses (Oehtrman, 2008), we expect to extend this study in subsequent years to credit courses such as intermediate algebra, pre-calculus algebra, and pre-calculus trigonometry, all of which presently incorporate computer-assisted instruction together with one weekly lecture meeting, and all in the course trajectory leading to calculus.

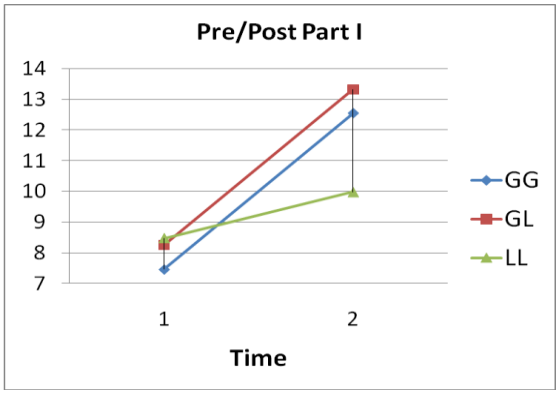


**Figure 1.** Open-Ended Pre/Post-Test 2010 Cohort: N=272, GG =85, GL=93, LL=94.

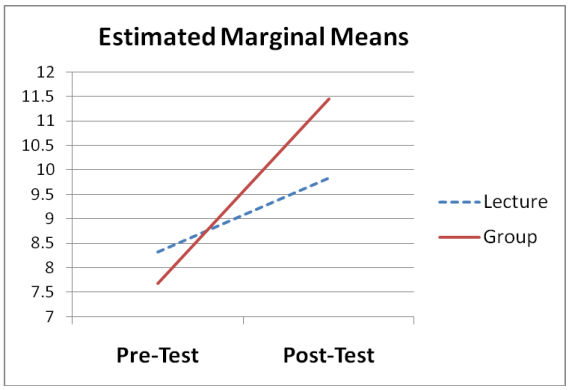


**Figure 2.** Open-Ended Pre/Post-Test 2009 Cohort: N=234, Lecture=115, Group=119.

| Fall 2010 Cohort: IDEA Ratings of Instruction | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Treatment** | **GG** | | | **GL** | | | **LL** | | |
| | Raw Average | | Conver-ted Score | Raw Average | | Conver-ted Score | Raw Average | | Conver-ted Score |
| | Excellent Teacher | Excellent Course | | Excellent Teacher | Excellent Course | | Excellent Teacher | Excellent Course | |
| **Instructor1** | 2.4 | 2.2 | 25 | 3.7 | 3.6 | 45 | 4.3 | 3.9 | 51 |
| **Instructor2** | 4.2 | 4.0 | 52 | 4.3 | 4.0 | 52 | 4.3 | 4.0 | 50 |
| **Instructor3** | 2.5 | 2.6 | 30 | 4.8 | 4.1 | 56 | 4.3 | 4.0 | 49 |

**Figure 3.** IDEA Survey: converted scores in the range 45-55 place instructor/course in the middle 40% of all IDEA mathematics student ratings; scores 37 or lower, in the lowest 10%.

| Fall 2009 Cohort: grades for subsequent course | | | | |
|---|---|---|---|---|
| Treatment | N | Mean | Standard Deviation | Significance (2-tailed) |
| Lecture (L) | 132 | 3.6591 | 1.0101 | 0.244 |
| Group (G) | 129 | 3.5116 | 1.03166 | |

**Figure 4.**

| Fall 2009 Cohort: grades for subsequent course | | | | | |
|---|---|---|---|---|---|
| Next Course | Treatment | N | Mean | Standard Deviation | Significance (2-tailed) |
| MA110 | Lecture (L) | 54 | 3.7407 | 0.91497 | 0.679 |
| | Group (G) | 56 | 3.8036 | 0.64441 | |
| MA102 | Lecture (L) | 77 | 3.6234 | 1.06424 | 0.067 |
| | Group (G) | 72 | 3.2778 | 1.21287 | |

**Figure 5.**

| Fall 2010 Cohort: grades for subsequent course | | | | |
|---|---|---|---|---|
| Treatment | N | Mean | Standard Deviation | Significance (2-tailed) |
| Lecture (LL) | 75 | 3.6267 | 0.94115 | 0.594 |
| Group/Lecture (GL) | 73 | 3.7123 | 1.00664 | |
| | | | | |
| Lectures (LL) | 75 | 3.6267 | 0.94115 | 0.381 |
| Group (GG) | 65 | 3.4923 | 0.86824 | |
| | | | | |
| Group/Lecture (GL) | 73 | 3.7123 | 1.00664 | 0.17 |
| Group (GG) | 65 | 3.4923 | 0.86824 | |

**Figure 6.**

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# Factors influencing students' propensity for semantic and syntactic reasoning in proof writing: A case study

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*Abstract. We present a case study of an individual student who consistently uses semantic reasoning to write proofs in calculus but infrequently uses semantic reasoning to write proofs in linear algebra. The differences in these reasoning styles can be partially attributed to his familiarity with the content, the teaching styles of the professors who taught him, and the time he was given to complete the tasks. These results suggest that there are factors, including domain, instruction, and methodological constraints, that researchers should consider when ascribing to students a proving style that have been ignored in previous research.*

Keywords: Proof; Proving styles; Semantic proof productions; Syntactic proof productions

## Introduction and research questions

In recent years, mathematics educators have noted that there are two qualitatively distinct ways to produce formal mathematical proofs (e.g., Raman, 2003; Vinner, 1991; Weber & Alcock, 2004). A prover can concentrate on the formal and logical aspect of proving, starting with appropriate definitions and hypotheses, carefully formulating what needs to be proven, and applying theorems and other valid rules of inference to these starting points until the desired conclusion is reached. This is sometimes referred to as a *syntactic proof production* and reasoning in this way is referred to as *syntactic reasoning.* Alternatively, a prover can try to represent relevant mathematical objects, explore their properties, and see why the theorem is true using informal representations of mathematical concepts, such as exploring prototypical examples, diagrams, or graphs, and using this insight as the basis for constructing a formal proof. This is referred to as a *semantic proof production* and reasoning in this way is referred to as *semantic reasoning*.

Researchers have recently advanced a number of intriguing hypotheses about these constructs (and related constructs). First, based primarily on case studies, some researchers have hypothesized that some students rely predominantly on one form of reasoning in most of their proof production tasks—that is, we can reasonably refer to some students as *syntactic provers* or *semantic provers* (e.g., Alcock & Inglis, 2008; Alcock & Simpson, 2004, 2005; Alcock & Weber, 2010a, 2010b; Burton, 2004; Moutsios-Rentzos, 2009; Pinto & Tall, 1999, 2002; Weber, 2009). From hereon, we refer to students' propensity to use semantic or syntactic reasoning as their *proving* style. Second, again based on case studies, some researchers have speculated that there is not a strong correlation between proving styles and success in proof writing in advanced mathematics (e.g., Alcock & Simpson, 2004, 2005; Pinto & Tall, 1999; Weber, Alcock, & Radu, 2005). We have recently begun a NSF-funded large-scale study on examining

the proving processes of 100 mathematics majors to assess the viability of these hypotheses.

A primary purpose of this preliminary report is to gain feedback on research goals and methodologies of our project from the undergraduate mathematics education research community. However, we also want to present a research finding. While numerous case studies have illuminated a consistency in individual students' propensity to use syntactic or semantic reasoning in proof writing (e.g., Alcock & Inglis, 2008; Alcock & Simpson, 2004, 2005; Alcock & Weber, 2010a, 2010b; Burton, 2004; Pinto & Tall, 1999, 2002; Weber, 2009), we note these studies have investigated students' reasoning within a single mathematical domain. Further, a general finding from the learning science literature on learning styles is that researchers and teachers alike are frequently too quick to assign students with learning styles based on limited evidence. In this talk, we present a case study of a student who displayed a strong semantic reasoning style when working on proving tasks in calculus, but only limited semantic reasoning when completing proving tasks in linear algebra. From his interview comments, we conjecture how the nature of the task, the way in which his courses were taught, and his familiarity with the material strongly influenced the proof processes he used. We then argue that there are factors researchers should consider before assigning proving styles to individual students.

Hence, our research question is: (1) Do students' proving styles depend on the domain in which they are working? (2) What factors influence the proving processes that an individual student uses on a proof construction task?

**Methods**

As a pilot study for our grant, we interviewed 12 undergraduate mathematics majors or recent graduates to participate in our study. These students met individually with the first or fourth author for two 90-minute interviews. During each interview, participants were instructed to "think aloud" while completing seven proof production tasks. They were permitted to write scratch work, but also told that they should write up their final solution as if it were to be graded as an exam question in a mathematics class. During their proof construction, participants were given access to definitions of every question involved in the proof and access to a graphing application on a computer. Participants were given ten minutes to write each proof. Afterwards, participants were asked general comments about their proving processes.

Seven proofs came from linear algebra and seven came from calculus. Proofs varied in terms of difficulty (one easy, three medium, three hard) and in terms of how accessible they were through semantic and syntactic reasoning (two *semantic tasks*, two *syntactic tasks*, and three *neutral tasks*). The labeling of tasks came from interviews with mathematicians and piloting the materials with roughly 20 students.

We coded participants tasks based on the nature of the semantic reasoning that was used. We coded every instance in which a participant created, referred to, or reasoned from an informal representation of a mathematical concept. We coded each as representing a concept, understanding a concept, recalling a definition, illustrating reasoning to an interviewer, verifying that a claim or the theorem was true, and seeing why the theorem was true. (We hope to receive feedback on this coding scheme from the audience).

This analysis focuses on the reasoning processes of one student, with the pseudonym Kevin, as he was unique in showing a strong propensity for semantic reasoning in calculus, but did not often use this reasoning in linear algebra.

**Results**

In all seven of his calculus tasks, Kevin used semantic reasoning. They often played an important role in his reasoning. For instance, in five of his proof productions, he used graphs to see why the statement was true. Further, in Kevin's comments on these tasks, he remarked that he viewed the task of proving as essentially "translating my intuition", or to use the language of Raman (2003), he viewed the task of proving as generating key ideas.

In the linear algebra tasks, Kevin's use of semantic reasoning was more sparse. For only two of the tasks did Kevin generate a graph or example. For these tasks, the role of these informal representations was relatively minor. For instance, in one task, Kevin tested his recall that det(AB)=det(A)det(B) with two sample matrices (a fact that was not used in Kevin's final proof of the statement), and sketched an example of a non-singular matrix before immediately crossing it out. When describing his proof processes, Kevin spoke of carefully reasoning from definitions to reach desired conclusions.

In his post-task comments, Kevin indicated that his differences in reasoning in calculus and linear algebra was *not* due to the conceptual differences in the two domains. Rather, he cited three differences. First, he noted that he understood calculus better so he had more access to graphical interpretations of relevant concepts. Second, his real analysis professor illustrated every concept both graphically and with diagram, while his linear algebra professor focused more on procedures. Third, the time constraints of our study prevented Kevin from exploring concepts in linear algebra conceptually. He indicated, that given unlimited time, he preferred to explore all concepts conceptually, but given the time constraints of our study (and the time constraints of most classroom examinations!), he had to rely on syntactic reasoning to get a solution efficiently, even though he valued the understanding engendered by a semantic proof production more.

**Discussion**

These results illustrate that students' proving styles may be a function of the domain in which their proofs are situated. As most research assigning proving styles to individual students asks them to construct proofs within a singular domain, we suggest that researchers either qualify students' proving styles to that domain or ask students to ask write proofs in multiple domains. Also, the results provide an existence proof that, at least in some cases, students' proving processes are directly influenced by how they are taught, their familiarity with the content being studied, and how the external constraints (such as time) placed upon them.

**Questions for the audience**

Methodologically, how can we determine if students are semantic or syntactic reasoners? How can we improve our coding scheme for inferences based on semantic reasoning? Are there any suggestions for how we can improve the direction of our project?

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# Using Community College Students' Understanding of a Trigonometric Statement to Study Their Instructors' Practical Rationality in Teaching

Preliminary Research Report

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**Abstract**

This preliminary research report documents work in progress from a study that seeks to understand community college trigonometry instructors' practical rationality regarding instructional decisions, using students' understanding of trigonometry notions as a trigger for the conversations about those decisions. Students' answers to one set of tasks were used to prompt discussions between two full-time instructors. We describe the task, the students' responses, teachers' anticipations of students' difficulties, and their reactions and interpretations of students' understanding of the task. The process provides insights into the nature of the obligations that instructors respond to, and instructors' impressions of the role of curriculum and the demands that it imposes on teachers and students when pressures for increasing transfer rates are high. As a preliminary research report, we seek guidance from the audience on furthering the analyses of teachers' data.

**Keywords:** practical rationality, teaching, trigonometry, community colleges, students' conceptions

The purpose of this study is to investigate the practical rationality for decisions that teachers make in teaching (Herbst & Chazan, 2011) in the context of community college mathematics (Mesa & Herbst, 2011). We sought to create a dissonance between what instructors thought their students understood about trigonometry and what the students revealed through questionnaires and in-depth interviews and use the dissonance to generated discussions between teachers that would allow us to answer the following questions:

1. What are the obligations that teachers experience as they teach trigonometry?
2. How do teachers manage those obligations in real time?

The study was not designed to alter teachers' practices but rather as an opportunity for us to understand how teachers make sense of the decisions they make when they are confronted with information about what their students understand about topics they teach in their courses. We describe the task, the students' responses, teachers' anticipations of students' difficulties, and teachers' reactions and interpretations of students' understanding of the task. Data collection is described within each section.

## The Task

We gathered students' interpretations and knowledge of various trigonometric ideas, in particular those about the statement below, which appeared in a trigonometry textbook that was being used by several of our participating instructors:

*Cosine - Inverse Cosine Identities*

$$\cos(\cos^{-1}(x)) = x \quad -1 \le x \le 1$$
$$\cos^{-1}(\cos(x)) = x \quad 0 \le x \le \pi$$

We collected questionnaire data from 45 trigonometry, pre-calculus, and calculus students, taught by two instructors, Elizabeth (trigonometry, pre-calculus) and Emmet (calculus). We asked the students to (1) explain what the $x$ in the intervals meant and (2) why the first statement had the values -1 and 1 and the second the values 0 and pi. Because of the complex nature of this statement, we anticipated that students would need to coordinate several foundational notions in order to be able to answer the two prompts successfully (Thompson, Carlson, & Silverman, 2007) (a detailed concept map illustrating the various concepts involved will be provided during the presentation). In follow-up interviews with ten students (4 trigonometry, 2 pre-calculus, 4 calculus), we had students watch Elizabeth's explanation of the meaning of the general statement, $f^{1}(f(x)) = x$, and then gave the students the task of explaining the statement in the box.

## Student Responses

The students interviewed were recruited from the high- and low-achievement bands as defined by the two teachers in the third week of class, but only three of the ten participants were in the low-achievement range. Thus, the interviewed student sample includes mostly high-achieving students (mean age = 21.7, sd = 6.57). Our analysis of the students' interviews using Balacheff's model of conceptions (Balacheff, 1998; Balacheff & Gaudin, 2010) revealed that their understanding of the statement is based on particular conceptions about composition, inverse functions, injective (one to one) functions, domain, range, and angle measures. In particular, we have evidence of students' difficulties in:

1. Identifying composition as an operation between functions (including interpretations of inverses under composition as multiplicative inverses). (Charlie[1], Cathy, Thomas, Tony)
2. Recognizing that the *identity* for the operation of composition is $f(x) = x$, and thus that a bijective function composed with its inverse results in that identity. (Tina, Cathy)
3. Interpreting the inverse of trigonometry functions, in particular the need to restrict the function so that it is one-to-one so and can have an inverse. (Carl, Tracy)
4. Recognizing the nature of the statement as a statement of truth and the role of the restrictions for making that statement true. (Peggy, Carl, Corey)
5. Managing multiple representations. (Carl, Peggy, Paul)
6. Choosing examples to justify a statement, without attending to the correctness of the example. (Cathy)
7. Using and interpreting radians, degrees, angles, axis, and periods. (Carl, Peggy, Corey, Paul)

The following excerpt illustrates some of these issues, regarding the identification of composition, restricting input values, and the selection of examples:

Cathy: This [line 1] is saying that the domain for the inverse cosine is in between negative one and one and this [line 2] is saying that the domain of the cosine is between zero and pi because this is the one that we are evaluating first here and this is the one that we are evaluating first here. The inverse cosine is giving you like one over cosine where $x$ is the inverse cosine of $x$ [writes $1/\cos(x) = \cos^{-1}(x)$]. So in doing that you end up with like an indeterminate function if you have your value outside of this [the intervals].

Notice that although Cathy states that "this is the one we are evaluating first," she means to calculate $\cos(x)$ first, then take the reciprocal. In this case, restricting $x$ was associated with avoiding a value [1/0] that "does not exist… it gives you error messages because you can't divide by zero".

In cases in which the students recognized the composition, they used notions of domain to interpret their meaning of the restrictions in the statement:

I: What happens when $x$ is not between one and negative one?
Tina: Then it's not a function. I don't think. I don't, no, it's not a function. Doesn't work.
I: And between zero and pi?
Tina: Same with, yeah.
I: And when you say it's not a function.
Tina: that, that equation doesn't work. Mathematically, it's not provable (pause) Like that number isn't a possible answer. Whatever number is plugged in.
I: And why is it not possible?
Tina: Because it doesn't fall between the negative one and one. If it fell, like, if it was two, whatever is plugged in wouldn't be a possible answer.

**Teachers' Anticipations of Students' Responses**

In an individual interview prior to collecting the student interview data, Elizabeth commented on her own explanation of the meaning of $f^{1}(f(x)) = x$ and the connection to the statement in the box. In a joint interview, conducted after we collected the student interview data, we asked

[1] Pseudonyms were chosen so that the first letter would identify the course in which the student was enrolled, Trigonometry, Precalculus, or Calculus.

Emmet and Elizabeth to anticipate students' answers to the prompts in our task. Both instructors produced quite complete explanations involving several ideas: (1) this statement is a particular case of $f^{-1}(f(x)) = x$; (2) trigonometric functions are periodic, and not one-to-one; so one must restrict the functions to obtain inverses; (3) when dealing with inverses, "one function undoes the other" which is why one obtains an $x$; (4) the different values in the two intervals stem from the different order in which the functions are composed. Neither of the instructors, however, indicated that the restrictions in each line operated differently: While the restriction in the first line is needed to ensure that the inverse function can be calculated, the restriction in the second line is needed in order to ensure that the equality holds (one does not need to restrict $\cos(x)$). In our sample of 10 students interviewed, only one student had this realization.

**Teachers' Reactions and Interpretations of Students' Understanding of the Task**

At the time of the data collection Elizabeth had seven years of college teaching experience, while Emmet had 16. We presented the teachers with summaries of data from their students' written questionnaires and the student interviews. Both teachers were surprised to read the students' responses and engaged in a search for explanations for why students could have such conceptions. The teachers thought that the results were indicators of larger issues with the curriculum. In particular with when and how functions are introduced and how inverses are taught. The instructors did not deal with the possibility that the curriculum may be set up to obscure that composition can be seen as a binary operation between functions and that inverses are functions that give the identity function, when composed with each other. Both instructors suggested that the problems might lie in the college algebra course, which is a pre-requisite for both trigonometry and calculus but indicated hope that a recent change of textbooks and course organization would better address this in the future. By suggesting these interpretations, the teachers appear to recognize institutional obligations more easily than disciplinary, individual, or interpersonal obligations (Herbst & Chazan, 2011). In other words, instructors did not produce justifications that could be tied to the complexity of the mathematics (for them the mathematics is too simple), differences among their students, or to their shared space during teaching. Instead the externally imposed curriculum is seen both as root of the students' difficulties and also as solution to address it.

**Questions for the audience**

1. The analysis of the practical rationality can be complemented with an analysis of teachers' mathematical knowledge for teaching. What types of analyses could we perform of our teacher data to address that? We have thought about using the scholarship on noticing (Sherin, Jacobs, & Philipp, 2011) and on teachers' knowledge (Ball, Thames, & Phelps, 2008), and will bring some initial analyses. We have also thought about pursuing a linguistic analysis about the way in which teachers position themselves vis a vis the students and the content (Martin & White, 2005; Mesa & Chang, 2010; Wagner & Herbel-Eisenmann, 2008). It is still unclear to us what would be the nature of the claims we could do with these analyses.
2. What are possible ways to use this research to inform a faculty development program that would push teachers into thinking about students' understanding and the role it plays in teaching?

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# Investigating Instructors' Concerns as They Learn to Teach with Inquiry-Based Learning Methods

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Preliminary Research Report

*We report initial findings of a study that seeks to investigate the changing nature of instructors' concerns as they learn to teach mathematics courses using inquiry-based learning approaches. Using year-long data from interviews with faculty and bi-monthly teaching logs, we seek to describe the concerns of instructors teaching with this method. Our initial analysis of pilot data with four instructors new to the method suggests that these concerns are organized into five major themes: Student preparation, motivation, and engagement; Coverage; Rigor; Difficulty of the material; and Student Learning. Additionally, the nature and relative frequency of these concerns seem to suggest that these faculty are more preoccupied with managerial aspects of teaching and less with student learning, consistent with a proposed developmental model of professional expertise in teaching. We seek input on the instrument used to gather the data as current results might be consequential to the organization of the instrument.*

**Keywords:** inquiry-based learning, teaching expertise development, instruction

With this project we seek to fill a gap in the knowledge that exists about how mathematics faculty members new to teaching with inquiry-based learning [IBL] methods learn to use these approaches. Specifically, we seek to produce accounts of the process of learning to teach using IBL from faculty who are new to the method and contrasting that process with faculty who consider themselves advanced users of IBL. At the undergraduate level, inquiry-based learning in mathematics finds its roots in views of R. L. Moore of the University of Texas. Moore believed students should build their own understanding and work through the course material individually. As peer collaboration and group work have come to be valued (National Council of Teachers of Mathematics [NCTM], 2000), Moore's insistence on individual work has fallen out of favor. Instructors have adapted Moore's values with time, and now, dubbed IBL, the method refers to a spectrum of instructional styles that allow students to work in small groups, consult outside resources, or pose and seek answers to their own questions. Present in all IBL classrooms, however, is an emphasis on student presentations and active student participation with very limited lecturing (Coppin, Mahavier, May, & Parker, 2009). A growing body of research has been documenting the positive impact of IBL methods on students' gains in the cognitive and social domains (Hassi, 2009; Laursen & Hassi, 2009, 2010; Laursen, Hassi, & Crane, 2009; Laursen, Hassi, Crane, & Hunter, 2010). In particular, Laursen and colleagues (Laursen, et al., 2010) report that students in IBL courses report higher cognitive gains than students in non-IBL courses, in terms of mathematical thinking, understanding of concepts, and application of mathematical knowledge. Students in IBL courses, and in particular future teachers, reported higher cognitive gains about teaching. Given that lecturing is a predominant model of instruction in college mathematics classrooms (Lutzer, Rodi, Kirkman, & Maxwell, 2007) we asked: How do faculty learn to teach with these new methods? What kinds of concerns

do they have? And how do resources such as experience, other colleagues, books, or conferences and workshops help them in developing a better sense of what it means to use IBL methods in teaching mathematics?

The literature on teacher learning to teach mathematics is extensive at the K-12 level, but is more limited in the post-secondary level. Investigations at the post-secondary level suggest a developmental path in the process of learning to teach (Nardi, Jaworksi, & Hegedus, 2005; Nyquist & Sprague, 1998). Nyquist and Sprague (1998) suggest that teaching assistants' concerns, discourse, and relationships with students and colleagues progress through a series of stages. Initially, teaching assistants focus on themselves ("Will my students like me?") and on their own survival; next, they worry about managing discussions or handling classroom participation; and in later stages, they start to focus on students' understanding and learning outcomes. These shifts from concerns about the self, to concerns about managing teaching, and finally to concerns about students' learning and understanding, determine a path that we might expect as instructors teach with a new method. Nardi and her colleagues (2005) worked with tutors at the University of Oxford over an 8-week period doing individual interviews in which they were prompted to reflect on aspects of their teaching. The researchers identified four stages of pedagogical awareness—naïve and dismissive, intuitive and questioning, reflective and analytic, and confident and articulate (p. 293)—which, they propose, reveal a spectrum of awareness about students' difficulties, strategies to overcome those difficulties, and self-reflection about teaching practices. Because they claim that instructor awareness can feed into other teaching formats (p. 293), we could anticipate comparable stages of awareness as teachers face a new instructional method for the first time. Other accounts of teaching with inquiry-oriented curriculum (Marrongelle & Rasmussen, 2008; Speer & Hald, 2008; Stephan & Rasmussen, 2002) point at specific dilemmas that instructors face, in particular navigating the need to stay away for lecturing and moving toward more discussion-based classes. This literature is informative and allows us to think that there might be common concerns faculty have when they start teaching using IBL methods, and that these concerns may change and evolve as faculty teach other IBL courses.

## Methods

There are two primary sources of data collected over a one-year period: on-line teaching logs filled every other week and three interviews with faculty, at the beginning of the year, half way through, and the end of the year. In the pilot phase of the study, we worked with four instructors, all new to the method, having been through one week-long workshop the previous summer.

The on-line teaching logs request information on time spent on various types teaching activities (homework review, lecturing, large-group discussion, small group work, student presentations, assessment, class preparation, mathematical content, and pacing); challenges faced and concerns about these activities, solutions found to resolve these challenges, and resources used. The initial interview seeks to get baseline information about their understanding of IBL, what are necessary and sufficient conditions for a successful IBL course, and their anticipated learning goals for the students. The intermediate interview seeks to get information on the students, the curriculum, the instruction, and their assessment practices; in addition we explicitly ask instructors to tell us what they have learned about themselves, the students, teaching, and mathematics through teaching with IBL. The interview also asks for information on specific entries in the logs. The

final interview asks a combination of questions from the initial interview (e.g., their understanding of IBL) and the intermediate interview (e.g., students, curriculum, assessment).

The log data have been analyzed by finding themes across all the comments (N=36) submitted by the four instructors over a one-year period, attending first to the type of teaching activity. The themes were then used to code across the comments and refined into five categories: Student preparation, motivation, and engagement; Coverage; Rigor; Difficulty of the material; and Student Learning. We are currently analyzing the interview data.

## Findings

The instructors most frequently reported concerns about **Student Preparation, Motivation, and Engagement** with the material (14/36). For example they mentioned that the students would come to class with incomplete homework or with no evidence of having worked on some of the assignments (e.g., "The only challenge was in the most recent class when none of the students had a proof of Euler's Theorem." Instructor 4). Instructors were concerned that student motivation waned towards the end of the term, presumably due to other commitments the students had (e.g., "My students are starting to feel the end of the semester, and they all seem quite worn down. I'm worried that their lack of enthusiasm will have a detrimental effect on their ability to keep being productive in the class." Instructor 1) or that they appeared, at times, to be less engaged than they should be (e.g., "well overall, it is good, but I guess before spring break, their mind were somewhere else." Instructor 3). **Coverage**, (8/36) was a concern shared by all instructors. As the method relies on students' discovering the material, this theme is not unexpected, of course, and the instructors tended to compare time with their experience with non-IBL courses (e.g., "We didn't get to the division algorithm until day 5, and usually this is covered by day 2 when I'm lecturing!" Instructor 1). Departments were mentioned as a source of the pressure to cover the material (e.g., "pressure from the department to reach a level of content (namely reach the fundamental theorem of calculus), at this point it seems impossible unless I switch to a lecture format." Instructor 3). But the pressure also came from the time that it takes to go through the discovery process (e.g., "I designed this course for prospective secondary math teachers to end with the proof of the three impossible constructions of Euclidean geometry: doubling the cube, squaring the circle and trisecting the angle. Everything was set up to get us there; it ties into the course we've taken in math history, and the 2-quarter sequence in geometry. And, we aren't going to make it. It's a disappointment to me." Instructor 2). **Rigor** and **Difficulty of the material** were each mentioned with the same frequency (6/36). Rigor referred to instructors' dissatisfaction that the students were not learning to be careful in writing proofs (e.g., "Students were getting a little too informal in class, particularly when it came to giving proofs by induction. I struggled with how to get them to write out formal proofs by induction." Instructor 1). Instructors also mentioned the difficulty of the content or assignments as a challenge ("The material that we are currently covering is a notch or two up in difficulty from what we have been doing all semester." Instructor 4), which tied to students' waning interest in some cases, led to disengaged classes. Finally, the two comments that we classified as **Student Learning** referred to the areas of assessment. Instructor 2 showed concern that in spite of designing a test that was quite similar to the homework assigned, students' scores were around 78% with 2 students failing. This instructor adds: "I was disappointed to see scores as low as they were when the students weren't asked to do anything that was significantly new." Instructor 4 showed concern about finding ways to assess students' knowledge using other means beyond homework and presentations. We took these comments as referring to student earning because

they appear to indicate worry about the measures we use (through assessments) of what students know.

## Discussion

It is interesting that these instructors voice concerns that are focused on whether students like them or the method but more about managing instruction: keeping students engaged, ensuring that they are prepared for class, regulating the difficulty of the material and the rigor of students' productions, and handling pressures to cover material. It is less evident that the instructors worry about students' learning. Although not definitive, this analysis gives us information about what types of concerns to expect from the larger sample. Up to now we have collected 131 teaching logs from a new sample of 28 instructors and we are in the process of analyzing these to identify trends over time and trends by instructors' experience.

## Questions for the audience

1. What types of questions could be added to the logs so that student learning can become more visible? 2. We propose a developmental path but other possible interpretations are viable. What could be other frameworks that could be used to analyze these data?

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# Investigating the Teaching Practices of Professors When Presenting Proofs: The Use of Examples

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**Abstract**

This study uses ethnographic methods to investigate the teaching practices of mathematics faculty members when presenting proofs in class. Four case studies of faculty members at a large research institution who are teaching in different mathematics content areas are used to describe the ways in which examples are used in proof presentations in upper-division proof-based undergraduate mathematics courses.

**Keywords:** proof presentations, examples, teaching practices, ethnographic methods

## Introduction

There have been few studies addressing the teaching practices of university teachers, although there have been calls for such studies (Harel & Sowder, 2007; Harel & Fuller, 2009; Speer, Smith, & Horvath, 2010). In particular, there has been very little research addressing the teaching practices of faculty members in upper-division proof based courses (Weber, 2004). This study will contribute to our knowledge of teaching practices of mathematics faculty members as they teach courses in which students are expected to construct original proofs. Interview data and video data from four different faculty members teaching abstract algebra, analysis, number theory, and geometry will be analyzed to determine the ways these instructors use examples to motivate and support their presentations of proofs in class.

## Research Question

In what ways are examples used to motivate and support proof presentations in an upper-division proof-based mathematics course? What is the pedagogical motivation of the instructor for the use of particular examples in proof presentations? How does the instructors' usage of examples contribute to their overall presentation style?

## Literature Review

At the collegiate level, there are few studies focusing on teaching practice, i.e. "what teachers do in and out of the classroom on a daily basis" (Speer, et al, 2010). A foundational understanding of teaching practice contributes to our understanding of the phenomenon of teaching and learning. In particular, there is value in focusing in on small, meaningful aspects of practice that mathematicians already use in the classroom (Speer, 2008). Many studies have emphasized the importance of using examples in teaching, particularly when the examples are generated by the students themselves (Watson & Mason, 2005; Watson & Shipman, 2008). Examples serve to make connections between what students already know and the new material that is being presented. When the content involves mathematical proofs, the use of examples may be even more important. Exploration of examples is often part of the process for constructing an original proof (Alcock & Inglis, 2008), and the ability to generate a

specific example of a proof strategy is an important facet of proof comprehension (Mejia-Ramos, Weber, Fuller, Samkoff, Search, & Rhodes, 2010).

Attending to proof presentations in class is one of the primary ways in which students construct their understanding of what constitutes a proof (Weber, 2004). There is evidence that instructors spend large portions of their class time (between one third and two thirds) presenting proofs (Mills, 2011). Several recent studies have used faculty interviews to investigate the pedagogical views of faculty members concerning proof presentations in class (Weber, 2010; Yopp, 2011; Alcock, 2009; Harel & Sowder, 2009; Hemmi, 2011). Some of these studies discussed a relationship between proof presentations and the use of examples. Several instructors mentioned that they often accompany a proof with an example (Weber, 2010). Alcock (2009) identified 'instantiation of definitions and claims' as one of the four proof-related skills that instructors are trying to teach. Observations of a particular professor throughout the course of a semester revealed that he modeled the mathematical behavior of 'example exploration and generalization' when presenting lectures in class (Fukawa-Connelly, 2010). Fukawa-Connelly, Newton, & Shrey (2011) focused on the use of examples in a proof based course by describing in detail how a faculty member used examples to instantiate the definition of a mathematical group in an abstract algebra class.

The contribution of the present study is that it combines faculty interviews with observation data to investigate what faculty members think about the pedagogy of proof presentations as well as to catalog their actual behaviors in the classroom. This will allow us to investigate their teaching practices, which is more in line with the type of studies that Speer, et. al. (2010) called for.

**Methodology**

Faculty members at a large comprehensive research university who were teaching proof-based upper division mathematics courses during between August 2010 and August 2011 were asked to participate in the study. Three instructors agreed to participate in a one hour interview and agreed to allow their lectures to be video-taped approximately every two weeks throughout the semester. All of the faculty members taught in a lecture style, with the instructor was primarily teaching from the board, and the students were listening, taking notes, and sometimes answering questions and participating in class discussions.

Interviews were transcribed and analyzed using the constant comparative method (Glaser & Strauss, 1967) to determine the pedagogical views of the participant concerning proof (Weber, 2010). The analysis of the video data occurred in several phases. First, I viewed the videos and took notes about what was happening in each time interval. Then all of the instances of proof presentation in the observation data were transcribed. For this study, I have pulled out all of the instances in the data when examples are used to support the proof of a claim in different ways. A careful search of the literature provided initial categories. These categories are:

1. *Start-Up Examples* – Motivate basic intuitions and claims (Michner, 1978)
2. *Generalization of a Pattern* – Examples are used to help the students generalize the statement of a claim from a small number of numerical computations (Bills & Rowland, 1999; Harel, 2001; Inglis, Mejia-Ramos, and Simpson, 2007)

3. *"Generic" Example* – When an example is used to go through the steps of a proof, and then the general method of the proof can be extracted from the example, or vice-versa (Rowland, 2001; Weber, 2010) Similar to Michner's (1978) "Model" Examples.

3a. Pictorial "Generic" Examples – When a diagram or picture is used to organize the proof of a statement, and the proof is based on the diagram (Weber, 2004)

4. *Instantiation* –

4a. Instantiation of Claims – An example to help students understand the statement of a claim, or the necessity of given conditions in a claim (Michner, 1978; Alcock, 2009)

4b. Instantiation of Definitions – An example of a mathematical object that satisfies a definition (Alcock, 2009) Pictorial examples are often used for this purpose.

4c. Instantiation of Notation – When an example is used to introduce a new notation

Each example will be categorized and the usage of examples as well as comments about example usage in the interviews will contribute to the construction of the characteristic style of the instructor. Since the instructors are teaching different content areas, it makes sense to consider these as separate case studies. Similarities and differences among the faculty members will be highlighted, however, the goal of this study is to describe and catalog, not to evaluate the methods used.

**Preliminary Results**

Though they were not explicitly asked about how they use examples in class, three of the faculty members in this study mentioned the use of examples in the interviews. They gave several reasons for using examples, and described different ways in which they use examples. One professor said that he gives simple examples to warm the students up for the statement of the theorem. Another participant said that if it is a proof of a pattern, he would emphasize computation to try to get the students to figure out what the pattern is. In other words, he would use the examples to get the students to conjecture the statement of the theorem. He also said that he uses examples to help the students know how the proof of the theorem should go. The use of pictorial examples as a guide to organize the proof was also mentioned. One participant said that he takes the statement of the theorem and produces examples from the statement, which is similar to what Alcock (2009) calls 'instantiation of claims.'

Initial analysis showed that examples were used by the instructors in 27% to 67% of the proofs they presented in the observation data (Mills, 2011).

There are many other dimensions that contribute to a professor's proof presentation style. In my dissertation work, I will investigate other aspects of the instructors' characteristic style, and follow-up interviews will explore the pedagogical reasoning behind each instructor's moves when presenting proofs in class.

**Questions**

1. How can these ways to use examples help us understand undergraduate teaching of mathematical proof?
2. How can this be linked to student learning? What's the next step?

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# Tackling Teaching: Understanding Commonalities among Chemistry, Mathematics, and Physics Classroom Practices

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Abstract

Education research in chemistry, mathematics, and physics tends to focus on issues inherent to the discipline, most notably content. At this time, little literature evidence exists that documents fruitful collaborations between education specialists across the STEM disciplines. This work seeks to unite the disciplines by investigating a common task: teaching. This study explores how discipline-specific practices influence the common act of reformed teaching pedagogy with a focus on the use of inquiry. We seek to identify commonalities among classroom teaching practices in these disciplines and contribute to the development of analytical tools to study STEM teaching.

List of Keywords

interdisciplinary, inquiry, RTOP, teaching

## Theoretical Perspective and Purpose

What makes good teaching good teaching? To what extent are the qualities of good teaching in science the same as the qualities of good teaching in mathematics? How much is the nature of good teaching influenced by the discipline? Questions about the general or subject-specific nature of high-quality teaching have confronted educational researchers for decades (e.g., Gage, 1963; Lortie, 1975; Leinhart, 2004). Some researchers acknowledge that 'good teaching' involves both disciplinary orientations as well as general principles (Gresalfi & Cobb, 2006). Recently, Grossman and McDonald (2008) lamented that, "…the field of research on teaching still lacks powerful ways of parsing teaching that provide us with the analytical tools to describe, analyze, and improve teaching" (p. 185).

We have begun to investigate teaching practices in three disciplines: chemistry, mathematics, and physics. We initially observed (in the literature and via conversations with colleagues) that chemists, mathematicians, and physicists use the word "inquiry" to describe specific classroom practices. Our work is now proceeding with two caveats: (1) the idea of "inquiry" in the three disciplines has common roots (e.g., Dewey, 1997; Freire, 1984; Piaget, 1964; Vygotsky, 1962) and (2) despite having common roots, "inquiry" has somewhat different operationalized meanings in each of the disciplines. Our intention is to honor the distinctions of "inquiry" among the disciplines and how these distinctions potentially arise from discipline—specific practices, but focus on the commonalities of inquiry-based teaching across the disciplines. **We are investigating the role of the teacher both in the research literatures and in three inquiry-based, disciplinary classrooms in order to identify commonalities among classroom teaching for the purpose of developing a common language to describe inquiry-based**

**teaching practice and contribute to the development of analytical tools to study teaching across the STEM disciplines**.

Methodology and Preliminary Results

First, we reviewed the research literatures in chemistry, mathematics, and physics education to better understand disciplinary perspectives on "inquiry" classrooms. **We identify here a few of what we consider to be exemplary STEM inquiry instruction at the University level**. While we recognize that this list is not exhaustive, we feel it is consistent with how "inquiry instruction" is described within our three distinct discipline-based education research literature bases.

**Process-Oriented Guided Inquiry Learning (POGIL) is an exemplary curriculum from the field of chemistry.** POGIL curricula exploit the notion that scientific discoveries are made using standard inquiry practices and models the "discovery" of chemistry knowledge through the presentation and analysis of data/models that undergird the phenomena being studied. POGIL promotes active engagement of students through a series of small group activities that incorporate guided inquiry as well as necessary processing skills such as information processing, critical and analytical thinking, problem solving, communication, teamwork, management, and assessment (Farrell, Moog, & Spencer, 1999). A POGIL activity assists students in develop understanding by employing the *learning cycle* in guided inquiry activities. The learning cycle is a pedagogical paradigm for enhancing student learning that first originated from a 1960s elementary curriculum project and consists of three stages: *exploration, concept introduction/ formation* and *concept application* (Karplus & Their, 1967). Published POGIL materials are available for general, organic, physical chemistry, and GOB (general, organic, and bio-chemistry) courses. The effectiveness of POGIL in general chemistry has been previously described (Farrell, Moog, & Spencer, 1999; Lewis and Lewis, 2005).

**Inquiry–oriented differential equations (IO-DE) is an exemplary curriculum from the field of mathematics.** The IO-DE curriculum capitalizes on advances within the disciplines of mathematics and mathematics education. From the discipline of mathematics, the IO-DE projects draws on a dynamical systems point of view and treats differential equations as mechanisms that describe how functions evolve and change over time. Interpreting and characterizing the behavior and structure of solutions are important goals, with central ideas including describing the long-term behavior of solutions, the number and nature of equilibrium solutions, and the effect of varying parameters on the solution space. From the discipline of mathematics education, IO-DE draws upon two complementary lines of K-12 research: the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991) and the social production of meaning (Cobb & Bauersfeld, 1995). RME is an instructional design theory that puts at its center the design of instructional sequences that challenge the learner to organize key subject matter at one level to produce new understanding at another level. This is referred to as mathematization. The process of mathematization is actualized in the core heuristics of guided reinvention and emergent models. Guided reinvention deals with locating appropriate instructional starting points that are experientially real to students and take into account students' mathematical ways of knowing. The heuristic of emergent models deals with the need for instructional sequences to be long-term, connected, and for which student engage in problems to create and elaborate symbolic models of their own informal thinking. Regarding the social production of meaning, an explicit intention of the IO-DE project is to create learning environments where student routinely offer

explanations of and justifications for their reasoning. In particular, the constructs of social and sociomathematical norms (Yackel & Cobb, 1996) are central in IO-DE classrooms.

**Tutorials in Introductory Physics is an exemplary curriculum from the field of physics.** The term "tutorial" was first coined within the Physics Education Research community by Lillian McDermott at the University of Washington. (McDermott, *et al.*, 2002) There have been a number of other groups contributing to the general paradigm of the University of Washington model (e.g. Activity-Based Tutorials (Wittmann, *et al.*, 2004, 2005)). The general idea of a "tutorial" is a highly structured series of questions that force students to reason through what are often contradictory models of physical phenomenon. At their most effective level these tutorials take into account an extensive amount of evidence about students reasoning and/or understanding of a given topic in order to present students with accessible but challenging scenarios that force them to reconcile any conflicting aspect of their thinking. They are commonly described as guiding students to realize that their understanding needs revision and provides an accessible path for completing that revision into coherent understanding.

**We then examined videos of these three different types of inquiry-based classrooms to extract commonalities and differences that collectively define how inquiry is operationalized in these classrooms**. The Reformed Teaching Observation Protocol (RTOP) was employed to identify specific elements of a classroom that we felt were essential, or visually indicative of an inquiry classroom. The RTOP is designed to measure the degree to which classrooms have been aligned with science and mathematics reforms. In particular, the strong relationship between the items and various content and pedagogy standards outlined in documents such as the NSES (NRC, 1996) and the Benchmarks (AAAS, 1993) demonstrates the face validity of the RTOP (Sawada et al., 2002). The RTOP lists twenty-five criteria under three subsections: lesson design and implementation; content and process knowledge; and classroom culture. We chose this instrument because we felt it framed our discussion of the behavior that should be observed in an inquiry classroom.

We identified a number of elements from the RTOP that were deemed "non-crucial" to an inquiry classroom. Most of these elements focus on the "best practice" of the instructor. While we do not mean to diminish the teacher's role within an inquiry classroom, we felt (based on Piaget's and Vygotsky's theoretical underpinnings of inquiry) that very often the success of inquiry instruction depends more on the behavior of students rather than the behavior of the instructor. In extreme cases, we feel an instructor need not be present, a unique identifier for certain "open inquiry" activities. Still, in the exemplar inquiry practices of our disciplines, the instructor plays a considerable part in the learning activity. Instructors are often expected to engage students in Socratic dialogue or ask questions to help train their thinking. In special cases instructors can guide a conversation based on their selection of groups to present. For instance, within IO-DE curriculum, students are asked to construct ideas within smaller groups and the instructor takes on the role of identifying certain groups who can present their work to the whole class in order to facilitate discussion.

We identified a few RTOP criteria that especially resonated with the exemplar inquiry practices of our disciplines. These criteria included student exploration prior to instructor presentations, students making predictions estimations and/or hypotheses and deriving means for testing them, students engaging in activities that involve assessment of procedures, and to what extent were

they reflective about how their thinking had changed. Essentially, these activities reflect common practice within a community of professionals within each of our disciplines.

Questions for Discussion

1. We have identified common theoretical underpinnings of exemplary inquiry instruction across our three disciplines (Karplus, Piaget, Vygotsky, etc.). Because of our somewhat limited scope (investigating only three instructional methodologies), have we bypassed any significant contributors to the understanding of teaching by inquiry in any of our disciplines?
2. Do our preliminary results (use of the RTOP with exemplary inquiry practices to uncover classroom behavior associated with inquiry) resonate in any way (good or bad) with this audience?
3. Is our methodology (use of the RTOP with exemplary inquiry practices to uncover classroom behavior associated with inquiry) appropriate for beginning to construct an assessment instrument for inquiry instruction across the STEM disciplines?

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# A Longitudinal Study of Mathematics Graduate Teaching Assistants' Beliefs about the Nature of Mathematics and their Pedagogical Approaches toward Teaching Mathematics

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**Keywords:** Graduate teaching assistant education, calculus, theories of learning and teaching.

**Abstract:** The purpose of this research study was to explore mathematics graduate teaching assistants' (GTA) beliefs about the nature of mathematics, their pedagogical approaches toward teaching mathematics and how these evolve over a span of a year. The GTAs participated in four open-ended interviews designed around the *planning, performing and assessing* framework of Speer and Kung (2009). Our preliminary analyses revealed hierarchical stages of GTA knowledge of their students as well as a separation between their ontological and pedagogical stances.

**Introduction:** It cannot be denied that communication of a subject is in part a reflection of the individual's view of that subject. Many have documented pre-service K-12 teachers' beliefs and have gone on to study how these beliefs influenced classroom practice. (For example: Cooney, T., Shealy, B. and Arvold, B., 1998; Day, R., 1996; Thompson, A., 1984; Thompson, A., 1992; Vacc, N. and Bright, G., 1999.) The harmony between the two is crucial. Austin (2002) discusses a number of issues related to professional development for future faculty members in general and further points out the importance of biography in understanding how beginning graduate students develop. An important aspect of doing so is understanding how GTAs' beliefs about mathematics might be similar to or different from those of their students.

**Methodology and framework:** The five participants of this study were graduate students pursuing Ph.D.s in mathematics at a top research university in the Unites States. At the beginning of this study, the participants were in the first or second year of their program and all held degrees in mathematics. Except one participant all other had no prior experience in teaching college mathematics. The participants primarily led discussion sections and followed a standard curriculum while reserving the autonomy in how they structure the discussion sections or write the quizzes they give.

Data were collected in a series of clinical interviews, the first one being recorded during the GTA orientation week prior to the start of fall classes. The participants were interviewed three more times approximately at intervals of one semester. The open-ended interviews were designed following the framework of Kung and Speer (2009) of stages of planning, performing and assessing. Our premise was that initially GTA expectations of their students' view of mathematics were formed by their own experience and ideas about how mathematics is (ontological stance) and how a teacher of mathematics is (pedagogical stance). These expectations drove their practice (planning, performing, assessing). The practice generated assessments of their students (in-class/exam).

**Research Question:** Our goal was to answer the following question. Can the assessment results portraying in part how *students view mathematics*, reform the expectation which is formed by how the *GTAs view mathematics* (and thereby the practice that follows), if those are at a contradiction with one another? And what do the GTAs do faced with the reformed expectation? Do the GTAs separate their pedagogical stance from their ontological stance, in that they present a different view of mathematics to their students while keeping their own view for themselves?

A preliminary analysis of the results of the first three interviews led us to gather further clarification in the following categories:

1. GTA's ontological stance versus pedagogical stance

    a. GTAs' view of mathematics and the view they portray to their students.
        i. Theory-based (abstract) versus example-based (concrete)
        ii. Connected (deep) versus stand-alone (shallow)

    b. GTAs' preferred style of teaching (when they were taught) and the style they adopt as they teach, including

        i. GTAs' preferred definition (of the concept of limit) and the definition they offer to their students
        ii. GTA's preferred type of questions and the questions they ask of their students.

2. GTAs' views of struggles and rewards of teaching as well as remedies they suggested.

**Preliminary results:** The preliminary analyses of this interview confirmed the following levels in GTAs' knowledge of their students

Stage A →Stage B1→ Stage B2→Stage C1→ Stage C2

i. **Stage A** Begin with egocentric model of students (e.g., students care only for their grades, not the knowledge)
ii. **Stage B1** Move to behaviorist observations (e.g., students react to task *x* by exhibiting behavior *y*)
iii. **Stage B2** Move to refined behaviorist observations (e.g., students can't think abstractly, for example in the case of negative T/F statements)
iv. **Stage C1** Move to cognitive explanations (e.g., students have difficulty coming up with counterexamples to justify a F response for a negative T/F statement)
v. **Stage C2** Move to cognitive theories (e.g., students have difficulty with negative T/F statements because of no training interpreting logical statements)

We classify the stages B1 and B2 as behaviorist, and the latter two as constructivist.

Stages B1, B2: Teacher centered-knowledge of how students react to tasks (behaviorist)
Stages C1, C2: Student-centered- knowledge used to create cognitive theories (constructivist)

Below we present a glimpse of the GTA thoughts and practices with the example of Clara.

*Example of Clara*

Clara, a GTA who has been teaching for four semesters, is a graduate student pursuing her doctorate in the field of logic. She likes math for its abstract and rigorous nature, "Like if you prove something, you know it's true and there is no discussion about it", prefers theory and proofs to examples and applications for herself but not abashed about offering the students quite the opposite, because she knows "they don't care about it" (the theories and the proofs) and that she doesn't want to "make them feel confused about it". She holds a rich and connected view of mathematics for herself admitting that challenging questions enriched her own understanding as a student, whereas provides her students with a user's manual approach to calculus and straightforward questions so that "they don't feel the pressure" to understand. She calls her approach as "adapting to her students" based on her knowledge that "they are not like me".

At first we see her claiming how she is not bothered by this dichotomy of her as a student and her as a teacher, as she offers dismissive sentences such as "They pay me, so I do it" or the excuse that it is "harder to bring these features in Calculus". However, as we go further the inner conflict becomes apparent by her disappointment that her students are not like her, that those "really amazing cool results" she learned in linear algebra are not the same for her students, when she finds that "most of the students probably don't see or don't care about this at all". Or her admittance that if all her students exhibited the curiosity or cared about the material, she would "definitely" change her questioning patterns to include more challenging questions, "questions they can think about". It gets further reconfirmed by the end when she chooses her one wish if she could change anything to make it more exciting for her as a teacher as having students who "want to be in the course", who "want to understand"; perhaps not to the extent of her as a student, perhaps not to the level of a guaranteed understanding of mathematics, but as a step merely necessary in that direction.

Clara clearly demonstrates different ontological and pedagogical stances. But what stage is she at when it comes to her knowledge of her students?

We will present further results along with more details about the framework for the behaviorist and the constructivist stages at the meeting.

**Questions for audience:**

1. How do you classify Clara?
2. Can this phenomenon be viewed as dichotomy or is it a duality?
3. Did you experience this as a GTA?

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# What's the big idea?: Mathematicians' and undergraduates' proof summaries

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**Abstract**
*In this study, seven mathematicians and seven undergraduates were asked to read and summarize mathematical proofs that they read to investigate which ideas they consider to be important in a proof. Mathematicians' ideas were generally a) important equations, theorems or facts used in the proof, b) general methods used in the proof, c) diagrams or graphs, or d) overarching goals of the proof. Additionally, mathematicians and students sometimes included details or computations in their summaries that were unfamiliar, subtle, or not routine for them.*

Key words: Proof, key ideas, proof reading, proof summaries

## Introduction

A large part of lectures in advanced undergraduate mathematics courses consists of professors presenting proofs of mathematical theorems to their students (e.g., Weber, 2001). Many mathematics educators contend that the purpose of presenting these proofs goes beyond convincing students that theorems are true; these proofs should also communicate to students some form of explanation or insight (deVilliers, 1990; Hanna, 1990; Knuth, 2002). However, as Raman (2003) and Weber (2010) noted, there is not a shared standard as to what this explanation and insight means.

The purpose of this paper is to investigate what mathematicians and undergraduates think are the important and useful ideas in the proofs that they read. We explore this broad issue by addressing these specific research questions: How do mathematicians and undergraduates create personal summaries of the proofs that they read? What aspects of these proofs do they choose to include in their summaries?

The decision to focus on summaries has practical importance. When reading any text, including mathematical proof, readers rarely remembers every word or sentence of that text. Rather, most readers remember main ideas from the text and use these to reconstruct the details, if possible and necessary. Knowing how mathematicians and undergraduates summarize text is there useful for two reasons. First, exploring how undergraduates summarize a proof provides insight into what insights undergraduates gain from the proofs that they read. Second, knowing how mathematicians summarize proof illuminates what aspects of proofs that they find important, and suggests aspects of proofs that could be emphasized to students.

## Theoretical perspective

Mathematical proofs are written so that each assertion that is not an acceptable premise (e.g., a hypothesis or a previously established fact) is a logical consequence of previous assertions in the proof. One way that a proof can be understood is at a line-by-line level, where the reader of the proof identifies the mathematical reasons for how new

assertions follow from previous ones (e.g., Weber & Alcock**,** 2005). However, many mathematicians and mathematics educators argue that a proof can be understood in terms of its global ideas, and that this understanding can be as valuable, if not more valuable, than understanding the proof at a line-by-line level (e.g., deVilliers, 1990; Hanna, 1990; Leron, 1983; Thurston, 1994; Mejia-Ramos et al, 2012). In this paper, we aim to investigate the different global mathematical ideas that mathematicians and students find important enough to include in their summaries.

Mathematics educators have posited different types of main ideas that may be present in a proof, which we summarize below:

*Explanation by characteristic property:* Building on the philosophical work of Steiner (1978), Hanna (1990) argued that proofs can explain why a theorem is true by revealing a crucial property that a mathematical object has that causes the theorem to be true.

*Mental models:* Thurston (1994) and Weber (2010) suggested that mathematicians are sometimes less concerned about the step-by-step logic in a proof but more concerned with inferring the mental models that the author used to explain why the theorem was true and support the construction of the proof.

*Key ideas:* Raman (2003) contended that a proof can be understood in terms of its "key ideas", where a key idea is a mapping from a "private" informal way of understanding why the theorem is true to a "public" formal rigorous proof.

*High-level ideas:* Leron (1983) argued that a proof can be summarized by its high-level ideas that describe the proof in terms of a few major steps while not including the logical details needed to support the proof.

*Central equation or theorem:* Weber (2006) claimed that a proof can be understood in terms of the central theorem or mathematical principal being applied.

We use these constructs as a preliminary means for analyzing our data.

## Methods

*Participants.*

Seven mathematicians and seven undergraduate mathematics majors were interviewed at a large university in the northeastern United States.

*Materials.*

Two proofs that were used in this study are attached in the Appendix. Two additional proofs, one based on ideas from elementary geometry, and another from elementary number theory, were also included in the study, from elementary geometry, were also used but were not included in this submission due to space restrictions.

*Procedure.*

Participants were interviewed in one-on-one, hour-long sessions. Participants were told to think aloud while reading each proof and write a summary on a separate sheet of paper. They were told that this summary would be for themselves and should include the important ideas of the proof.

*Analysis.* We used a semi-open coding scheme to characterize each of the following: (a) any statement in a participant's summary that was not merely a translation from the text of the proof, (b) any statement in a summary that was written in a manner to highlight its importance or accompanied by an oral comment of the participant that indicated its purpose or importance, and (c) any assertion that a participant made when the interviewer asked him or her what the main idea of the proof was. Each of these assertions was, if possible, coded using the themes listed in the theoretical perspective: As an explanation

by characteristic property, mental models, key ideas, high-level ideas, or central equation or theorem. Additional categories were formed to account for assertions or comments that did not fit these categories.

**Results**

The ideas mathematicians included in their summaries largely fell into one of the four categories: a) important equations, theorems or facts used in the proof, b) general methods used in the proof, c) diagrams or graphs, or d) overarching goals of the proof. Additionally, mathematicians and students sometimes included details or computations of parts of the proof that were unfamiliar, subtle, or not routine for them.

**Discussion and significance**

The data in this paper illustrate is that there is no single way to characterize the "big ideas" in a proof. The mathematicians in our study highlighted four major types of ideas in their summaries. Further, the data suggest that mathematicians and undergraduates did not only value global properties of the proofs that they were asked to summarize; they also sometimes included specific details, such as a tricky calculation, that was not routine or obvious to them. A significant finding is that participants' findings were related to their knowledge of the content being studies. Participants' highlighted techniques that were novel to them, but also synthesized proof methods that were routine to them. In total, these findings illustrate that there is no simple way to characterize a summary of a proof, nor is this there a way to objectively say what constitutes a good proof.

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## Appendix

**Claim.**
The only solution to the equation $x^3 + 5x = 3x^2 + \sin x$ is $x = 0$.
**Proof.**
Clearly, $0^3 - 5(0) = 3(0)^2 + \sin(0)$, so $x = 0$ is a solution to the equation. We need to show there are no other solutions.
Let $f(x) = x^3 - 3x^2 + 5x - \sin x$.
Roots of $f(x) = 0$ precisely correspond to solutions of $x^3 + 5x = 3x^2 + \sin x$.
Suppose $f(x) = 0$ has a nonzero root; that is $s \neq 0$ and $f(s) = 0$.
$f'(x) = 3x^2 - 6x + 5 - \cos x = 3(x^2 - 2x + 1) + 2 - \cos x = 3(x-1)^2 + 2 - \cos x$.
Since $3(x-1)^2 \geq 0$ and $2 - \cos x > 0$ for all real numbers *x*, $f'(x) > 0$ for all real numbers *x*.
Since $f(0) = f(s) = 0$ and, $s \neq 0$ by Rolle's theorem, there exists *c* between 0 and *s* such that $f'(c) = 0$.
However, this is a contradiction because $f'(x) > 0$ for all *x*.

**Note:** Rolle's theorem states that if *f* is a differentiable function, $a < b$, and $f(a) = f(b)$, then there is a *c* such that $a < c < b$ and $f'(c) = 0$ .

*Figure 1.* Proof 1, using ideas from calculus.

We say that a number is monadic if it can be represented as $4j + 1$, and triadic if it can be represented as $4k + 3$, for some integers *j* and *k*.
**Claim.**
There exist infinitely many triadic primes.

**Proof.**
Consider a product of two monadic numbers:
$(4j + 1)(4k + 1) = 4j \cdot 4k + 4j + 4k + 1 = 4(4jk + j + k) + 1$,
which is again monadic.
Similarly, the product of any number of monadic numbers is monadic.
Now, assume the theorem is false, so there are only finitely many triadic primes, say $p_1, p_2, \ldots, p_n$.
Let $M = 4p_2 \cdots p_n + 3$, where $p_1 = 3$.
$p_2, p_3, \ldots, p_n$ do not divide *M* as they leave a remainder of 3, and 3 does not divide *M* as it does not divide $4p_2 \cdots p_n$.
We conclude that no triadic prime divides *M*.
Also, 2 does not divide *M* since *M* is odd.
Thus all of *M*'s prime factors are monadic, hence *M* itself must be monadic.
But *M* is clearly triadic, a contradiction.

*Figure 2.* Proof 2, using ideas from elementary number theory.

# What do mathematicians do when they reach a proving impasse?

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*I report how two mathematicians came to impasses while constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. By an impasse, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the length of time but its significance to the prover and his or her awareness thereof. I point out two kinds of actions these mathematicians took to recover from their impasses: one relates directly relates to the ongoing argument, while the other consists of doing something unrelated to the ongoing argument which can be mathematical or non-mathematical. Data were collected using a new technique being developed to capture individuals' autonomous proof constructions on tablet computers in real-time.*

Key words: university level, proof, mathematicians, impasse, tablet PC

This preliminary report presents part of an ongoing larger study of mathematicians and graduate students constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. During separate data collection sessions, each of two mathematicians came to an impasse in proving certain theorems. This study investigated what actions these mathematicians took to try to recover from those impasses. Data were collected in real-time using a new technique being developed to capture individuals' autonomous proof constructions on tablet computers.

## BACKGROUND LITERATURE

While there has been research on mathematicians' actions during proof validation (Weber, 2008), on how mathematicians learn new mathematics (Burton, 1999; Wilkerson-Jerde & Wilensky, 2011), and on how mathematicians use diagrams to construct proofs (Samkoff, Lai, & Weber, 2011), to date there appears to have been little or no research on what mathematicians do when they reach an impasse during proving. This may have implications for helping students with proving.

To date, research on university students' proving has been concerned with a variety of topics including: difficulties they encounter during the proving process (Moore, 1994; Weber & Alcock, 2004), with their validations of proofs (Selden & Selden, 2003), and with their comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010). Such research is helpful in teaching proving. In the same way, it would be interesting to know what students do when they are actually in the process of proving, and in particular, what they do when they come to an impasse. This study is a start in that direction.

## THEORETICAL FRAMEWORK

By an *impasse*, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the length of time but the significance to the prover and his or her awareness thereof. Mathematicians themselves often colloquially refer to impasses as "being

stuck" or "spinning one's wheels." This is different from simply "changing directions," when a prover decides, without much hesitation, to use a different method, strategy, or key idea. I will point out two kinds of mental or physical actions a prover may take to recover from an impasse. One kind of action directly relates to the ongoing argument. The other kind of action is doing something else unrelated to the ongoing argument which can be mathematical or non-mathematical. Examples of both will be provided.

Computer scientists working on automatic theorem provers have considered how machines overcome impasses, noting that "when an expected progress does not occur or when the proof process gets stuck, then an intelligent problem solver analyzes the failure and attempts a new strategy" (Meier & Melis, 2006). However, this is different from my description of an impasse because it does not have a time component and for a person, analyzing the failure, can be considered as a continuation of the proving process.

## DATA COLLECTION TECHNIQUE

Several mathematicians agreed to participate in this study. They were provided with notes on semigroups containing definitions, requests for examples, and theorems to prove. The notes were a modified version of the semigroups portion of the notes for a Modified Moore Method course for beginning graduate students. This topic was selected because the mathematicians would find the material easily accessible, and because there are two theorems towards the end of the notes that have caused substantial problems for beginning graduate students.

Data on the mathematicians' written work, with time-stamps, were collected electronically on a tablet PC. I explained how to use the stylus that came with the computer, the CamStudio screen recording software, and Microsoft OneNote, which was the space in which the mathematicians wrote their proof attempts. All mathematicians, including the two described here, kept the tablet PC for 2-4 days. After the computer was returned, I analyzed the screen captures (like small movies in real time) and the mathematicians' proof writing attempts. One or two days later, I conducted an interview during which I asked each mathematician about his proofs and proof-writing. The two mathematicians offered that the choice of semigroups was judicious, because they were able to grasp the definitions and concepts quickly, and because at least one of the theorems had been somewhat challenging to prove.

## WHAT THE MATHEMATICIANS DID

In this paper, I focus on just two mathematicians: Dr. A, an applied analyst, and Dr. B, an algebraist.

In his proofs, Dr. A encountered an impasse on the final theorem in the notes: "If $S$ is a commutative semigroup with minimal ideal $K$, then $K$ is a group." He first attempted a proof by contradiction. After two and a half minutes, he moved on to the final part of the notes containing a request for examples, which he provided quite quickly. Dr. A then spent another 8 minutes, during which time he scrolled up on the OneNote program in order to view his first contradiction proof attempt, which he then erased. He then unsuccessfully attempted another proof, trying to utilize his previous correct proof of the penultimate theorem. The screen-capturing of these unsuccessful proof attempts started at 3:48 PM, and the session ended at 4:17 PM. The next screen capture started the following day at 11:07 AM, with Dr. A again attempting the proof, this

time using mappings and inverse mappings of elements. Then he wrote, "I don't know how to prove that $K$ itself is a group," thereby acknowledging that he was at an impasse. After that, there is a 30-minute gap between screen-captures until he finally proves the theorem successfully.

Dr. A indicated in his post-interview where he had had an impasse, noting "One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck." Dr. A also indicated generally how he consciously recovers from impasses: he prefers to get "un-stuck" by walking around, but distractions caused by his departmental duties also help.

Dr. B got to an impasse on the penultimate theorem, "If $S$ is a commutative semigroup with no proper ideals, then $S$ is a group." Unfortunately with Dr. B, I did not get any screen captures, but his written proof attempts are very detailed and the exit interview was very informative. He wrote, "Stuck on [theorem] 20. It seems you need $1 \in S$, but I can't find a counterexample to show this." Dr. B then moved on to the final theorem on which Dr. A had had an impasse, proved it correctly, and then crossed his proof out. He then moved on to the final request for examples, explaining in his exit interview, "I moved on because I was stuck...maybe I was going to use one of those examples...I might get more information by going ahead." Dr. B's next approach was to create counterexamples. After considering his counterexamples for some time and taking his family to lunch, Dr. B proved both theorems correctly.

In the exit interview, Dr. B stated that he had created a property that had confused him, and thought that he needed to assume that there was an identity. Also, he said, "I probably spent 30 minutes to an hour trying to come up with a crazy example. I went to lunch and while I was at lunch, then it occurred to me that I was thinking about it the wrong way. So I went back then and it was quick."

## RESULTS

The actions directly-related to the ongoing argument that these two mathematicians took to recover from impasses were: utilizing semigroup proof techniques that they had used earlier in the sessions, utilizing prior knowledge from their own research areas, and generating examples and counterexamples. The second kind of action involves doing something else. In the data, these were doing subsequent problems in the notes and coming back to their unfinished proof attempts, and engaging in other “non-proof” activities (such as walking around the office, doing other tasks, going to lunch). The first one of these is mathematical, whereas the remaining are non-mathematical diversions. Most of these actions were more or less automatic and not consciously noted by the mathematicians either during the session or in the exit interviews. In analyzing an action, it is sometimes difficult to distinguish between a conscious intention to recover from an impasse or a serendipitous action later recognized as having been helpful.

Doing other activities and coming back to an unfinished problem might be considered an example of *incubation*, which is the process by which the mind goes about solving a problem subconsciously and automatically, and which happens best when one takes a break from creative work (Krashen, 2001). While there are many reports of experiments on incubation in the psychology literature (Sio & Ormerod, 2009), they typically allow only a short time for incubation. However, both mathematicians stated that when they received the notes, they immediately glanced at them to estimate how long the proofs might take, but both started proving the next day. It is difficult to know whether there was an incubation effect due to actually commencing their proving the next day. How can we gain information on when and how

incubation is used in mathematics? Is it important to let students know about incubation? How can we collect all actions that mathematicians use to recover from impasses? Also, can we encourage students to take some of these actions to recover from impasses?

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# First Semester Calculus Students' Understanding of the Intermediate Value Theorem

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In our calculus courses, we often see students perform poorly on problems involving the Intermediate Value Theorem (IVT), despite being a fairly basic concept. Thus, we designed a study to analyze students' conceptual understanding of the IVT and their ability to express the theorem in their own words. Two groups of students were video-taped while working on an activity designed to guide their construction of an initial understanding of the IVT, and fifty-four students were later asked to state the IVT in their own words. Both video data and student responses on the written work were analyzed to identify common themes. It was found that even though students were able to understand the concepts behind the Intermediate Value Theorem, they were unable to correctly describe the IVT in their own words, largely due to confusing the independent and dependent variables and issues with the if/then structure in a theorem.

Keywords: Calculus, Intermediate Value Theorem, mathematical language

The Intermediate Value Theorem (IVT) is typically the first theorem introduced in a first-semester calculus course, and quite possibly the first formal mathematical theorem that many students encounter. In Stewart's *Essential Calculus*, this theorem is introduced in Section 1.5, which discusses an informal notion of continuity. Recall that the IVT states that if a function $f$ is continuous on the closed interval $[a,b]$ and $N$ is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number $c$ in $(a,b)$ such that $f(c) = N$. (See Figure 1.)

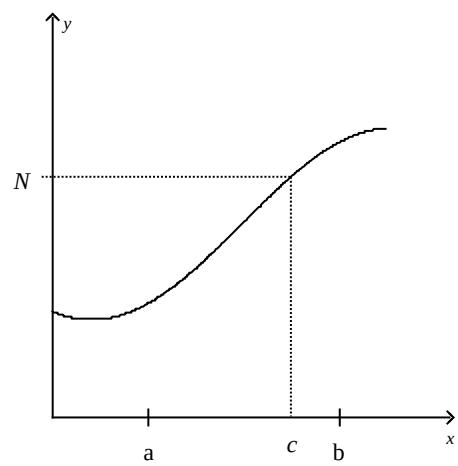


**Figure 1: Illustration of the Intermediate Value Theorem**

While much research has been conducted on student understanding of some fundamental mathematical concepts and theorems, very little work has been done to investigate student understanding of the IVT. For this study, we have identified 3 aspects of understanding the IVT: a conceptual understanding, the ability to state the hypotheses and conclusion of the theorem correctly (written language), and the ability to apply the theorem to a problem (finding zeroes, etc.). This preliminary report focuses on the first two aspects of IVT understanding. From prior teaching experiences, we have seen that students may appear to have a conceptual understanding of the Intermediate Value Theorem but are still unable to apply or express this idea when necessary. In this study we specifically investigate whether or not students are able to

conceptualize the meaning of the IVT and whether or not they are able to express the IVT in written form.

## Literature Review

In an exploratory study, Monk (1992) found that students conceptualized functional situations in two distinct ways, termed *point-wise* and *across-time*, and that if the manner in which they conceptualized function did not meet the demands of a given task, student difficulties arose. We found this to be very similar to the ways in which our students initially interpreted the Intermediate Value Theorem. Instead of determining if a $y$-value of $N$ existed on the entire function, the students focused on one particular $x$-value, and evaluated if that input produced an output of $N$. Carlson (1998) found that even mathematically talented students still have misconceptions about functions, specifically with respect to the language of functions.

Another fundamental mathematical concept that relates to the Intermediate Value Theorem is that of limits and the struggles students have in understanding them. Cottrill et al. (1996) provide a genetic decomposition of the limit concept and posit that a more complete development of a dynamic view of this concept will promote a better understanding in students. Oehrtman (2009) describes various metaphors students use when understanding and describing the concept of limit, and he advocates for promoting an approximation metaphor when teaching students, since it is easy for students to understand and also closely aligned with formal mathematics. On the other hand, Williams (1991) found that students held fast to their models for understanding limit and were "extremely resistant to change" (Williams, 1991, p. 219). This emphasizes the need to be careful and deliberate about the ways in which we first introduce these ideas to our students.

Researchers have also investigated how students understand theorems such as the Extreme Value Theorem and Rolle's Theorem. Abramovitz et al.(2007, 2009) developed a process for learning theorems (the self-learning method) to help students better understand the hypotheses and conclusions of the Mean Value Theorem and Rolle's Theorem. Much work has also been done on students' ability to prove theorems, but our work does not address proving the IVT, only understanding the statement.

## Theoretical Perspective

Piaget's structuralism (1970, 1975) is used as the theoretical perspective throughout this study. Structuralism is a type of constructivism wherein it is believed that students construct an understanding of mathematical concepts not at free will, but within certain constraints. In this particular study, students worked in groups on an activity that guided them to construct an understanding of the hypothesis and conclusion of the Intermediate Value Theorem.

## Methods/Subjects

Participants in the study were first-semester calculus students at a large, public, research university. Two sections of students participated, both of which were taught by one author. In each class, a group of four students was videotaped while working on the activity mentioned above. This activity was given before the instructor formally introduced the IVT to the class. Students were asked to draw a series of functions which satisfied some of the conditions given in the IVT. Two class periods after completing the activity, all students ($n$ = 54) were given a pop quiz which asked students to state the Intermediate Value Theorem in their own words. Written responses were collected and analyzed using Corbin and Strauss' (2008) open and axial coding.

## Results

As mentioned earlier, we examined two aspects of student understanding of the IVT: conceptual understanding and the ability to state the hypotheses and conclusion correctly (written, language). At this point, we have not, yet, studied students' ability to apply the theorem to a problem (finding zeroes, etc.), but will do so in a future study. Analysis of the video tapes shows that students in this study do, in fact, understand the concept of the IVT, although they have significant difficulty with formal mathematical language.

In the first question on the in-class activity, students were asked to sketch the graph of a function such that $f(c) = 13$ does not exist. Seven of the eight students sketched a graph where the *x* value of 13 produced no *y*-value, instead of avoiding a *y*-value of 13. Both groups of students needed assistance to recognize their mistake, but all students were easily convinced of their mistake. One student said, "Oh yeah, *f* of *x*" with a strong emphasis on *x*. Some students, who happened to initially graph a function that was one-to-one realized that they could simply "rotate" their graph so that a *y*-value of 13 would not exist, instead of avoiding an *x*-value of 13. However, students did not seem to be aware that functions that were not initially one-to-one would not produce a function when rotated. (See Figure 2.) The instructor or a teaching assistant eventually pointed out the problem to the students, and they were able to fix their graphs to produce appropriate functions.

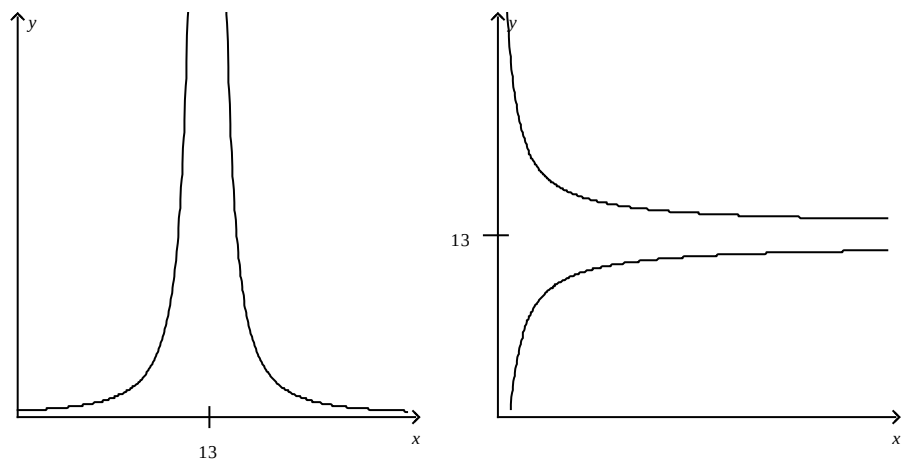


**Figure 2: Illustration of a rotation that does not produce a function**

Another mistake that was prevalent in the video data is the tendency for students to avoid one specific *y*-value of 13, namely the point (0, 13). Students seemed to be attending only to the place on their graph where they had labeled *y* = 13, instead of attending globally to *any y*-value of 13. This aligns with Monk's (1992) classifications of point-wise versus across-time reasoning. The graphs that our students drew did not cross through the point (0, 13), but sometimes had a *y*-value of 13 elsewhere (See Figure 3). Other students were able to draw a graph that never had a *y*-value of 13, but it was unclear in the video if that was by chance or if it was a purposeful decision. Additional data is needed to more fully understand students' beliefs.

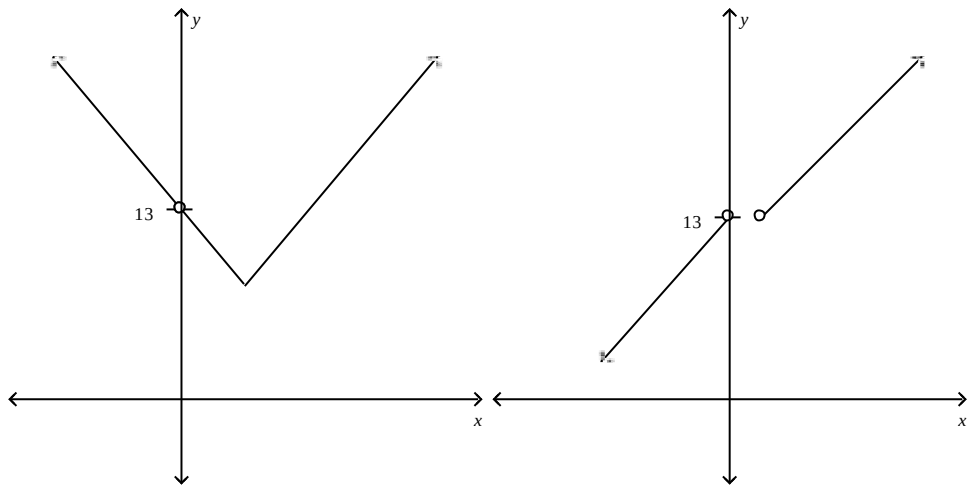


**Figure 3: Considering the point (0,13) instead of any *y*-value of 13**

Even though the students had difficulty with basic function notation, they were able to understand the ideas behind the Intermediate Value Theorem. Students had no trouble believing that a continuous function *must* pass through a *y*-value of 13 if there were a *y*-value less than 13 and a *y*-value greater than 13 somewhere in the function. Throughout the semester, students told the instructor that they "get the idea" but have difficulty expressing it. Their verbal descriptions of the theorem often included gestures, which made it easier for them to express. In the written work, the students greatly struggled with function notation and the overall structure of an "if-then" statement.

Fifty-four responses to the pop-quiz question were collected, and mistakes were categorized according to common themes. One noticeable error was in the students' attempts to use the standard if/then wording of the theorem. Common errors in this category included the presence of a hypothesis with no conclusion statement or switching the 'if' and 'then' statements (yielding in an incorrect assumption that the IVT proves that a function is continuous). Often, students would omit one or more parts of the theorem (e.g. not stating that the function must be continuous), resulting in a statement of a theorem that was not always true.

Another common problem in the student responses on the quiz dealt with issues in the *x* and *y*-values. A few students used a non-standard notation, but still produced a mathematically correct statement. He wrote, "The Intermediate Value Theorem states that if a function is continuous and there is a point *a* with *y*-value *x* and a point *b* with *y*-value *z*...". As mathematicians and teachers, we would never consider labeling a *y*-value with *x*, but it is not mathematically incorrect. Other students were less clear about whether the variables they used referred to *x* or *y* values, making it unclear whether or not their statements were correct. Still, other students were clearly wrong in their labeling. For example, one student stated, "The Intermediate Value Theorem is proving that *N* (*y*-value) exists by finding an *a* and *b* (*x*-values) on a continuous graph both greater than and less than *N*." In this example, the student clearly labeled *N* as a *y*-value and said that this *y*-value should between two *x*-values.

What we find most interesting in our data is the difficulty students have with the IVT, even though they seem to understand the concepts behind it. Much of the research in our RUME community supports that if the conceptual understanding is present, then the rest of the work (application problems, etc.) should follow without too much difficulty. We realize we are over-

simplifying the research, but we want to stress that, in this case, a conceptual understanding of the theorem was not enough to allow our students to move forward. We still have more work to do to determine what exactly the deficiency is and what solutions to these problems are.

**Implications for Teaching and Future Research**:

As teachers, identifying our students' misconceptions in understanding fundamental mathematical theorems and concepts will help us to better teach these concepts in ways that address the common misconceptions, thereby improving student understanding. Not only will this help us to reach future students when presenting the IVT, but it will also help us to reach our current students in other topics in the class, such as the Mean Value Theorem. Later this semester, we plan to collect similar data regarding student understanding of the Mean Value Theorem. Based on results from our initial study of the IVT, we already know some of the common underlying issues students will have.

**Questions for the Audience**

1. ***We*** believe that it is important for our students to be able to express their ideas using correct mathematical language and notation, but our ***students*** do not always see the need. How do we convince our students to buy into this idea and understand the importance of using mathematical language correctly?

2. One weakness that we see in our results is convincing the reader that students ***do*** understand the concept of the IVT, even though they cannot express it well using the appropriate mathematics. What data could we collect to convince readers of this?

3. What changes or additions should we make in the next round of data collection?

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# Examining Students' Mathematical Transition Between Secondary School and University – The Case of Linear Independence and Dependence

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To understand the mathematical transition students make between secondary school and the university requires an in-depth look at the mathematical topics students learn at the time of this transition and the contextual, institutional changes that simultaneously occur. This preliminary presentation explores how linear algebra students at both the secondary school and university in Germany understand vectors and linear independence and dependence in the course of video-recorded, think-aloud problem-solving interviews. Analysis of these interviews indicate not only differences in mathematical content and sophistication between secondary school and university students, but also in students' disposition, particularly towards new mathematical experiences. A look at more informal data about the various institutional environments, secondary school and university, provides a potential reason for these differences. This report concludes with a discussion on how to create a blended analysis of these individual understandings and dispositions and their relationship with the institutional context as a better means of understanding the transition to university-level mathematics.

**Keywords:** transition to university mathematics, linear algebra, conceptual understanding, institutional environments

## Examining Students' Mathematical Transition Between Secondary School and University – The Case of Linear Independence and Dependence

The gap between secondary school mathematics and university mathematics has proved to be a particularly difficult challenge for students (cf. De Guzmann, Hodgson, Robert, & Villani, 1998; Tall, 1991). To understand the mathematical transition students make between secondary school and the university requires an in-depth look at the mathematical topics students learn at the time of this transition and the contextual, institutional changes that simultaneously occur. In particular, there are certain courses that fall exactly during this transition. In Germany, linear algebra is one such course, with foundational linear algebra topics like vectors and linear independence being introduced in the last years of secondary school then revisited and built upon in the first year at the university.

This study begins by asking how do students think about and work with the ideas of vectors, linear independence, and linear dependence at the secondary school and university levels and what differences these two distinct groups of students have in viewing and working with these concepts.

The initial results of this analysis suggest differences not only in how these distinct groups view these concepts, but also in how students approach tasks that require the students to work with these concepts in novel or more unfamiliar settings and their disposition towards these new mathematical experiences. This begs the question: how do we account for the differences between the secondary school students and university students? The study conjectures that these differences come from not only the level and sophistication of the mathematical content of their courses, but also from the differences in the institutional settings.

### Literature

There is a growing body of work regarding student reasoning in the context of linear algebra, the most comprehensive of which is an edited volume by Dorier (2000). Within this volume, Hillel (2000) observes that at the US university linear algebra is often the first mathematics course that students encounter as a mathematical theory, with formal definitions and proofs and built up systematically. Furthermore, Hillel details some of the difficulties students have in terms of understanding different but related modes of description of mathematical objects in linear algebra such as abstract, algebraic and geometric notions of vectors. More recent work regarding student difficulties in learning linear algebra include students' conception of the equal sign in matrix equations and early notions of eigenvalues (Larson, Zandieh, Rasmussen, & Henderson, 2009) and connections students make between fundamental concepts in linear algebra (Selinski, 2010). However, in each of these contexts, because students often have encountered linear algebra for the first time at the university and only university students are addressed, these reports do not touch directly on the transition from secondary school to university mathematics nor do they examine students understanding of vectors and linear independence in depth.

This study aims to build from these works by examining how students think about vectors, linear independence and linear dependence. In terms of these key linear algebra concepts, we focus on the importance of a flexible understanding of mathematical concepts as detailed by Tall and Vinner (1981). Furthermore, as with Dahlberg and Housman (1997), we explore the significance of example generation for student reasoning and concept understanding. This report uses this strong foundation on students understanding of concepts as a means for exploring the difficulties in transitioning from notions of mathematical concepts from the secondary school to the university level.

One piece of literature that may help us understand how these more individualistic understandings relate to the institutions and learning environments, and thus to this transition, would be Cobb and Yackel's (1996) elaborated interpretive framework for the emergent perspective. This framework sees the "individual students' activities… located in the broader institutional setting" (p. 181) and create a framework for understanding the reflexive relationship between individual psychological and more sociocultural perspectives.

**Methods**

Data for this report comes from a year-long project examining how students learn linear algebra at the end of their secondary schooling at the German *Gymnasium* (upper-level high school) and in their first year at the German university. As a part of this project, six secondary school students and five university students participated in individual, semi-structured, think-aloud problem-solving interviews (Bernard, 1988) that were approximately 60 to 90 minutes long. The interviews were video-recorded, and the analysis of the data involves repeatedly reviewing these videos, selective transcriptions of the videos, and copies of students written work created during the course of the interviews. This report will focus on the two questions posed in these interviews, which asked:

1. How do you think about what a vector is? (Follow-up questions to get at geometric, algebraic and abstract understandings)
2. For each of the following, please create an example that fits the given criteria:
    a. A set of vectors in $R^2$ ($R^3$, $R^4$, $R^n$, not in $R^n$) that is linearly dependent
    b. A set of vectors in $R^2$ ($R^3$, $R^4$, $R^n$, not in $R^n$) that is linearly independent.

Additional follow-up questions were also asked to clarify how students thought of vectors, linear independence and linear dependence, and how these understandings were reflected in their creation of examples.

These interviews were then reviewed and selectively transcribed. The initial analysis paid extra attention to the differences between these distinct groups of students.

Furthermore, in order to account for the different environments in which the students learned linear algebra, more informal data was collected about the Gymnasium and German university. Data about the institutional environments comes from notes completed while observing classes at the secondary school and lectures, homework sessions, and informal study groups of students at the university. Further data comes from notes while discussing the expectations of learning in these environments with instructors from both institutions.

**Preliminary Results:**

Preliminary results suggest that most students at the *Gymnasium* had well-established geometric and algebraic notions of vectors and linear independence and dependence. Similar understandings were given by all university students, which the students cited originated or built from their studies of linear algebra at the *Gymnasium*, before university. Surprisingly, despite the strongly formal approach to instruction of linear algebra at the university, few university students were able to cite or work with abstract notions of vectors or linear independence, rather opting for algebraic and geometric descriptions first seen at the *Gymnasium*.

A more surprising result was not initially seen until the students were pushed into more unfamiliar mathematical situations. For example, when *Gymnasium* students were asked to generate an example of linearly independent vectors in $R^4$ or the university students were asked to create a similar example but not in $R^n$. About half of the *Gymnasium* students struggled with sets of vectors in $R^4$, citing that they could not see $R^4$, so a set of such vectors does not exist. This reasoning came quickly and without question in the course of the interview. Compare this to the three university students who could not produce an example not in $R^n$. Each of these students paused and struggled with the problem, and when they could not produce an example, the students reasoned that did not mean such a thing did not exist. Rather, it meant that they had not previously seen such an example before or could not understand how to create an example with their personal understanding of vectors, vector spaces, and linear independence.

This difference is one of many that indicates not only a mathematical but a dispositional difference between *Gymnasium* and university students. Whereas university students left open their understandings of these concepts for future sophistication, and most suspected that with this additional knowledge, they could then generate such an example, *Gymnasium* students did not consider mathematical possibilities or representations beyond their own experience.

It should be noted that in the course of an interview, the students were occasionally asked to generate an example that could not and did not exist (e.g., three linearly independent vectors in $R^2$) and they often correctly identified these situations as not possible. As such, both the *Gymnasium* and university students knew that "no example" was a possible valid solution.

The students' disposition towards applying their knowledge in a new unfamiliar mathematical context correlates with what one might expect given the different institutional norms in which the students had recently encountered linear algebra. At the *Gymnasium*, students were asked to solve tasks that mimicked or varied slightly the examples previously worked by the teacher or students in front of the entire class. Emphasis was placed on mastery of solving specific mathematical problems, ones that would be later encountered on student exams in the classroom and for the end degree, the *Abitur*. Successful completion of these exams is essential for students to "graduate" from the *Gymnasium* and go on to a university.

Compare this with the university setting, where courses are often given in a large traditional lecture format – providing students with the same information that might be available in a textbook (note that textbooks are not commonly used in German universities), complimented by smaller homework practice sessions and weekly problem sheets. These problem sheets often asked students to work through novel problems using the concepts introduced in the lecture or had students unpack concepts that were introduced in the lecture but had been previously unfamiliar. Many students used a common space made available to them to collaborate with their peers, an approach to learning that is widely expected and encouraged by the faculty and students alike. As such, small groups of students would often work together to solve these novel problems, explain solutions and understandings they had, or question other students about their solutions and understandings. This corresponds well with the active, undeterred way in which the university students reacted to the novel example generation tasks in the interview.

What still remains to be explored how do we better use this data, particularly in conjunction with a theoretical framework, to account for these differences in mathematical experiences, individual disposition and institutional environments – and more importantly, how to relate these differences. This aspect of the research provides the main impetus for discussion.

**Questions for Discussion:**

- How can the present literature like that for the emergent perspective be best used to unpack the variety of data presented? Or, are there other frameworks (e.g., learning progressions or trajectories) that can better flush out these changes in the individual students mathematical conceptions, beliefs and dispositions, the institutional changes between the *Gymnasium* and university, and how these two relate or blend together?
- How do we best obtain data for and analyze data to understand student changes in disposition? Recommendations for literature or frameworks and new ideas encouraged.
- Germany's situation with linear algebra taught at the secondary school and university differs from the US, where linear algebra is primarily an undergraduate course. How does research coming out of different cultures contribute to research here – or rather, how can Germany and the US best learn from each other's experiences despite major differences in the mathematics covered (and how it is learned) in secondary schools and universities?

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# A First Look at How Mathematicians Read Mathematics for Understanding

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*Abstract*

As students progress through the college mathematics curriculum, enter graduate school and eventually become practicing mathematicians, reading mathematics textbooks and journal articles appears to comes easier and these readers appear to gain quite a bit from reading mathematics. Previous research has focused on what early college students do as they read and the difficulties they encounter that interfere with understanding what has been read. This preliminary study was designed to help us begin to understand how more advanced readers of mathematics read for understanding. Four faculty members and four graduate students participated in this study and read from a first year graduate textbook in an area of mathematics unfamiliar to each of them. The reading methods of the faculty level mathematicians were all quite similar and were markedly different from all the students the researcher has encountered so far, including the more advanced students in this study.

*Introduction*

Many would agree that reading is critical for gaining understanding within a discipline. Yet, most teachers of first-year college level mathematics courses are well aware that even if they ask or require their students to read from their textbooks, that few students do so with understanding. Students complain about how hard it is to read their mathematics textbooks, and it appears that even good readers in general do not read their mathematics textbooks well (Shepherd, Selden & Selden, in press). But as students continue in mathematics courses through undergraduate and graduate work, and eventually become mathematicians, somehow they "learn" to read mathematics textbooks and similar writings in journals with deep understanding.

Is there some "thing" or combination of things that mathematicians "do" as they read that helps them understand better? Maybe mathematicians are better at monitoring their own personal understanding and have confidence that they can "fix" any misunderstanding. And the questions that motivates this study: (1) Are there obvious differences in the reading strategies of mathematicians versus first year undergraduate students, and (2) If there are differences, which differences appear to be significant in learning from reading mathematical text in this situation?

*Literature & Theoretical Perspective*

Reading involves both decoding and comprehension. On the comprehension side of the coin, research has identified several strategies that good readers employ as they engage with text (Flood & Lapp, 1990; Palincsar & Brown, 1984; Pressley & Afflerbach, 1995). These strategies depend on the individual reader, the reader's goals and the material being read.

The theoretical perspective used herein is aligned with the view that reading is an active process of meaning-making in which knowledge of language and the world are used to construct and negotiate interpretations of texts (Flood & Lapp, 1990; Palincsar & Brown, 1984; Rosenblatt, 1994). Yet, it appears that for many students, a major factor in their ineffective reading is a lack of sensitivity to their own confusion and errors and an inappropriate response to them (Shepherd, Selden & Selden, in press).

*Research Questions*

There is considerable reason to believe that most mathematicians can read mathematics textbooks and other mathematical writing effectively. This must be done, not only to teach new courses, but to support a mathematician's mathematical research. However few mathematicians seem to have received any instruction in reading mathematics and seem to have tacitly learned effective reading. Although we would like to eventually know why mathematicians appear to be effective readers and first year college students are not, we limit our research question for this preliminary study to attempting to understand some differences that mathematicians have in approaches to reading mathematics versus both first-year and advanced mathematics students and whether any observed differences seem to contribute to mathematicians' apparent ability to learn from reading mathematical text.

*Research Methods*

The participants were four students and four faculty members at a large southwestern university. Each participant attended a single interview/reading session. One student was a masters level mathematics student, the other three were all pursuing PhD level work in mathematics education. The four faculty members were all experienced teachers and researchers. All participants read from *Lectures on Differential Geometry* (Chern, Chen & Lam, 2000) starting at the beginning of the book and were given instructions that they were to read to learn the material. Two of the faculty members had taken coursework in Differential Geometry (because it was required), but none had done research in the area. The students reading sessions were done first as a pilot. All the reading sessions were video recorded and initial questionnaires were given to assess background and teaching/research experience of each participant. Each reading session lasted about 45-60 minutes. At the end of each reading session, the faculty members were asked to create a homework set over the material they had read.

*Very Preliminary Results*

The advanced mathematics students used techniques and strategies similar to the first-year undergraduate students, although they were more sensitive to monitoring of their own comprehension. These students essentially read the material word for word as undergraduate students appear to do, and worked through the problems or examples on their own which undergraduate students appear to do only when encouraged to do. The mathematicians rarely read word for word. They frequently read "meanings" instead of the words or symbols that appeared on the page. They were very cautious about their own understanding and frequently adjusted their interpretation to match more closely that of the authors of the textbook. More results will be obtained as analysis of the data continues.

*Implications for Further Research and Teaching*

This research project is a preliminary step in understanding the broad scope what it means to read mathematical text for understanding. This is an initial pilot research project to understand the "expert" side of reading mathematical text. Previous research has focused on the "novice" or first-year undergraduate course student. As researching into reading mathematics textbooks continues, there are opportunities to understand not only what experts "do" differently, but how they learn to do this and what steps or phases of learning to read occur between novice

and expert. We can also anticipate the integration of reading for understanding with learning theories.

This current research has strong implications for teaching as we design tasks and textbooks, paper and online, what can we do to help our students move toward the "expert" end of the reading mathematics for understanding scale.

*Discussion Questions*

1. The text chosen was one on a topic unfamiliar to the readers. There were no theorems in the portion read. Would the reading strategies be similar for more familiar topics?
2. What would one expect a mathematician to do when "stuck" on understanding some topic or example while reading? Can we test this?
3. If one of the reasons mathematicians read more effectively is because they have had positive reinforcement that they can learn from reading, how do we achieve similar positive reinforcement with lower level students?

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# Summing Up Students' Understandings of Sigma Notation
## (Preliminary Research Report)

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*Within the context of an advanced calculus instructional design teaching experiment, four students encountered interesting difficulties with sigma notation. This report tells the story of those students' progress; it describes the nature of the difficulties encountered and the ways these difficulties were resolved. Specifically, we wish to answer the questions:*

*1) How do post-calculus students talk about and use sigma notation?*
*2) How do they handle the transition from discrete to continuous cases in their use of sigma notation? In particular,*
   *a) What challenges do students encounter when transitioning from sums involving the terms of a sequence to sums involving approximate area under a function?*
   *b) What skills or tools do students use to meet these challenges?*

**Key words:** *Sigma notation, calculus, analysis, concept image/definition, mathematical discovery, semiotics*

## Literature Review

Mathematics education research that focuses specifically on students' understandings and interpretations of sigma notation is scarce. Studies (Alcock & Simpson, 2004, 2005; Ferrari, 2002) have attributed student difficulties with mathematical topics that rely on sigma notation to a lack of *semiotic control* (Ferrari, 2002)--students' ability to properly interpret and manipulate symbolic expressions involved in tackling a mathematical task. However, the nature of these difficulties, and students' conceptions of sigma notation in general, have not been well documented. Some researchers (Arcavi, 1994, 2005; Hiebert, 1988; Pimm, 1995) have focused on notation/symbol use as a whole. This corpus of work, as well as general work on the connection between symbols and concepts, informs our work on students' understanding of sigma notation.

Much of the work on symbols highlights the importance between a symbol and its referent (e.g. Arcavi, 1994, 2005; Hiebert, 1988; Pimm, 1995; Tall & Gray,1994). Tall and Gray in particular highlight the connection between processes, objects, and the symbols used to represent them. They refer to this triad as procepts. For example, sigma notation is used to represent both the process of adding together the terms of a specific sequence and the resultant sum. Moving between process, object, and symbol with relative ease is an important part of fluency. Arcavi (1994, 2005) coined the term 'symbol sense,' to describe such fluency with symbols and their referents. The term is used analogously to how the term 'number sense' is used in relation to numerical reasoning (see, Sowder, 1992 for a review). He describes several categories of reasoning which exemplify various forms of symbol sense as they relate to algebra. This includes flexible strategic choices of symbolic referents, being able to smoothly transition from algebraic symbols to their referents when it is prudent, and noticing higher-order structure within algebraic expressions. Arcavi is not intended to be a complete catalog of types of symbol sense but instead paints a picture of the types of reasoning that symbol sense can encompass.

In the development of fluency with concepts and the mathematical-symbols and notation practices that are used to work with them, many things can go astray. Tall and Vinner (1981) studied the disconnect between mathematical formulations of concepts and students' uses and interpretations of these concepts in action. They refer to these student notions as concept images. Concept images are often inconsistent, context dependent, and removed from formal mathematical notions.

While Arcavi's (1994, 2005) work highlights the things that can go right with students' understandings of mathematical notations, Tall and Vinner's work helps illuminate the varied nuances that occur when students' conceptions are misaligned with formal mathematical notions. Both of these bodies of work provide useful tools with which we can describe students' interactions with and understandings of sigma notation. In our presentation we will give examples of what symbol sense looks like in relation to sigma notation and we will present a picture of what students' concept images look like in relation to sums.

## Background

In the Spring of 2011, we began an Advanced Calculus teaching experiment. The purpose of this experiment was to investigate the efficacy of a new instructional sequence for Advanced Calculus (Real Analysis). This sequence was designed to provide the students with tasks that would leverage their knowledge of calculus to motivate further investigation into its theoretical underpinnings. The first sequence of tasks had the students investigate notions of area, with increasing formality and rigor, in order to motivate the study of sequential limits. Along the way our students demonstrated some of the challenges they faced in using sigma notation to talk about area. Specifically, while the students were able to use sigma notation to denote the sum of odd integers without difficulty, they were *unable* to use it to accurately to reflect a rectangle-approximation to the area under a curve, at least initially. These challenges led naturally to the following questions:

1) *How do post-calculus students talk about and use sigma notation?*
2) *How do they handle the transition from discrete to continuous cases in their use of sigma notation? In particular,*
    a) *What challenges do students encounter when transitioning from sums involving the terms of a sequence to sums involving approximate area under a function?*
    b) *What skills or tools do students use to meet these challenges?*

This report will add to the body of knowledge of how students think about and use math symbols in general and sigma notation in particular, with potential application to improved instruction in calculus, statistics, and analysis.

## Method

For this initial investigation four students who had completed an introductory calculus (up through Sequences & Series) with high marks were recruited to work in pairs on the prescribed sequence of tasks. Both pairs of students worked in sessions of 60-90 minutes, with the first group participating in fourteen sessions and the second group in nine. Two researchers ran the interview/experiment sessions, with each session being video and audio recorded. The video and audio data were reviewed after each session by the researchers to facilitate ongoing modifications to the instructional sequence and to plan the next session.

Analyzing the students as individuals, we will use a grounded theory approach (Strauss & Corbin, 1990) to explore and explain the challenges faced by students in attempting to use sigma

notation in the context of area under a function, in addition to what eventually helped them overcome those challenges.

**Preliminary Findings**

Below is an excerpt from the teaching experiment. Early in the third session of the first teaching experiment, Betty and Kathy had worked out the sum of the first ten odd integers using sigma notation (Figure 2). We then returned the context to area. After drawing an arbitrary function on an arbitrary interval, they successfully wrote out a long-hand approximation for the area with eight rectangles using the left-hand rule (See Figure 1). When asked to do the same with sigma notation, they encountered difficulty.

> Betty: So our starting point in this case is, umm i equals x sub 'o' and we're going to go to x-seven? Well no, we're not saying where we're ending, we're saying how many times we're doing it, so, does thi-
> Kathy: Wait, see? Tha- that's what confuses me, this number [*referencing the top of the sigma*] whether it's where we're ending or how many times we're doing it
> Betty: It's how many times
> Kathy: [emphasis] THERE [*points to board with first ten odd integers added together. See Figure 2*], it's how we're ending [emphasis] AND how many times we're doing it, that's where it screws me up. And I have no clue which one it is. I think it's-
> Betty: I think it's where we're ending.

Though there is not room to present further excerpts, these two students also experienced difficulty when dealing with the varying sizes of ‘the change in x’ and its relation to the sigma notation representation of the approximate area. This was surprising given that the students were able to use sigma notation to deal with non- integration related sums during other portions of the teaching experiment. These difficulties led to an important insight into the behavior of the index variable in sigma notation, namely that the rule to increment that index by 1 each time is not an explicit part of the notation.

**Results and Applications**

Sigma notation is a useful and widespread standard for describing finite and infinite sums. This research makes inroads into mapping common student understandings (and misunderstandings) related to its use. Of particular interest was the difficulty that students experienced (which they demonstrated multiple times in the first teaching experiment) in making the transition from using sigma notation in discrete situations to continuous ones. This research is a worthwhile endeavor for two reasons: as a field, it adds to our knowledge base on this topic, and has the potential to inform improved instruction.

**Discussion**

As this is a preliminary report, we hope to receive constructive feedback and suggestions for future research. In particular,

- What are some steps the research team should take in developing this theory?
- How can this theory be used to inform instruction?
- Are there other theoretical frameworks that might be useful in analyzing this data?

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**FIGURE 1 - Sigma Notation and Area**

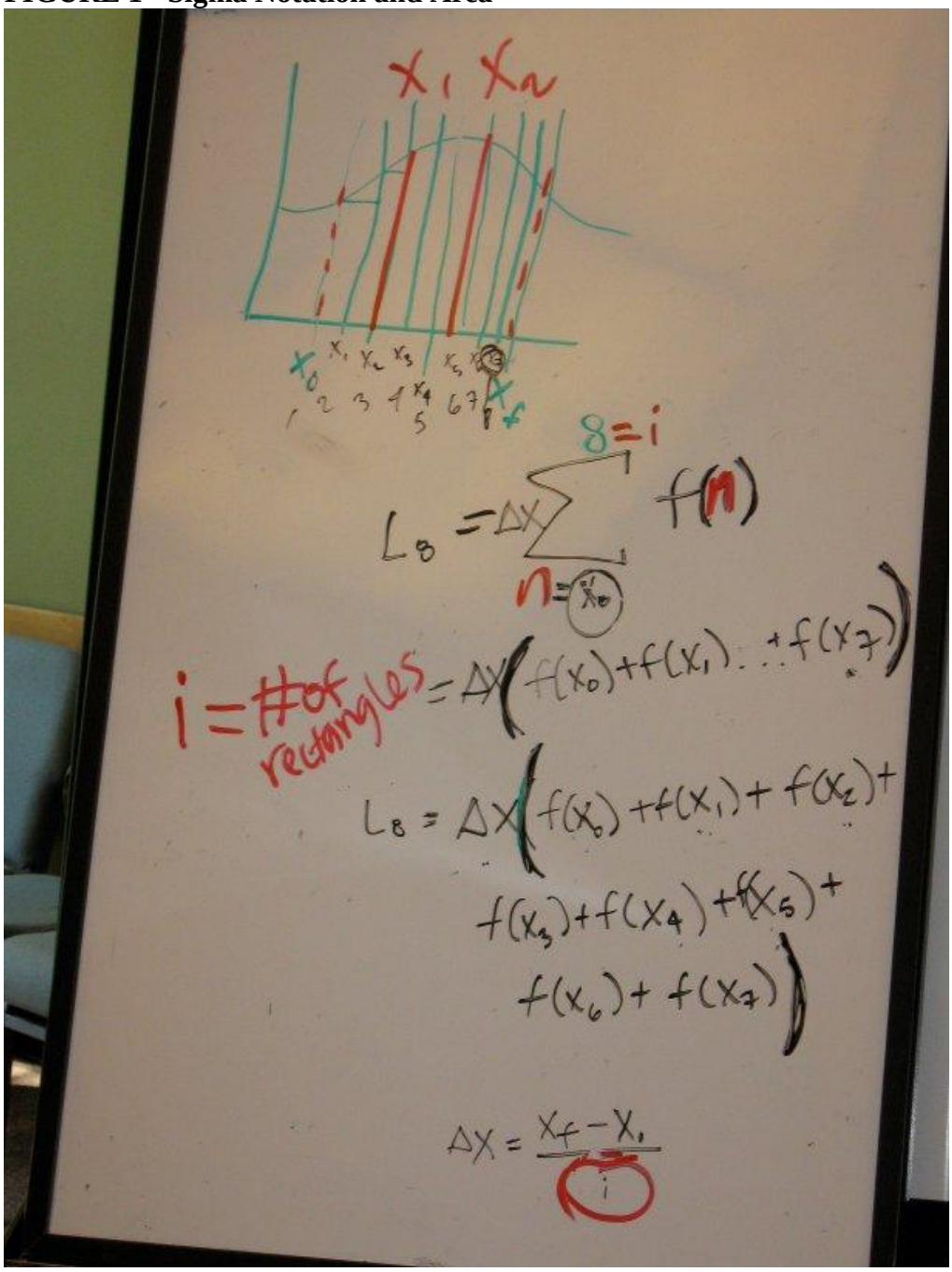


**FIGURE 2 - Sigma notation for a simple sum**

$$\sum_{i=1}^{10} 2i - 1$$

# Improving Undergraduates Novice Proof-Writing: Investigating the Use of Multiple Drafts

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**Abstract:**
Because proof-writing involves both understanding mathematical ideas related to the theorem, as well as structural norms of formal proofs, we hypothesized that students could improve both content and structure of their proofs using the drafting techniques common to English Composition research. Our research question is, "Does proof revision lead to improved proof-writing skills?" The intervention group revised their proofs and turned in up to three drafts of each formal proof. This pilot led to the development of a coding tool to categorize the types of student individual errors. In this proposal, we share the coding tool as well as the ongoing analysis of two sets of Linear Algebra student proofs. Preliminary results suggest the drafting group engaged with the work more often than the control group during the semester and on the final, while the control students were more likely to skip proofs rather than attempt them.

**Key Words:**
teaching experiment
proof-writing
transition to proof
undergraduate mathematics

# Improving Undergraduates Novice Proof-Writing: Investigating the Use of Multiple Drafts

## Introduction

Teaching undergraduates to write proofs involves much pulling of hair for both students and professors. We know quite a bit about the issues students typically face while proving such as whether a proof is convincing, understanding quantifiers, etc (e.g. Weber (2007); Dubinsky & Yiparaki (2000)). However, proof-writing techniques of instruction for the undergraduate student is under researched.

In contrast, the field of English Composition pedagogy has long considered the question of how to teach essay writing. One school of thought, offered by John Bean (2001), is that students learn to think critically by revising their own writing. Often, a student struggling with difficult ideas will make basic grammatical mistakes with surprising frequency, due to a sort of cognitive overload. But, though drafting—reorganizing the essay multiple times— the student develops a more sophisticated mastery of the content, and automatically corrects the grammar and spelling errors on their own. (Bean, (2001)). Instructors are advised to help the student clarify her ideas, but not to focus on the mechanics of the paper.

Inspired by colleagues in English Composition, we wondered whether this drafting technique could be imported for teaching proof-writing. Specifically, we elected to explore whether having students submit the same proof multiple times (drafting) would help them better learn to write proofs. Our guiding research question was: "Does revising proofs lead to improved proof-writing skills for undergraduates in introductory proof-writing settings?"

We hypothesized that proof-writing is analogous to essay writing. That is, when students are struggling with difficult mathematical content, their communication of their ideas often becomes unintelligible. Since drafting helps English students develop their ideas, and the writing mechanics automatically improve alongside, perhaps proof drafting would help math students develop their understanding of content, and the mechanics of writing a clear proof might naturally emerge.

## Methodology

During the fall and spring of 2009 we conducted a pilot study with Linear Algebra students: a control and drafting group both taught by the same professor under otherwise similar conditions. The control group was assigned approximately 15 proofs but none were formally revised. The spring section included 10 proofs (a subset of the control's) and students were allowed to resubmit each proof up to 3 times total with instructor comments between submissions. Both sections' (closed-book) finals included the same proof questions, none of which the students had encountered before. This allowed us to determine whether the drafting group could craft better proofs "spontaneously". Had they learned to write better proofs overall or were any improvements limited within the drafts themselves? The setting was a small liberal arts college in central Texas where Linear Algebra serves as the introductory proof-writing course.

We required a coding method that could adapt to several different contexts: allowing us to compare the end of semester output of one class of students with a different class of students, to analyze individual student growth, and to determine if the types of errors differed between the two Linear Algebra sections. Such a coding scheme would provide a more fine-grained assessment of errors, rather than only measure the relative strength of an attempt.

We initially consulted coding schemes published by Selden & Selden (2003) and Andrew (2009), and began to classify our proofs. At times neither coding scheme seemed to fit our need. For example, Selden & Selden's classification was drawn from an Abstract Algebra course and at times the error categories were overly specific to that context. Andrew's work was well-suited to a classroom grading context but not to a research comparison project. We attempted to create a coding scheme that would both fit our Linear Algebra context, but possibly apply to a broader range of courses as well.

Our basic process was to jointly categorize each error according to the codes from Selden & Selden's or Andrew's work, and to note errors that seemed to not fit anywhere. We then looked for clusters of outliers, or ways in which existing categories might be modified to include these. Then we reviewed whether any originally categorizations might fit better into the newer categories. This was a heavily iterative process.

Once we felt we had created a system that was neither too narrowly specific nor too broad we gathered proofs from other courses (a Modern Geometry and a Number Theory course) and individually coded them. We then came together and compared our individual assessments. At this stage we did not find any errors requiring new categories, but we did refine our descriptions of the errors types to create better reliability between us. Our final coding scheme is attached but space restrictions prevent us from also including the more narrative guide to the codes themselves.

What emerged was a coding matrix. Rows are assigned to types of proof errors, such as "Misusing Theorem" and "False Implication". Columns are assigned to a possible attribution of the source of the error: If evidence suggests that the student understands the key ideas, but is incorrectly communicating their ideas, the error would be coded as a Rhetorical error. If it appears that the student misunderstands the content of the statement(s), definitions, or related math, then the error would be coded as a Content error. If the student seems not to understand logical implications, or has grave misunderstandings of what makes a proof "prove", then the error would be coded as a Fundamental Error.

**Preliminary Results**

Unfortunately, the classes had very few students in any one semester (13 control and 8 drafting) making statistical comparisons difficult. Therefore we continued the research during the 2010-2011 academic year with two more Linear Algebra courses using the same protocols. Results provided in this proposal are from 2009-2010, although by the conference, we will have results for both years.

The most basic result was that in the drafting course more proofs ultimately were fully correct than in the control group. This is in some ways obvious as the drafting group had three attempts for each proof. While the drafting group engaged with fewer overall proofs, they actually turned in their assignments far more often than in the control group. That is, the students in the control group often just skipped the assignment, while the students in the drafting group turned in each assignment at least once, and usually three times. Therefore engagement with proof writing was increased for the drafting group. In the final exams, we did not measure an overall difference in the nature of the errors, in part because of the small sample size. However, there is some suggestion that the drafting group were more likely to attempt the proofs on their final, mirroring what we saw throughout the semester. The control group skipped proofs at four times the rate of the drafting group on their final. Overall, the drafting group did better because they did more, but when the skipped proofs are controlled for, the results are more similar

between the groups. Unrelated to the comparison, it was discovered that students made content errors that derailed proofs far more often than fundamental errors. On the final, for example the ratio of content errors to fundamental was 31:1 (control) and 24:1 (drafting).

**Discussion**

At this stage in the work, it is unclear that either group performed better than the other in any of the categories with the exception of actually doing the work, where the drafting group outperformed the control. From a teaching perspective, the grading burden was roughly the same between the two groups. Drafting appears to lead to higher engagement, but higher engagement has not been shown here to produce better results. The drafting may then improve something else - possibly perseverance, self-confidence, respect/understanding of the general process - but these qualities were not measured in the study and remain an important area to further investigate. Course evaluations did suggest that students enjoyed the drafting approach; it would be a nice result if students found pleasure in proof-writing. Finally, in both groups, content errors were far more frequent than structural errors on the final exam. This suggests that students' weak knowledge of definitions and theorems is an underlying cause of inadequate proofs.

**Questions for the Audience**

1) We are currently in the process of conducting more formal inter-rater reliability information and seek the RUME audience's opinion and advice related to our tool.
2) We seek suggestions for the describing of results that would be most helpful to both other researchers and teachers.
3) We seek suggestions for other issues related to proof-writing where we might use our tool for research.

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| Error Type | | Fundamental | Content | Rhetorical | Ambiguous |
|---|---|---|---|---|---|
| Basic Problems - Acategorical | Wrong Problem | | | | |
| | Wrong Method | | | | |
| | Scratch Work | | | | |
| | Other | | | | |
| Omissions | Error-caused Omission | An error caused large parts of the proof to be omitted. Must be used in conjunction with another code specifying the triggering error. | | | |
| | Assertions | Entire result is asserted | Local assertion | Context requires more details | |
| | Omitted Sections | Did not address one or more sections of the proof. (Trailed off halfway, skipped cases, etc.) | | | |
| | Local Omission | Omitted a single isolated step | | | |
| Importing known material | Misusing Theorem | Ex - applying the converse of a theorem | Misunderstanding known results or theorems | Misphrasing a known theorem, while using it correctly | |
| | Vocab & Grammar | | Misuse of definition or vocabulary | Prose needs editing or student fails to follow mathematical conventions | |
| Flow of proof | Logical Order | Didn't proceed through the proof in a linear fashion, or ideas were not in logical order | | | |
| | Circular Argument | Uses conclusion in the course of the proof | | | |
| | False Implication | A does not imply B | | | |
| | Extraneous Detail | Attempted or proved irrelevant results, in addition to stated problem | Extraneous details or steps that did not really contribute to the proof. | | |
| Miscellaneous | Locally Unintelligible | A single statement within the proof is completely unintelligible | | | |
| | Proof by Example | | | | |
| | Notation | Notation is awkward/confusing/misleading | | | |
| | Weakened Result | Proving a weaker result (eg - a result that only applies to a subset of the scope of original statement) | | | |
| | False Statement | | Made a false statement or incorrect computation | Apparent typo | |

# Definite Integral: A Starting Point for Reinventing Multiple Advanced Calculus Concepts

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*While recent Realistic Mathematics Education (RME) studies have shed light on students' abilities to formalize limit conceptions, it remains to be seen how students might make similar progress with other fundamental advanced calculus concepts like continuity, derivative, and integral at a depth required for success in upper-division courses. To address this gap in the literature, we conducted a fourteen session teaching experiment geared at students' reinvention of the formal definition of definite integral. Our presentation will address the following research questions: 1) Do students' efforts to formalize the concept of definite integral motivate a need (for them) to formalize the notion of convergence/limit?; and, 2) Once students reinvent a formal definition of definite integral, can they use this formalization as a tool for formalizing other advanced calculus concepts? If so, which concepts?*

**Keywords:** Advanced Calculus, RME, Guided Reinvention, Definite Integral, Limit

## Introduction and Motivation

The limit concept is one of the most fundamental ideas in advanced calculus, serving as a conceptual foundation for derivative, integral, and continuity, among other mathematical notions. Until recently, the vast majority of research on students' understanding of limit focused on informal misconceptions (e.g., Davis & Vinner, 1986; Monaghan, 1991; Tall & Vinner, 1981; Williams, 1991) possessed by introductory calculus students, and little was known about what challenges students face in reasoning about limits more formally. In the past few years, however, Realistic Mathematics Education (RME) studies (Oehrtman, Swinyard, Martin, Hart-Weber, and Roh, 2011; Swinyard, 2011) have employed the heuristic of guided reinvention to provide insights into students' reasoning about limit in the context of reinventing formal definitions. Swinyard looked at limits of a function and found that students with robust concept images can construct formal concept definitions of limit at infinity and limit at a point. Similarly, students in a teaching experiment conducted by Oehrtman et al. constructed formal concept definitions of sequence convergence, series convergence, and Taylor series convergence. In both teaching experiments, the students' success in constructing precise limit definitions appears to have been supported by their: 1) ability to shift their reasoning from an $x$-first perspective to a $y$-first perspective; and, 2) use of an arbitrary closeness perspective to operationalize what it means to be infinitely close to a point (Swinyard & Larsen, 2011).

In both of the aforementioned teaching experiments (Oehrtman, Swinyard, Martin, Hart-Weber, and Roh, 2011; Swinyard, 2011), reinvention was supported by first having the pairs of students generate examples and non-examples of limits. For instance, the two students in Swinyard's study were given the following prompt: "Please generate as many distinct examples of how a function could have a limit of 2 at $x$=5." Student-generated examples and non-examples of limits subsequently served as tools for motivating definition refinement. Additionally, in both studies the pairs of students used their formal concept definition of one idea as a template for generating a formal concept definition for a more sophisticated idea. For example, in Swinyard's study, the students used the formal definition of limit at infinity as a template for constructing a formal definition of limit at a point, and similar findings emerged in Oehrtman et al.'s work.

The studies discussed above suggest that guided reinvention may be a productive means by which to: 1) gain insight into students' reasoning about foundational Calculus concepts; and, 2) support students in transitioning to more advanced mathematical thinking. While these studies have shed light on students' abilities to formalize limit conceptions, it remains to be seen how students might make similar progress with other fundamental concepts like continuity, derivative, and integral at a depth required for success in upper-division courses. Further, the students in the studies conducted by Swinyard and Oehrtman et al. reinvented formal definitions of limit/convergence in an attempt to precisely characterize the functional behavior of the examples and non-examples they had constructed. In other words, the students' reinvention of these formal definitions was not motivated by a mathematical need in a separate context. Though the students' reinvention of limit was ultimately successful in both studies, the following question arose: Is there a mathematical context conducive to motivating a need (for the students) to formalize the notion of convergence/limit? To answer this, we conducted a study geared at students' reinvention of the formal definition of definite integral. Our work aims to address the following research questions:

1) Do students' efforts to formalize the concept of definite integral motivate a need (for them) to formalize the notion of convergence/limit?
2) Once students reinvent a formal definition of definite integral, can they use this formalization as a tool for formalizing other advanced calculus concepts? If so, which concepts?

## Methods

Because the intent of our research was to learn how students can leverage their informal notions of Riemann approximation and bounded area to formalize their understanding of definite integral, we adopted a *developmental research* design. Gravemeijer (1998) describes the goal of developmental research as follows: "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Instead of presenting students with a formal definition of definite integral and asking them to interpret the definition based on their informal understanding, we utilized the *guided reinvention* heuristic, using the students' informal knowledge as a starting point for constructing a formal definition. Thus, the students' reinvention of definite integral more closely resembled the historical development of the idea, avoiding what Freudenthal (1973) critically referred to as the anti-didactical inversion in which the end results of mathematicians' efforts (formal definitions) are taken as the starting points for students' learning.

The teaching experiment consisted of fourteen 60-90 minute sessions, occurring roughly once a week. The study was conducted with two students (Betty and Kathy) at a large, public university in the Pacific Northwest. Both students had completed an introductory calculus sequence, earning high marks in each course, and demonstrating strong conceptual understanding on written assignments and exams. We purposely chose to work with students possessing robust concept images because we felt doing so increased the likelihood of us gaining insight into how students reason about definite integral (and other Calculus concepts) as they formalize their intuitive understandings.

Ongoing analysis between sessions consisted of analyzing video, creating content logs designed to describe the students' mathematical activity, and making initial conjectures about

students' reasoning. Analysis also included weekly research team meetings, during which time an outline of the following week's session was constructed based on analysis of the students' reasoning to date. In the coming months, we will conduct a retrospective analysis of the data corpus, with the intent of better understanding the students' reasoning related to all of the mathematical concepts that arose: area, definite integral, sigma notation, sequence convergence, derivative, limit at a point, limit at infinity (for a function), continuity, and the Fundamental Theorem of Calculus (FTC). Specifically, our goal will be to identify the challenges the students encountered in their reinvention of these concepts, as well as what supported the students in overcoming said challenges.

## Initial Findings

We found the concept of definite integral to be a promising starting point for exploring and formalizing advanced calculus concepts. For instance, as the students attempted to characterize precisely what it means for a definite integral to exist on a closed interval [*a*, *b*], they recognized a need to clarify their language. Specifically, they were struggling to operationalize what might be meant by Riemann approximations getting "closer and closer" to an actual sum. This spurred an exploration of what sequence convergence could mean. Unlike previous research (Oehrtman, Swinyard, Martin, Hart-Weber, and Roh, 2011; Swinyard, 2011), the reinvention of sequence convergence in this setting was purposeful from the students' perspective – it served to help them better understand what is involved for a definite integral to exist for a function *f* on a stated interval [*a*, *b*]. Despite not having been introduced to a formal definition of sequence convergence prior to the teaching experiment, Betty and Kathy were able to construct the following definition: A sequence converges to X if for any distance ε from X there exists a K such that for all k>K, $a_k$ satisfies $|X-a_k|< \varepsilon$. They then used that definition to help them formalize the notion of definite integral. After doing so, they wondered aloud why evaluating an antiderivative of a function *f* at the endpoints of an interval [*a*, *b*] results in the exact same area one would get by taking the limit of approximating rectangles under *f* on that same interval. This curiosity led Betty and Kathy to explore why the FTC works. Although their exploration was never fully resolved (the FTC was not "reinvented" from a mathematician's perspective), it did lead the students to also wonder what it means for a function to have a derivative. A subsequent exploration included the students formalizing what it means for a derivative to exist at a specified point, using their experience of approximation in the integral context as a model. In our talk, we will provide an overview of what concepts Betty and Kathy reinvented, as well as some conjectures for why definite integral served as such a fruitful starting point.

## Questions for Audience

With the intent of furthering our research, we provide the following questions for consideration from the audience

1) What advantages/disadvantages can you think of for beginning an advanced calculus inquiry based curriculum with definite integral?
2) Do the audience members have insights from their experiences teaching and learning advanced calculus that might suggest other points of contact/starting points for motivating students toward thinking deeply about definite integral?

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# STUDENT THINKING OF FUNCTION COMPOSITION AND ITS IMPACT ON THEIR ABILITY TO SET UP THE DIFFERENCE QUOTIENTS OF THE DERIVATIVE

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**Abstract:** *While the limit in the definitions of the derivative is troubling to many students, a difficulty that preceded this confusion was observed: students were not able to correctly set up the difference quotients as required in the definitions. The purpose of this study is to investigate the cognitive processes involved in setting up the difference quotients and the associated errors. This explanatory case study seeks to explain why these particular errors occur from the perspective of student thinking of function composition. At the end of the study, a framework that aggregates criteria used (by past studies and this study) to assign student membership into a function conception category will be produced in an attempt to move towards a systematic classification of students' cognitive processes. Implications from this study can inform teaching practices by exposing students to expected errors. As observed from the data, this can lead to rich discussions on the concept of function itself.*

**Keywords:** case study, function composition, difference quotient, APOS Theory, Pre-Calculus

## Introduction

The purpose of this study is to investigate the cognitive processes involved in setting up the limit definitions of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \text{ and}$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Calculus students often have difficulty finding the derivative of a function using the limit definitions (Zandieh, 2000). While the concept of limit is troubling to many students, I observed a difficulty that preceded this confusion: students were not able to correctly set up the difference quotients as required in the definitions. Success with computing the derivative requires correctly completing this crucial initial step.

In an exploratory study to document the prevalence of student difficulty with setting up the difference quotients, I examined student work on quizzes from two large calculus courses. Students were generally quizzed on content a week following its introduction. For each of the limit definitions above, students were asked to "Set up the difference quotient[s] for the following functions: $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sin x$."

Regarding the first limit definition, of the 101 quizzes examined, 17 students failed to correctly set up the difference quotient for the quadratic function, 19 students for the rational function, and 22 students for the trigonometric function. In every single case, this was due to failure to compute or correctly compute $f(x+h)$.

For the second limit definition, of the 107 quizzes examined, 58 students – 54% – failed to correctly set up the difference quotient for the quadratic function, 60 students – 56% – for the

rational function, and 58 students – 54% – for the trigonometric function. The most frequent error made was failure to compute or correctly compute $f(a)$. Astonishingly, students also had quite a bit of difficulty recognizing what to place in the $f(x)$ portion of the difference quotient. These are alarming observations that warrant further investigation.

Even though setting up of the two difference quotients seems to be at different levels of mathematical difficulty, the errors students made in the respective first terms of the numerators, $f(x+h)$ and $f(x)$, were very similar in nature. The aim of this study is to investigate the errors made in the difference quotients and to connect them with the cognitive processes of students when setting up the difference quotients.

**Literature Review**

Functions are foundational building blocks of mathematics. The importance of functions is well known within the mathematics community and documented and acknowledged among mathematics education researchers (Dubinsky, 1991; Dubinsky & Harel, 1992; Tall, 1991; Thompson, 1994). The concept has been widely explored both theoretically (Eisenberg, 1991; Harel & Kaput, 1991) and empirically (Ayers, Davis, Dubinsky, & Lewin, 1988; Carlson, Oehrtman, & Engelke, 2010; Cottrill, 1999; Sfard, 1992). Although more recent research on undergraduate mathematics education has moved away from functions and towards upper level topics such as Linear Algebra and Mathematical Proof, student difficulty with function has not disappeared; as indicated in the title of the Gooya & Javadi (2011) paper, "Unversity Students' Understanding of Function is Still a Problem!"

We can see from the results of the exploratory study that indeed, functions still pose a problem for students. Similar to computation of $f(x+h)$, Carlson (1998) briefly documented difficulty with computing $f(x+a)$ for a quadratic expression as part of a larger cross-sectional study. While the most common error that surfaced in Carlson's study was adding $a$ to the expression, other more common errors were found in the exploratory part of my study. Research from my study seeks to further Carlson's study by describing the additional errors and explaining why these errors occur with respect to student cognitive processes while setting up the difference quotients.

Unlike Carlson (1998), this study looks at computing $f(x+h)$ from a function composition perspective; it explores the possibility of using the concept of function composition as a learning tool to evaluate $f(x+h)$. The few studies on composition focus on the topic itself, its relation to chain rule, or its place in secondary and post-secondary curricula (Ayers et al., 1988; Cottrill, 1999; Horvath, 2011). At the time of this study, no studies view composition in relation to evaluating $f(x+h)$, even though it is recognized as a composition (Horvath, 2011).

**Comparative Framework**

In Carlson (1998) and in other studies (Ayers et al., 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Zandieh, 2000) that investigate student cognitive processes, researchers categorize students as having action, process, or object conceptions of function by using the operational definitions of the conceptions. During data analysis, raw student interview data are fit to these definitions at the discretion of the researchers. As a result, student function conception classification can vary from study to study. Recent studies (Carlson et al., 2010; Oehrtman, Carlson, & Thompson, 2008) attempt to mitigate this problem by breaking the conceptions into specific function topics such as domain, inverses, etc., and providing operational definitions for them. Some

definitions are so specific that they detail raw student interview observations.

In an attempt to move towards a systematic classification of students' cognitive processes, this study seeks to produce a framework to aid researchers in assigning student membership into a function conception category. This data analysis tool will be created from criteria used by this study and by past studies.

**Research Methodology**

This explanatory case study (Yin, 2009) seeks to explain why these particular errors occur. There are many trajectories leading to student error: students' interactions with past texts, teachers, curricula and other learning resources, and their thinking as a result of the complexities of their mathematical background. This study looks at reasons for error from the perspective of student thinking.

Five cases from the previously identified errors were chosen to be studied in-depth. To learn about the complexities of these errors, ten student sources were interviewed. These ten student volunteers came from the pool of participants from the two large lectures of the exploratory study. For each error, the students were split into two groups: those whose answers were in the error category and those whose answers were not. Triangulating data from these two groups is necessary to create a robust study.

Interviews were used as a research tool in this case study. The purpose of the interviews was to understand student cognitive processes when discussing functions and function composition in relation to setting up the difference quotients. Over the course of two interviews, students were asked performance questions, unexpected "why" questions, and reflection questions (Zazkis & Hazzan, 1999). They were also asked to complete "give an example" tasks and construction tasks (Zazkis & Hazzan, 1999). For example, students were asked to construct a $g(x)$ such that $f(x+h) = f(g(x))$. Some of the questions were developed in an interview guide beforehand, while others came up during the interview as a result of individual student response. These were generally the unexpected why questions or questions posing counter-examples to student claims.

Students were also shown samples of other students' work and asked to comment on the procedures the students took in the samples and why s/he believed those students performed those procedures. After the culmination of the interviews, data analysis began. Analysis of this data is still underway.

*Data Analysis Plan*

Analysis of student interviews will be focused on student thinking of functions and function composition as revealed by interviews and the errors when setting up the difference quotients. I will analyze the instances of errors made by the interviewees, answers to pre-developed interview questions, answers to particular topics of function and function composition, and interviewee descriptions and explanations of student sample work. One error will be chosen at a time and data from the two aforementioned groups of students will be analyzed for trends. Data will be analyzed in this way to make claims about why these errors are occurring.

The framework for the data analysis tool will be developed iteratively. That is, data from the interviews will be used to inform the framework and in turn the framework will help classify students into a conception category. The conception categories will be correlated with instances of error within the interviewees, interviewee descriptions and explanations of student sample work,

and interviewee conceptions of function composition.

Preliminary findings on the effectiveness of using function composition to evaluate $f(x+h)$ shows the concept can be a valuable learning tool to help students transition out of an action conception into a process conception; that is, they no longer view evaluation of $f(x+h)$ as an action of replacing or substituting $x$ for $x+h$. For instance, the student who took the longest (the times ranged from 6 seconds to 9.5 minutes) to complete the construction task described above said, "[Before] I just said you just replaced [$x$ with $x+h$]...Because I didn't know–or you just substitute in. It's actually a composition. I get it." In addition, findings from this study on errors can be used to inform teaching practices, curriculum development, or further research. Presenting errors found in this study to students learning composition can be used as a teaching method or part of the curriculum to open discussion with students. As observed during the interviews, this can lead to rich discussion on the concept of function itself.

**Questions for the Audience:**

1. What other implications can you see from this work?
2. What other criteria for function conception classification have you seen or used?

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# Opportunities to Develop Understanding of Calculus: A Framework for Analyzing Homework Exercises

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Abstract

The purpose of this study was to construct and apply a framework to examine opportunities to understand calculus deeply, as informed by prior research. I applied this framework to analyze opportunities to learn derivatives in two calculus texts: Hughes-Hallett et al. (2009) and Stewart (2012). These tests were chosen to represent different points on a continuum between conventional and reform calculus materials. An analysis of both texts suggests that they are more similar than might be expected with respect to the amount of context given in problems, their attention to position, velocity and acceleration, and opportunities to use multiple representations – algebraic, numeric, graphical, and descriptive. There were differences between graphical and descriptive problems between texts and opportunities to make connections between representations. The framework presented here illuminated degrees of variation and similarity between opportunities to understand calculus in these texts and could have further utility for examining additional calculus texts.

Keywords: Calculus, Textbook analysis, Multiple representations, Derivatives

## Opportunities to Develop Understanding of Calculus: A Framework for Analyzing Homework Exercises

What does it mean to learn and understand calculus, and how are these opportunities to learn represented in mathematical tasks in college curriculum materials? In this study, I have constructed a framework that integrates various perspectives on calculus learning, as informed by research literature. These perspectives were chosen because they represent what a range of scholars value about calculus understanding and opportunities to understand calculus deeply through multiple representations (Sofronas et al, 2011; Knuth, 2000; Aspinwall and Miller, 2001; Porter and Masingila, 2000; Lithner, 2004; Cunningham, 2005; Roth and Bowen 2001; Bossé, 2010). The primary purpose of this study is to examine the degree to which curriculum materials support opportunities to learn calculus in these ways.

Why study textbooks? Outside of classroom instruction, students spend significant amounts of their time interacting with their text and solving problems and exercises in a typical college course. According to Rezat (2007), "The mathematics textbook is one of the most important resources for teaching and learning mathematics" (p. 1260). Additionally, textbook analysis is an area where research is lacking (Love and Pimm, 1996). In sum, this area is important because in mathematics learning in higher education, texts may play a large role.

Decisions about the content of mathematics curriculum, or what students should learn, must ultimately rest upon value judgments (Hiebert, 1999). Different value systems are likely to be represented in the design of particular curriculum materials. Comparing textbooks is one way to examine the opportunities that students may have to engage in various ways with calculus, but deciding which text is *better* from the analysis depends on the type of learning one would like to foster in calculus students. Using the framework I developed to examine opportunities to learn calculus, I compared opportunities to learn derivatives, a central idea in calculus (Sofronas et al, 2011), in these two sets of curriculum materials. Such an analysis could reveal whether the values of textbook authors differ, in what ways, and whether there is common ground. To this end, I investigated the following question: How do Stewart (2012) and Hughes-Hallett et al. (2009) calculus texts compare in the opportunities that students have to engage with mathematics in the homework exercises?

### Conceptual Framework

The framework for this analysis was developed around components that could help students develop a robust understanding of calculus in which students see connections between calculus topics and connections between mathematics and the world outside of their classrooms (See Table1). Previous work by Sofranos et al. (2011) indicated that, with the emergence of many different 'types' of calculus classes for different audiences it may be important to agree on common elements of calculus that *are* vital. Three components identified in Sofranos et al. and other research include the importance of context, the importance of attending to the relationship between position, velocity, and acceleration, and the importance of including derivatives that are not the "typical" rates of change in regard to $x$, $t$, or $\theta$. These components, such as context (Boaler, 1993), may be important for developing connections between classroom mathematics and real-world problems and the ability to use mathematics flexibly. These components may help to foster breadth and flexibility of students' knowledge of calculus.

Drawing connections between and deeply understanding the meaning of different representations is also a worthy goal for calculus students. For this reason, texts ought to give students opportunities in homework exercises to deal with different mathematical representations, and both texts analyzed in this study explicitly state in the preface that

representing mathematics in multiple representations is one of their goals (Hughes-Hallett et al, 2009; Stewart, 2011). The framework divides representations into four types: 1) Algebraic representations consist of algebraic equations, algebraic proofs, and most symbolic notation. 2) Numeric representations consist of tables and numeric approximations. 3) Graphical representations consist of graphs on the Cartesian or polar coordinate plane. 4) Descriptive representations consist of describing mathematics in 'plain language,' in one's own words, or integrating a real-world context into the problem.

This framework analyzes the representations that are present in the problems given to students, and the representations that are asked for in the expected student solutions. Both of these elements have merit. Research has shown that students do not typically form effective connections between multiple representations unless they have experience solving problems that ask them to transfer knowledge from one representation to another (Cunningham, 2005). Additionally, calculus has traditionally focused on algebraic representation (Hughes-Hallett et al., 2006), but there are reasons to believe that other representations are important. In real-world mathematics, calculus problems will not always take this form, and other studies have shown the benefit of writing in calculus classes (Aspingwall and Miller, 2001) and in mathematics more generally (Bossé, 2010; Porter, 2000). Additionally, research has shown the benefit of understanding graphs in technical occupations (Roth and Bowen, 2001; Knuth, 2000). For these reasons, examiningthe degree to whichdifferent texts use and ask for representations is worthwhile.

## Methods

The two particular texts analyzed in this study were Hughes-Hallett et al.'s *Calculus: Single Variable, 5th Edition* (2009) and Stewart's *Single Variable Calculus, 7th Edition* (2012). The most recent editions of these texts were chosen to represent different standpoints on a continuum between conventional materials and the reform calculus materials which arose from the Tulane Conference of 1986 and the Harvard Calculus Consortium. Calculus instruction has undergone changes and criticisms in the last 25 years, in response generating interest in developing new reform textbooks (like Hughes-Hallett et al.) and sparking changes in existing texts (such as Stewart). The criticisms varied: not enough students were involved in higher mathematics, technology was not being implemented in ways that maximized its potential benefits, and procedures trumped problem solving and modeling. The biggest concern, however, was in developing a conceptual understanding of calculus which would allow students to use what they had learned in class in ways in unfamiliar territory (Hughes-Hallett, 2006).

The inclusion of Stewart's textbook in this analysis is justified because of the popularity of Stewart's texts. These textbooks are in many ways canonical across university introductory calculus classrooms. In 2009, Stewart's textbook outside all other curriculum combined in the North American market of calculus texts (Peterson, 2009). Its widespread use and popularity make it a strong sign-post for comparison to other works. Though Stewart's text may be labeled 'conventional' because of its widespread popularity before the calculus reform, Stewart has been influenced by reform thought, and he describes in the beginning of his textbook ways in which the reform movement has influenced the direction of new editions of his work (Stewart, 2012). In this way, Stewart may be more of a hybrid between conventional and reform texts than firmly instanced in either trend. The framework presented here helps to illuminate potential similarities and differencesthat exist between the two texts, as well as point out areas where both texts have relatively similar opportunities for students to learn through homework exercises.

Because of the importance given to derivatives in first-year calculus, I chose to analyze the homework problems in chapters of Hughes-Hallett et al. and Stewart that introduced this topic. This corresponds to three chapters in Hughes-Hallett et al. and two chapters in Stewart. In these chapters, I analyzed 1111 problems in Hughes-Hallett et al. and 1072 problems in Stewart for each of the components listed in the conceptual framework. In order to check for the consistency coding, I randomly selected 99 problems from Hughes-Hallett and Stewart texts. These problems were re-coded by one instructor and one graduate student from a mathematics education program. The reliability for all codes exceeded 85%, ranging from 86.9% to 97.0%.

## Results

The most surprising finding from the study was that Hughes-Hallett et al. and Stewart have very similar distributions along many of the properties for comparison (See Table 2). Both texts have a similar percentage of problems with context and a similar distribution of representations across problems. Both texts are unlikely to give numeric data in a problem (3.2% for Hughes-Hallett et al. and 1.6% for Stewart). Both texts attend to position, velocity and acceleration around 5% of the time, and both texts have similar distributions of representations in the expected student solutions with the exception of graphical and descriptive representations. The clearest difference between tests happened among these representations. Whereas Hughes-Hallett et al. expected students to describe or explain their solution 28.8% of the time and to form a graph for 15.3% of problems, Stewart expected students to describe only 17.8% of the time and to construct a graph for 26.5% of problems. Another difference between the two texts was the percentage of problems which called for students to convert information from one representation to another, such as from algebraic to graphical, etc. The Stewart text was more likely than the Hughes-Hallett et al. text to ask students to make this kind of transfer (46.9% to 36.1% of problems were transfer tasks in each text). Although instructors have influence students' opportunities to engage with these components in the framework, the analysis helped to showcase the degree to which the textbooks provide instructors with opportunities to engage students that would not have been apparent without a systematic study.

Applying the conceptual framework to an analysis of these two textbooks provided the following insights. I found the textbooks to be more similar than I had anticipated, given that Hughes-Hallett was formed out of the reform movement and editions of Stewart existed before the movement began. The components of this framework more generally can help reveal a better picture of the ways in which calculus texts give opportunities for students to engage in homework problems. These components are based on recommendations from experts and from research, and while they are not an exhaustive list of the important aspects of calculus texts, this framework provides an illuminating method of examining mathematics problems; this analysis suggests that the framework can be applied with high reliability. This framework could be used to track some of the changes that have occurred within calculus textbooks since the call for reform in the 1980s and to notice general trends or exceptional texts if applied to a range of textbooks representing periods in time. The framework also has practical implications, in that it could be used by members of mathematics departments to compare and contrast textbooks when school districts or universities are selecting textbooks.

**Questions:** 1) During this analysis, I became concerned about the authenticity of contexts given in problems. How could I analyze whether contexts given are relevant for making sense of the mathematics in the problem or whether they are "psuedocontexts" (Boaler, 1993)? 2) How could I extend this study to examine ways that students use their textbooks, i.e. the degree to which students take up these opportunities to learn?

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**Table 1**:

*Components of the Framework*

| Presence of Components that Support Connections Outside of Mathematics | | Presence of Components that Support Connections Within Mathematics | |
|---|---|---|---|
| Context | Whether the problem contains a real-world scenario | Representations in the problem | Whether the problem statement includes *Algebraic*, *Numeric*, *Graphical*, or *Descriptive* representations |
| Rates of change concerning position | Degree to which texts address the rate-of-change between an object's position, velocity, and acceleration | Representations in the expected student solution | Whether the expected student solution calls for *Algebraic*, *Numeric*, *Graphical*, or *Descriptive* representations |
| Uncommon independent variables | Whether the problem contains an independent variable that is not $x$, $t$, or $\theta$ | Transfer between representations | Whether the expected student solution asks for students to convert information into a different representation than is present in the problem |

**Table 2:**

*Number (and Percent) of Problems with the Given Components*

| | **Hughes-Hallett et al.** (N=1111) | **Stewart** (N=1072) |
|---|---|---|
| Context | 240 (21.6%) | 211 (19.7%) |
| Rate of change concerning position | 27 (2.4%) | 65 (6.1%) |
| Uncommon independent variables | 220 (19.8%) | 98 (9.1%) |
| Representations in the problem | | |
| Algebraic | 909 (81.8%) | 856 (81.0%) |
| Numeric | 35 (3.2%) | 17 (1.6%) |
| Graphical | 177 (15.9%) | 64 (6.1%) |
| Descriptive | 329 (29.6%) | 266 (25.2%) |
| Representations in the expected student solution | | |
| Algebraic | 807 (72.6%) | 816 (77.2%) |
| Numeric | 62 (5.6%) | 73 (6.9%) |
| Graphical | 170 (15.3%) | 280 (26.5%) |
| Descriptive | 320 (4.8%) | 188 (17.8%) |
| Transferring between representations | 357 (32.1%) | 496 (46.9%) |

**For Educational Color Work: Diagrams in Geometry Proofs**
Preliminary Research Report

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***Abstract.*** Historically grounded in Oliver Byrne's reworking of Euclid's Elements, and based on a student-generated proof, we investigate the use of coloring to enhance geometry proofs. Charlotte Knight, an undergraduate mathematics major enrolled in Modern Geometry, regularly employed coloring techniques as a tool in her proof-writing. We met for a single semi-structured, task-based interview to discuss Charlotte's use of coloring in her organization and understanding of geometry proofs. Preliminary results indicate that Charlotte's use of diagrams is closely related to her construction of a proof. In particular, her use of color serves several purposes: (1) as an organizational tool to connect her diagrams to the content of her proofs, (2) to enhance her understanding of the proof she is writing, and (3) to illustrate relationships within her diagrams and proofs. We feel this small study has particularly interesting pedagogical implications.

***Keywords:*** modern geometry, proofs, diagrams, color

***Background &Theoretical Framework***

In 1847, Oliver Byrne published his reworking of Euclid's *Elements*. He used colored diagrams so extensively that the visual representations were inseparable from the proofs they were intended to support. Published during a period when geometer's had their attention focused on non-Euclidean investigations, Byrne's work was not taken seriously, and was "regarded as a curiosity" (Cajori, 1928, p. 429). However, Byrne did not intend his work for mere entertainment. Instead, he proposed that the book enhanced pedagogy by appealing to the visual and encouraging retention of the ideas. He suggested that by communicating Euclid's ideas through a colored, visual means, instruction time could be used more efficiently and student retention is more permanent (Byrne, 1847).

Students' transition to formal proof is a well-covered area of research in mathematics education (e.g., Moore, 1994; Selden & Selden, 2003; Weber 2001). However, students' use of representations to support their arguments is still an emerging field of research at the post-secondary level. Where there is considerable research available about calculus students' use of visual representations (e.g., Hallet, 1991; Tall, 1991; Zimmerman, 1991), there is still little research available about students in advanced undergraduate mathematics. Additionally, the National Council of Teachers of Mathematics (NCTM, 2000) asserts that creating and using representations is an essential component to mathematical understanding. As a result, the use of visual representations in K-12 mathematics (and, in particular, K-12 geometry) is well-documented (e.g., Christou, Mousoulides, Pittalis, Pitta-Pantazi, 2004; Hanna, 2000; Ye, Chou, & Gao, 2010).

In his research investigating students' use of visual representations in an introductory analysis course, Gibson (1998) found that students implement diagrams to (1) understand information, (2) determine the truthfulness of a statement, (3) discover new ideas, and (4) verbalize ideas. Yestness and Soto (2008) used Gibson's results to frame their study of 7 students who used diagrams in the development of their understanding of abstract algebra concepts. They found students most commonly employing (1) and (4) in their diagramming. In particular, they discussed students who explained that their drawings were merely for personal use and not for proof or explanation. However, when asked to explain their proof, many drew a diagram to support their explanation.

The primary goal of this small research study is to investigate how students in an undergraduate modern geometry class use diagrams as proof-writing tools. In particular, we noticed a growing number of students employing the use of color to support their diagrams in our advanced undergraduate mathematics classes. We used the framework proposed by Gibson (1998) and reinforced by Yestness and Soto (2008) to guide our small phenomenological research study into a single geometry student's use of color-enhanced diagrams as a proof-writing tool. The question guiding our research is: *What is the nature of students' use of color as a proof-writing tool in college geometry?*

### *Methods*

The research took place at a medium-sized public university in the southeast. To address the research question, we met for a single 75-minute semi-structured, task-based interview with Charlotte Knight. We purposefully identified Charlotte, an undergraduate mathematics major with a concentration in teacher licensure, as a participant because of a "colored" proof she provided on an in-class exam. Very similar to the proofs Oliver Byrne presented in his reworking of Euclid's *Elements*, we were curious about Charlotte's reasoning. The audio-recorded interview focused on a discussion of Charlotte's original proof and the construction of a new "colored" proof.

We are using the constant-comparative method of analysis as outlined by Corbin and Strauss (2008). That is, using the transcription of the interview, we are systematically open and axial coding the data to identify emergent themes in Charlotte's interview, while regularly revisiting the theory identified in Gibson (1998) and supported by Yestness and Soto (2008).

### *Results and Future Work*

Preliminary results indicate that all four aspects of diagramming offered by Gibson (1998) and supported by Yestness & Soto (2008) are apparent in Charlotte's colored proof. Additionally, she appears to use color (1) as an organizational tool to connect her diagrams to the content of her proofs, (2) to enhance her understanding of the proof she is writing, and (3) to illustrate relationships within her diagrams and proofs. It may be the case that these are, in fact, embedded within Gibson's categories.

We feel that this small study will have some particularly interesting pedagogical implications. Byrne (1847) asserted that using color-coded proofs allows a whole class to see the key parts of the argument rather than having to mentally connect what the letters refer too, and thus reducing opportunities for confusion. Early passes through Charlotte's interview transcription support and expand upon this argument. Extending this research to include other participants who utilize diagrams (and, in particular, utilize colored diagrams) may shed light onto how to reform instruction accordingly.

Necessary next steps for this research study include identifying additional participants who employ color in their proof-writing techniques. This will enable us to further investigate any conjectures that emerge as a result of this research. By the time of the RUME conference, we will be ready to report on our constant-comparative analysis of Charlotte's interview. Questions we intend to pose to the audience include the following:

- Charlotte was selected for an interview because of an isolated proof she provided on an exam. How might we go about identifying additional participants without creating an artificial environment?
- The aforementioned issue is a limitation to this research. What steps might we take to "beef up" the validity and reliability of our small study?

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# Student Understanding of Integration When Applied to Finding Volumes of Solids

Preliminary Research Report

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Past research has shown that students struggle when solving definite integral application problems, but little has been done to examine the sources of their difficulties. This study aims to more thoroughly examine student misconceptions about definite integrals and develop new curricula to address these issues. Participants were second-semester calculus students enrolled in a large, public university. Exam problems required students to sketch approximating slices of given solids, and set up a corresponding volume integral. Students' written work was analyzed for common mistakes and misconceptions. Although some students solved the problems correctly, a majority exhibited major deficiencies in their understanding of how to apply the definite integral. Most surprising was students' widespread failure to make a connection between the sketch and the set up of the integral. Further research is currently under way that aims to expose sources of students' faulty thought processes when using definite integrals to solve volume problems.

**Keywords**: Calculus, definite integral, visualization, conceptual understanding

Finding volumes of solids is an application of the definite integral that is routinely covered in a second-semester calculus course, but very little research has been conducted with the aim of understanding how students conceptualize these problems. The definite integral is typically introduced to students as a tool for determining the area of a region contained between a continuous, positive curve $y = f(x)$ and the $x$-axis on a closed interval. Since students generally encounter the area conception of definite integral first, this can lead to the definite integral being tied only to the physical quantity of area in students' minds (Bezuidenhout & Olivier, 2000; Gonzalez-Martin & Camacho, 2004; Sealey, 2006). This rigid association would almost definitely lead to difficulties in applying the definite integral in other applicable physical situations.

Previous research has found that when solving definite integral application problems, students often rely on previously encountered methods for setting up and evaluating integrals (i.e., mimicking methods encountered in class) (Grundmeier, Hansen, & Sousa, 2006; Huang, 2010). Yeatts & Hundhausen (1992) examined student difficulties in applying calculus concepts to physics problems and found that students relied "heavily upon memory and pattern to establish the integrals prior to routine manipulation."

A key component in successfully solving volume problems is visualization of the solid. Optimally, visualization of the solid and its constituent parts guides and dictates the construction of the corresponding volume integral. Unfortunately, it is possible to correctly solve many routine volume problems without the aid of visualization (we consider routine volume problems to be those in which the required function formulas are stated explicitly). In an early study on student understanding of integration (Orton, 1983), students were asked to give detailed explanations of their reasoning when solving integration problems. Orton observed that students had very little idea of the dissecting, summing, and limiting processes involved in integration.

Huang (2010) observed students focusing on "calculating correctly, while ignoring the true meaning of the concepts behind the calculations."

## Current Research Aims and Questions

The goal of this study is to more deeply explore student understanding of applications of the definite integral. The first phase involves identification and classification of common mistakes students make when setting up and solving volume problems. The second phase consists of a more in-depth analysis of student thinking via data collected from one-on-one interviews concerning past written work and novel problems, and small-group problem-solving sessions concerning novel problems, with the goal being identification of the sources of students' misconceptions. The third phase involves development and implementation of new teaching techniques and materials that will aid in greater student understanding of the definite integral.

Although area is involved in certain volume calculations, it is not the physical quantity that is being determined by the integration problems considered in this study. Because of this, we believe that volume problems can expose any underlying deficiencies students may have that may otherwise be concealed due to the relative simplicity of integral-as-area problems. Visualization is an important aspect of integrating to find volumes, so we want to examine the connections (or lack thereof) between students' visualizations of solids and their set-up of the volume integral.

## Conceptual Framework

Our research is built on the foundation of the constructivist learning theory (Piaget, 1970). We believe that students construct their own understandings of mathematical concepts given the information that is presented to them and the information that they extract from the learning materials. The theoretical perspective guiding our analysis of student thinking in this study is based on Dubinsky's (1991) Action-Process-Object-Schema framework. The portion of our study where we analyze student written work and subsequently interview students about their work will also be guided by Vinner's (1997) conceptual framework that describes and analyzes verbal (oral/written) mathematical behaviors of students. Vinner classified student mathematical behaviors as occurring within two different contexts – a conceptual context, which involves understanding of mathematical symbols, notation, and meanings of words; and an analytical context, which involves problem solving. *Conceptual behaviors* are a result of conceptual thinking, which arises from meaningful learning and correct conceptual understanding. *Analytical behaviors* are a result of analytical thinking, which involves accurate analysis of the type and structure of a problem, and selection of a valid solution procedure. When students act in ways that superficially resemble these types of behaviors, but lack the deep, proactive "thinking" aspects of each, they are exhibiting what Vinner calls *pseudo-conceptual behaviors* or *pseudo-analytical behaviors*. He explains that, in "mental processes that produce conceptual behaviors, words are associated with ideas, whereas in mental processes that produce pseudo-conceptual behaviors, words are associated with words; ideas are not involved" (p. 101). Similarly, in analytical mental processes that produce analytical behaviors, problem-solving strategies are associated with ideas, whereas in mental processes that produce pseudo-analytical behaviors, problem-solving strategies are associated with methods that have been previously encountered.

Application problems come in a variety of types and forms, and require a solid understanding of the underlying mathematical concepts. Optimally, when students begin solving application problems, they are familiar and comfortable with the relevant mathematical concepts – in other words, the concepts have been encapsulated into objects that can be used as problem-solving tools (Dubinsky, 1991). Incomplete or insufficient understanding of these concepts can

lead to pseudo-conceptual and pseudo-analytical behaviors in the classroom. It is the aim of the researchers to identify and examine these pseudo-behaviors for definite integral problems, and determine where and how students' misunderstandings occur. We believe that in the interview and problem-solving sessions, we will be able to uncover where in the action-process-object procedure students become stuck that requires them to resort to pseudo-strategies. We hope to create problems that better expose these inconsistencies, and develop teaching methods that discourage these types of student actions and foster more meaningful learning.

## Subjects/Methods

The first phase of data collection occurred during the Fall 2010 and Summer 2011 semesters at a large, public, research university. The participants were second-semester calculus students – a total of 40 in Fall 2010 and 57 in Summer 2011. After learning about applications of integration, and in particular, using integration to find volumes of solids, the students were tested on the material, and their written responses were analyzed for common mistakes and misconceptions. Each relevant exam question required that students: (a) set up (and possibly evaluate) an integral that represented the volume of a particular solid, (b) sketch the 2-dimensional region that was being rotated about a line to form the solid, and (c) sketch a typical approximating cylinder on the same graph as the 2-dimensional region.

Currently, we are in the process of recruiting volunteers to participate in video-taped, task-based interview sessions. We will examine participants' written work, and identify those whose mistakes fall into the categories that emerge from the phase data. This subset of participants will be interviewed about their thought processes and problem-solving strategies with respect to their written work, and they will also be asked to complete some non-routine definite integral application problems.

## Preliminary Results

Approximately one-fourth of the students were able to correctly construct volume integrals and sketch the corresponding approximating cylinders for each solid. The remaining students had extreme and varied mistakes and misconceptions in many different aspects of the problem-solving process. The errors that were most pervasive in student solutions were: incorrect variable of integration, incorrect bounds of integration, incorrect integrand, inability to sketch an approximating cylinder, inability to connect the integral set-up with the visualization of the solid, and failure to understand the ways in which the two "methods" for finding volumes (slicing vs. shell) differ.

There were no obvious patterns that emerged with respect to student misconceptions in students' written work. There were instances of correct sketches with incorrect integral set-ups. There were instances of incorrect sketches with correct integral set-up. And, of course, there were instances where both the sketch and the integral were incorrect.

One problem asked students to find the volume of the same solid in two different ways (via the slicing and the shell methods). Many students chose the same variable of integration for both methods (some choosing $x$, some choosing $y$), indicating a lack of appreciation for and understanding of the inherent differences between the two methods.

The broad range of errors and the lack of connection between students' sketches and integrals indicate that students may be thinking pseudo-analytically during the process of solving these problems. Vinner states that the "most characteristic feature of the pseudo-analytical behavior is the lack of control procedures" (p. 114). Due to the absence of obvious patterns in students' errors, the video-taped interviews will lend a great deal to our understanding of the source of their confusion.

## Future Questions/Research

After complete analysis of phase one data, we hope to arrive at a classification and categorization scheme that will act as a guide for the subsequent phases of research. We hope to come up with non-routine problems for interview sessions that will help us distinguish analytical from pseudo-analytical behaviors. We also want to investigate any pseudo-conceptual behaviors that students may exhibit when discussing their problem-solving processes/strategies during one-on-one interviews.

At this university, students are introduced to the concept of definite integral and a few elementary applications, and they then move on to investigate techniques of integration. After a full chapter of learning techniques, they return to the study of definite integral applications, but in more complex physical situations (volume, work, centers of mass, etc.). Since students do not seem to be making the connection between the volume of the small slice of the solid and the integral set-up, we believe it would be advantageous to actively maintain the Riemann sum-definite integral connection and not have it broken up by discussion of calculation-heavy integration techniques. Knowing techniques of integration definitely gives students more tools for solving a greater variety of application problems, but these tools are only useful if the student can set up the integral correctly in the first place.

## Questions for Audience

--What are examples of non-routine volume problems that will aid in uncovering students' underlying misconceptions about the applications of definite integrals?
--Is there computer software that could aid students in the visualization aspect?
--What are the implications of students continuing on through calculus and *not* truly understanding the definite integral as a limit of the sum of smaller constituent parts of the whole? (In other words, "So what?")

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# The Effects of Three Homework Systems on Student Learning in Intermediate Algebra: A Comparative Study

Preliminary Report

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Abstract: Online homework systems are designed to engage students with course topics while providing immediate feedback. Few studies have indicated a significant difference in student performance using online homework systems compared to traditional homework. Our study seeks to add to the growing body of research examining the effectiveness of online homework systems by investigating the performance of students taking Intermediate Algebra. This study will compare differences in student exam scores based on their homework medium: WebAssign, ALEKS, or traditional homework. Each instructor participating in the study taught at least one course with each homework system. Preliminary results indicate no significant difference in student learning between students using WebAssign or traditional homework. ALEKS data is currently being collected and suggests students are developing a thorough understanding of specific Intermediate Algebra topics.

Keywords: online homework, classroom research, ALEKS

Online homework systems, such as WebAssign, MyMathLab, WeBWorK, and ALEKS, have been designed to engage students with course topics while simultaneously providing immediate feedback. To this end, the systems are designed to better the delayed (or altogether absent) feedback of traditional paper-and-pencil homework. If a student has difficulty on a problem they usually have the option of seeing, either through text or a video-clip, an explanation of a similar problem.

Few studies have indicated a significant difference in student performance using online homework systems compared to traditional homework. In a study using WeBWorK in a calculus class, Hirsch and Weibel (2003), found that the final exam grades for students using WeBWorK were 4% higher, on average, than their non-WeBWorK peers. Hauk and Segalla (2005) found no significant difference between the performance of online and traditional homework sections of students taking college algebra. By designing online software that provides students with detailed feedback for incorrect responses and allowing several attempts at each assignment Zerr (2007) found that student learning in an introductory calculus course improved. Allowing multiple attempts can also be detrimental: not only can scores be quite high (over 85% is common) but students, seeing they received high marks, might develop a false sense of confidence and prepare less for their exams than their traditional homework counterparts.

ALEKS (Assessment and LEarning in Knowledge Spaces) was not designed as purely an online homework tool such as WebAssign but is based on Knowledge Space Theory (Falmagne et al., 2000). To form a knowledge space, one must first define a set of concepts. For an Intermediate Algebra class this consists of a list of specific algebraic topics (the software contains almost 400 topics in Intermediate Algebra alone.) Based on a student's performance on an initial assessment, ALEKS determines a subset of topics known by students and, taking into account prerequisite relations among topics, provides a list of topics students are ready to learn. In order to learn new topics students must correctly answer three similar randomly-generated problems sequentially. At any time a student can bring up a detailed explanation of their current problem; if a student answers incorrectly three times they are given an explanation and asked to try again. Due to the nature of the ALEKS software a student spends the most time on topics they find difficult. In addition, students are always working on topics near their current ability. This also means a classroom of students working on ALEKS can be working on a wide range and variety of topics and does not fit well within a typical structured class covering only specific topics on specific days.

Research on the integration of ALEKS is growing, though results are still varied. Stillson and Alsup (2003) found an almost 50% increase in drop and failure rate among Basic Algebra students using ALEKS. However, they believed ALEKS would benefit students more if they took the time to use it and recommended the course be offered in a classroom setting. Taylor (2008) found that Intermediate Algebra students using ALEKS had a better attitude and felt less anxious toward mathematics than a control group yet performed as well as students in a lecture-based class. Hagerty and Smith (2005) found that students using ALEKS performed significantly better (both short-term and long-term) in College Algebra than students in the control group. Oshima (2010) argued that ALEKS improves students' mathematical knowledge and skills as well as their passing rate of College Algebra. He also reasoned the success of ALEKS was due to how it was integrated into the classroom—that ALEKS was not simply an add-on homework tool.

Our study seeks to address the question of which (of three) homework systems had the greatest impact on student learning (defined according to performance on a common final exam) on students taking Intermediate Algebra at a large public university in the Midwest and to add to the growing body of research examining the effectiveness of online homework systems. Specifically we will compare WebAssign, traditional paper-and-pencil homework, and ALEKS. Data collection began in Spring 2010 with three instructors each teaching at least one section using Webassign and one section using paper and pencil homework and will continue through Spring 2012. Each participating instructor used each homework system at least once to aid in the comparison. Before the study began a comprehensive multiple choice exam was created using questions from the test bank provided by the publisher that was designed to cover all of the learning objectives of the course. This exam served as both the pre-test and the post-test and was

administered to all students participating in the study. For the ALEKS classes, a group of four instructors met and chose the 228 specific topics that would comprise the course.

At the current stage of our study we have collected three semesters of pre and post test data with instructors using both WebAssign and traditional homework. Currently analysis has consisted of t-test comparisons by treatment and as of yet there has been no significant difference in student learning (as defined above) between WebAssign and traditional homework sections.

We are currently collecting ALEKS data from three instructors. Along with further t-tests, an item analysis is planned with the inclusion of the ALEKS data. Following the direction of Stillson, Alsup, and Oshima we have allowed the ALEKS software to become the classroom. Classes meet in a computer lab and spend the entirety of classroom time working on ALEKS topics. Instead of "professing" the instructor serves as a learning guide by moving about the classroom individually helping students with topics they are working on. No additional "homework" is assigned. Preliminary ALEKS data is promising and suggests students are developing a thorough understanding of specific Intermediate Algebra topics.

Questions for audience consideration:

1. What do instructors who have used ALEKS think of the program?
2. What do instructors think of using ALEKS as the sole provider of course topics?
3. What are the benefits and downsides to online homework systems and traditional homework formats?
4. How could ALEKS be used in an activity-based, collaborative course where the focus is on conceptual understanding over procedural mastery?

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# RAISING STUDENTS' CALCULUS UNDERSTANDINGS TO THE SURFACE IN MULTIVARIABLE CALCULUS
# PRELIMINARY REPORT

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*This presentation will share initial observations and data collected from a pilot classroom study in which groups of multivariable calculus students made physical measurements and drew actual curves on real, tangible surfaces to construct geometric mathematical objects fundamental to the course and discover their properties. Students completed short group activities focusing on relationships between functions and level curves, properties of gradient vectors and directional derivatives, and solutions to optimization problems before other symbolic representations or procedures were discussed in lecture. Initial self-reported data suggests working with the surfaces helped students visualize functions. It also appears the activities helped students develop strong connections between the geometric, symbolic, and verbal representations of multivariable calculus concepts. Collected data suggests the surfaces helped uncover students' single variable misconceptions which hindered their new understandings. The goal of this presentation is to receive feedback for the design of a rigorous phase two study of this project.*

*Key words:* Multivariable Calculus Visualization, Classroom Research, Geometric Representation

The basic ideas of multivariable calculus, those of level curves, gradient vectors, directional derivatives, and optimization problems with constraints, can be seen as direct generalizations of fundamental single variable calculus ideas through the use of geometry. The author is aware of various studies addressing the effective use of visualization technology to help students understand multivariable calculus concepts, but the implementation of technology in this setting produces two problems: Due to the necessity of projecting a three-dimensional shape onto a two-dimensional screen, students are unable to interact with the real object in a way that mimics the way they worked with graphs of one-dimensional functions. Secondly, although visualization technology is wonderful, it can restrict the ability of students to conduct self-directed explorations of multivariable calculus ideas without additional technical help. The former issue is subtle, while the latter can prevent students from being allowed to explore and find answers to fundamental questions important for the full understanding of multivariable calculus concepts until such concepts like coordinate systems and multivariable functions have been defined.

## In Class Activities

In Fall 2009, the author of this paper used real, tangible surfaces and short mini activities to allow groups of students in a multivariable calculus course the opportunity to discover the important features and concepts related to multivariable functions. Pictures and descriptions of the surfaces are included in Figure 1. The first activity, occurring 10 minutes into the very first class, required students to identify the relationship between their surface and level curves. The second activity focused on understanding the dot product as a project. In the third activity,

students constructed the gradient vector and discovered its geometric properties by measuring *slope* on the surface in two perpendicular directions. Students also discovered the relationship between directional derivatives and gradient vectors in the fourth activity, and they discovered the geometric relationship between a constraint and the gradient of a function for problems typically solved by the method of Lagrange multipliers.

## Collected Data

The author of this paper has collected data from the in-class activities and student exams, as well as self-reported data from students collected at the conclusion of the pilot study. For each activity, students self-reported the main point of the activity. In addition, they described (a) something that was still not clear and (b) something which they better understood as a result of doing the activity. Despite using the activities to introduce ideas before formal lecture or discussion in class, very few students reported having trouble understanding the point of the activities. Furthermore, most students were able to describe the geometric properties of gradient, directional derivatives, and level curves at the start of the lecture intended to introduce those properties. As these activities were designed to help students discover these concepts, the author of this paper is very interested in designing assessment activities which can investigate the level of understanding of these students on later exams or in later activities.

The author has collected anecdotal evidence suggesting students uncover misconceptions about single variable calculus ideas, like derivatives and functions, using the surfaces. The reliance upon the $dy/dx$ notation for derivative is troublesome in a setting where $y$ and $x$ are now independent variables. In order to understand directional derivatives, one group physically changed the $x$ and $y$ coordinate system so that, instead of lying flat beneath the surface, the $x$ direction lay tangent to the surface and the $y$ direction was oriented perpendicular to the surface. (See Figure 2.) This group held firm to the notion that a derivative was $dy/dx$ , instead of a more general notion $dg/$dx for partial derivatives of the function $g$. Additional troubles occur when trying to generalize the notion of *negative slope*. Students are reluctant to recognize that a negative directional derivative indicates the surface function is decreasing in that direction. On a more positive note, most students are able to discover and explain the geometric relationship characterizing the solutions to optimization problems subject to a constraint, typically solved by the method of Lagrange Multipliers, after the 20 minute lab activity.

## Student Self-Reported Feedback

The surface activities and minilabs were used during the first five weeks of the course, after which students (n = 36) were asked to self-report on how the activities and surfaces influenced their learning. Students were asked questions about how working with the surfaces helped them visualize (10 questions) and understand (8 questions) various multivariable calculus concepts. Students were allowed to indicate that the surfaces (A1) provided no help , (A2) provided a bit of help, (A3) provided some help, or (A4) provided a lot of help. Of the 36 respondents, 25 indicated that working with the surfaces really helped them visualize gradient vectors while only 2 said the surfaces helped a bit. No students said the surfaces provided no help. In regards to visualizing solutions to Lagrange multiplier problems, 35 of the 36 students indicated that the surfaces helped some (16) or a lot (19). The surfaces were least helpful for helping students visualize second order and mixed partial derivatives, with only 24 of the 36 students indicating the surfaces helped some or a lot. Additional results are listed in Table 3.

In term of understanding concepts, students also indicated that working with the surfaces helped them understand the relationship between gradient vectors and level curves. Overall,

77% of students indicated that working with the surfaces and level curve boards helped them understand how to match level curves with surface features. 91% of the students said that working with the surfaces helped them understand slope in different directions on a surface, and 80% of students indicated that working with the surfaces provided some or a lot of help as they connected ideas of directional derivatives and slopes in various directions.

In general, students appreciated the design of the mini-labs and being able to explore the concepts using the functions. As one student said, working in groups with the mini-labs "was a good chance to bounce ideas off of each other and [get] us more involved in what was going on. You weren't just being told what to do. We had to figure it out on our own."

When asked "How did working with the surfaces help your ability to visualize and work with multivariable functions?", one student replied:

> "Being able to see all three dimensions at once rather than interpret the height from a 2D curve was really helpful. It allowed me to spend less time thinking about the vertical components of the graph and more time on solving and learning from the problem."

Many students indicated the best features of the surfaces were that they liked having a visual representation of what the graph looks like is very helpful. As one student said, "working with the surfaces helped with the learning of ideas in the course because it contained problems that are key ideas, and by doing the lab I better understand the ideas." Students repeatedly commented on the value of being able to see how vectors compared to the level curves for a surface, and how being able to draw on the surfaces helped them.

## Audience Questions

One student summed up the effect of using the surfaces by saying:

"I did not understand gradient until working with the surfaces."

The author of this paper would like to know how working with the surfaces actually changed the student's ability to expand upon calculus ideas into multivariable calculus concepts.

The author of this paper is not a trained mathematics education researchers, and is unfamiliar with resources and studies focused upon the student understanding of multivariable calculus concepts. The author is looking for feedback specifically on:

- How to make sense of the data collected during this pilot study in regards to the connection between a student's calculus and multivariable calculus understandings.
- How to design assessment activities which investigate the actual level of student comprehension of these concepts as a result of the in-class activities.
- How to design and implement a rigorous study as phase two of the project.

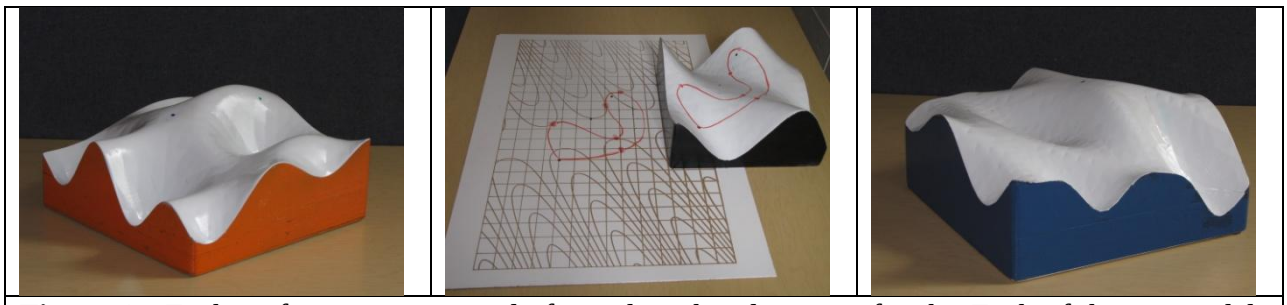

**Figure 1:** Each surface is constructed of wood, with a dry-erase finish. Each of the six models have a base of 10”x10”, and stands roughly 4”-6” tall. Two dry-erase whiteboards, one engraved with a rectangular coordinate system and the other engraved with level curves, are associated with each surface.

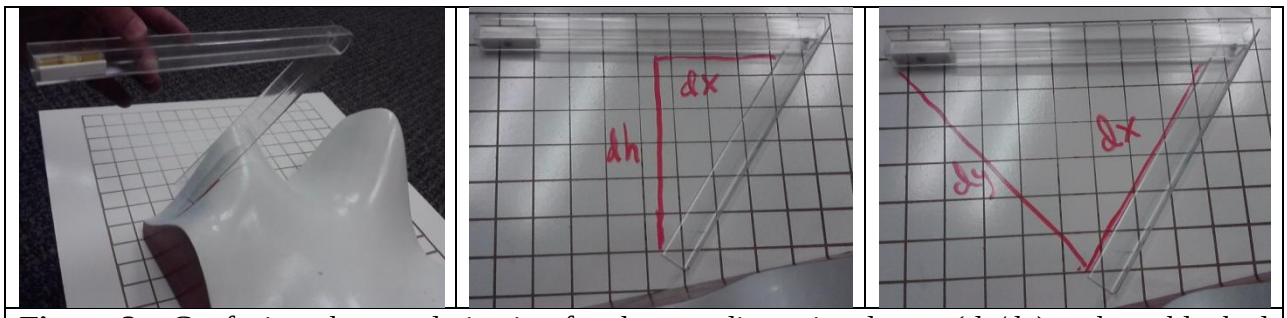


**Figure 2:** Confusion about a derivative for the one-dimensional case (*dy/dx)* perhaps blocked the ability of one group from connecting the partial derivative (*dg/dx)* with the correct quantities *dg*, *dy*, and *dx* when working with the surface. Instead of measuring vertical rise, *dg*, this group used *dy* to measure the rise perpendicular to the surface and *dx* to measure the run parallel to the surface.

| **Did working with the surfaces help you with any of the following ideas?** | Working with the surfaces … | | | |
|---|---|---|---|---|
| | provided no help. | helped a bit. | helped some. | helped a lot. |
| Understanding differences between 1-dimensional (curves) and 2-dimensional functions (surfaces)? | 0 | 4 | 14 | 18 |
| Visualizing different characteristics (maximums, minimums, saddle points) of multivariable functions? | 0 | 5 | 14 | 17 |
| Visualizing partial derivatives and slopes on surfaces | 1 | 8 | 12 | 15 |
| Visualizing second order or mixed partial derivatives. | 3 | 9 | 15 | 9 |
| Visualizing vectors in two dimensions | 1 | 4 | 16 | 15 |
| Visualizing vectors in three dimensions | 1 | 3 | 11 | 21 |
| Visualizing directional derivatives | 0 | 5 | 15 | 16 |
| Visualizing gradient vectors | 0 | 2 | 9 | 25 |
| Visualizing the maximum value of a surface restricted to a path | 0 | 1 | 16 | 19 |
| Visualizing maximum and minumum values on the boundary of a surface | 0 | 3 | 20 | 13 |

| **Did working with the surfaces help you with any of the following ideas?** | The surfaces provided… | | | |
|---|---|---|---|---|
| | provided no help | helped a bit | helped some | helped a lot. |
| Matching level curves with surface features. | 3 | 5 | 11 | 17 |
| Distinguishing between a function's domain (input) and its graph (surface) | 1 | 10 | 10 | 15 |
| Connecting the ideas of directional derivatives and slopes in different directions on a surface | 1 | 6 | 10 | 19 |
| Measuring and understanding slope in different directions on a surface. | 1 | 2 | 14 | 19 |
| Understanding the relationship between gradient vectors and level curves | 0 | 3 | 10 | 23 |
| Realizing gradient vectors are independent of their coordiante system description. | 1 | 8 | 10 | 17 |
| Understanding how the direction of gradient vectors would change at different points for a surface. | 0 | 2 | 11 | 23 |
| Understanding how the magnitude of gradient vectors would change at different points for a surface | 0 | 2 | 12 | 22 |

**Table 3:** Effect of using surfaces and activities on visualization and understanding.

# DO GENERIC PROOFS IMPROVE PROOF COMPREHENSION?

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*In undergraduate mathematics courses, proofs are regularly employed to convey mathematics to students. However, research has shown that students find proofs to be difficult to comprehend. Some mathematicians and mathematics educators attribute this confusion to the formal and linear style in which proofs are generally written. To address this difficulty, some researchers have suggested that students be exposed to generic proofs. We report preliminary results of a study that employs a recent model of proof comprehension to assess the extent to which reading a generic proof improves student understanding over reading a traditional proof.*

*Key words:* students understanding of proof, generic proof, proof at the undergraduate level.

## 1. Introduction

In advanced mathematics courses, proofs are a primary way that teachers and textbooks convey mathematics to students (e.g., Weber, 2004). However, researchers note that students find proofs to be confusing or pointless (e.g., Harel, 1998; Porteous, 1986; Rowland, 2001) and undergraduates cannot distinguish a valid proof from an invalid argument (Selden & Selden, 2003; Weber, 2010). Some mathematicians and mathematics educators attribute students' difficulties in understanding proofs to the formal and linear style in which proofs are written (e.g., Thurston, 1994; Rowland, 2001).

To address this difficulty, several mathematics educators have suggested alternative formats for presenting proofs, such as using generic proofs (e.g., Rowland, 2001; Malek & Movshovitz-Hadar, 2011), e-proofs (Alcock, 2009), explanatory proofs emphasizing informal argumentation (e.g., Hanna, 1990; Hersh, 1993), and structured proofs (Leron, 1983). These suggestions have an obvious appeal; if changing the format of a proof can increase students' understanding of its content, then these alternative proof formats provide a practical way to improve the effectiveness of lectures and textbooks in advanced mathematics courses. When Roy, Alcock, and Inglis (2010) attempted to see if Alcock's (2009) e-proofs improve students' comprehension of proofs in a pilot study, they found that students who studied an e-proof performed significantly worse on a post-test than students who studied the same proof from a lecture or textbook. Also, (year) investigated whether Leron's structured proofs improve students' comprehension. They found that students who read a structured proof were better than students reading a linear proof at identifying a good summary of the proof, but performed slightly (though not statistically reliably) worse on questions pertaining to justifications within the proof, transferring the ideas from the proof to another context, and illustrating the ideas of the proof using examples. Moreover, many students complained that structured proofs "jumped around," requiring them to scan different parts of the proof to coordinate information.

The goal of this study is to examine the extent to which generic proofs will improve student understanding of written proofs they read. A generic proof, also known as a proof by generic example, illustrates general steps of reasoning in terms of a particular mathematical object

without relying on specific properties of that object. Generic proofs are claimed to aid student comprehension, particularly in number theory, where proofs can be illustrated using appropriately chosen numbers (Rowland, 2001). Malek and Movshovitz-Hadar (2011) used the term "transparent pseudo-proof" (TPP) for the same idea, highlighting that it is not a formal proof but allows one to "see through" the particular case that is illustrated. They examined the impact of exposing linear algebra students to TPPs as compared with exposing them to formal proofs of the same theorems. They found that exposure to TPPs made no difference for "algorithmic" proofs, but for non-algorithmic proofs it improved students ability to (a) reconstruct a proof, (b) explain the main idea of the proof, and (c) construct a similar proof of a new statement. Malek and Movshovitz-Hadar posited that TPPs help students construct meaning for a proof by providing a concrete model of its flow of ideas. They acknowledge that their small sample size—ten students, only three or four of whom read each TPP—and particular domain limit the strength of their interpretations. Our study builds on these results by further investigating the performance of mathematics majors who see either a generic or a traditional proof of the same statement.

## 2. Theoretical perspective

Our model of assessing proof comprehension is based on Mejia-Ramos et al (2012). This model posits that students' proof comprehension can be measured along seven dimensions: (1) understanding terms and statements in the proof, (2) logical status of statements and proof framework, (3) justification of claims, (4) summarizing via high-level ideas, (5) identifying the modular structure, (6) transferring the general ideas or methods to another context, and (7) illustrating the ideas of the proof with examples.

## 3. Methods

Ten students were interviewed for this study—all were in their fourth or fifth (final) year of a joint B.A. and Ed.M. mathematics education program. Each student met individually with a co-author of this paper and was presented with two generic proofs. The first was a generic proof of the claim "There are $2^{n-1}$ ways to express $n$ as an ordered sum of natural numbers" and the second was a generic proof of the claim "There are infinitely many triadic primes [primes of the form 4k+3]." We will refer to these two proofs as the Partition proof and the Triadic Primes proof, respectively.

First, participants were given instructions on the format of generic proofs to reduce potential confusion due to lack of familiarity. Participants then read the Partition proof until they had studied it to their satisfaction. At this point, participants reported how well they felt they understood the proof (on a scale of 1 to 5), how convincing they found the argument (on a scale of 1 to 5), and whether they were confident that the proof would work in general. The participants then returned the proof to the interviewer and answered open-ended questions about the proof's content—these questions were based on the model of Mejia-Ramos et al (2012). Finally, students were asked to comment on the format of the proof, whether they preferred a proof in a traditional format, and if there was anything about the generic proof that aided or hindered their understanding of the content. This procedure was then repeated for the Triadic Primes proof.

Our analysis concentrates on (1) the participants' comments on the format of generic proofs, (2) the participants' self-reported levels of understanding, conviction, and confidence that the proofs work in general, and (3) the participants' performance on the assessment questions.

## 4. Results

Overall, participants appeared to have positive opinions of generic proofs. Of the ten participants, nine commented that they could see how generic proofs could improve student comprehension (their own and others'). Particular positive features of generic proofs that were mentioned include reducing abstraction and eliminating confusing notation and jargon. Five of the ten participants expressed some reservations about generic proofs, focusing in particular on whether these were true proofs, sufficiently general, or sufficiently rigorous.

For the Partition proof, the participants on average reported an understanding of 4.1, that they were convinced with a score of 3.89, and 8 of 10 participants were confident of the generality of the proof. Participants answered an average 6.1 out of 10 questions correctly (61%).

For the Triadic Primes proof, the participants on average reported an understanding of 3.1, that they were convinced with a score of 3.49, and 8 of 10 participants were confident of the generality of the proof. Participants answered an average of 3.2 out of 7 questions correctly (45.7%). In a similar study by Fuller et al (2011), a group of six mathematics majors answered assessment questions based on the same model after reading a traditional linear version of the Triadic Primes proof and answered an average of 2 of 7 questions correctly (29%).

## 5. Discussion

The above results provide preliminary evidence that generic proofs can increase comprehension, since students seeing a generic version of the Triadic Primes proof were able to answer more comprehension questions correctly than those seeing a linear version of the Triadic Primes proof. Moreover, students mentioned several ways in which they felt generic proofs were helpful for their understanding. However, more evidence is needed before formulating any conclusions.

We are currently conducting a larger-scale internet study in which math majors from various universities will be shown either a linear or generic version (chosen at random) of either the Partition or Triadic Primes proof. Following this, they will answer comprehension questions (a subset of those used in the interview study). By comparing the performance of students seeing the linear versus generic proofs, we can begin to answer the question of whether generic proofs improve comprehension.

## 6. Questions for the audience

Under what conditions might we see the benefits of generic proofs? Are there any series of studies or interventions that might convince you that generic proofs (or any alternative proof format) are not effective at improving student understanding? What setting and methodology might be appropriate for investigating a longer-term intervention involving generic proofs?

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# Children's Reasoning about Integers: Video Clips to Share with Preservice Teachers Preliminary Research Report

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Abstract

We have conducted interviews with children using integer-related tasks, and we have identified various ways of reasoning that children bring to bear on these tasks. One product of this work is a collection of compelling video clips. We will share examples of children's reasoning, and the audience will be engaged in discussions of children's reasoning and use of video in instruction. Attendees will receive a free DVD with video clips that can be used with preservice teachers.

Keywords: Children's thinking, integers, preservice teachers, video

We have conducted interviews with children using integer-related tasks, and we have identified various ways of reasoning that children bring to bear on these tasks. One product of this work is a collection of compelling video clips. We will share examples of children's reasoning, and the audience will be engaged in discussions of children's reasoning and use of video in instruction. Attendees will receive a free DVD with video clips that can be used with preservice teachers.

## Theoretical Perspective and Prior Research

We approach this research from a children's thinking perspective. That is, we seek to understand the mathematics through the lens of children's conceptions (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Research has yielded valuable information regarding children's mathematical thinking in the whole-number domain, including a framework describing developmental trajectories of students' strategies and conceptions related to multi-digit arithmetic (Carpenter et al., 1999). This work has benefited elementary teachers and their students. Teachers who participated in professional development focused on understanding children's mathematical thinking changed their beliefs about teaching and their teaching practices, and these changes were related to improvements in student achievement (Fennema, Carpenter, Franke, Levi, Jacobson, & Empson, 1996).

One compelling way of engaging preservice or practicing teachers with children's mathematical thinking is through the use of video. Often, those who choose to pursue careers in elementary education are not strong mathematically (e.g., Ball, 1990; Ma, 1999). However, they care about children, and their interest in helping children can be leveraged to get them interested in mathematics via children's mathematical thinking (Philipp, 2008). Studying children's mathematical thinking has been found to positively influence prospective teachers' beliefs about mathematics, teaching, and learning, as well as their mathematics content knowledge (Philipp et al., 2007). Prolonged involvement in professional development with a focus on children's

mathematical thinking can help teachers to develop expertise in *professional noticing of children's mathematical thinking* (Jacobs, Lamb, & Philipp, 2010).

Although much has been learned about children's reasoning in the whole-number domain, children's reasoning about integers has received relatively little attention (Kilpatrick, Swafford, & Findell, 2001). The introduction of the integers poses conceptual challenges for students, as they are required to expand their mathematical worlds to include negative numbers (Bruno & Martinón, 1999; Janvier, 1983; Vlassis, 2004). This extension contradicts students' previous conceptions, which often involve overgeneralizations of their experiences with the natural numbers, e.g., that addition makes larger and subtraction makes smaller. Children's difficulties with integers can be appreciated in light of the history of mathematics, wherein famous and accomplished mathematicians struggled with counterintuitive notions associated with negative numbers (Gallardo, 2002; Hefendehl-Hebeker, 1991; Henley, 1999; Thomaidis & Tzanakis, 2007). At the same time, researchers have reported on cases in which children reasoned productively about integers, even in the lower elementary grades (Behrend & Mohs, 2006; Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Hativa & Cohen, 1995; Wilcox, 2008). In contrast with the literature concerning children's reasoning about whole numbers, the literature concerning children's reasoning about integers is sparse. There is not an established framework for integer reasoning, and developmental trajectories have yet to be identified.

## Methodology

In 2010, we conducted more than 90 interviews with K-12 students, as we piloted a variety of integer-related tasks. We refined our interview protocol on the basis of these. In 2011, we conducted 160 interviews at seven school sites across three districts – 40 each with children at grades 2, 4, 7, and 11. We used a range of tasks in these interviews, but this report focuses on open number sentences (such as 5 – □ = 8), which were used extensively with students at each grade level. We developed codes for children's strategies via a process of constant comparative analysis (Strauss & Corbin, 1998), and we have organized these strategies into a framework based on the underlying number conceptions that they suggest. We have also used descriptive statistics to compare the relative difficulty of problems both within and across grade levels. Our ongoing analysis involves relating students' strategies and conceptions to problem difficulty.

## Results

We present examples of various ways of reasoning from elementary, middle school, and high school students. In grades K-4, we identify ways of reasoning about integers prior to formal instruction. Some of these children were entirely unfamiliar with the notion of negative numbers. Their responses reveal the counterintuitive nature of ideas related to negative numbers for children who live in a whole-number world (e.g., 3 – 5 is impossible). Other elementary children were familiar with negatives. Many of them were able to engage productively with our tasks, although they had received no formal instruction in integer arithmetic. At the middle-school level, we see the influence of instruction on children's reasoning. Many responses are indicative of attempts to follow school-learned procedures. Often the reasoning of these children is in contrast to the sense making approaches of their younger counterparts. At the high-school level, some students' approaches remain very procedural, while others employ a variety of productive ways of reasoning about integers. In this short proposal, we offer a few specific examples.

James, a first grader, had heard of negative numbers and could solve some of the open number sentences that were posed to him. For example, he was given the problem -5 + -2 = □. James wrote -7 in the blank and explained his thinking as follows: "Because like five plus two equals seven. So, like, if you're doing negatives, it's like the same as regulars." James applied an

analogy between negative numbers and "regular" numbers, which enabled him to obtain some correct answers. Essentially, if the given numbers were both negative, he thought about the problem the same way as he would if the given numbers were both "regulars," and then he simply wrote a minus sign in front of his answer and called the number "negative." James could not solve problems that involved both a positive and a negative number. He said that the numbers behaved like magnets that would repel one another. Thus, his reasoning about integer arithmetic was rather limited. On the other hand, James's way of reasoning enabled him to solve problems such as -5 – -3 = ☐, which were difficult for some seventh and eleventh graders.

Roland was in fourth grade. He had heard of negative numbers, and he knew the ordinal relationship between these and positive numbers. Although Roland was not familiar with addition or subtraction involving negatives, he was able to solve many of our tasks. For example, Roland solved -5 + -1 = ☐ by employing an analogy between negative and positive numbers: Since 5 plus 1 equals 6, -5 plus -1 equals -6. In contrast with James, however, Roland had a meaningful justification for his approach. He reasoned that combining two negative numbers would give a result "farther from the positive numbers," so that -6 made sense. In several instances, Roland reasoned productively by deducing whether the given operation should result in moving in the direction of the positives or away from them. He even solved -5 – -3 = ☐ by reasoning in this way. (The reader is encouraged to imagine the details of Roland's solution.)

Jane, a fifth grader, had received instruction in integer addition and subtraction. When she was given the problem -12 + 7 = ☐, Jane changed it to read -12 – +7 = ☐. She came up with two possible answers, +5 and -19, and she decided that -19 was correct. A song that Jane's teacher had taught her informed her thinking about the problem. Jane mentioned this song and recited it: "Same signs, add and keep. Different signs, subtract. Take the sign of the higher number. Then you'll be exact." Jane's reasoning contrasts starkly with Roland's. Whereas he made sense of problems on the basis of the ordinal relationship between positive and negative numbers, Jane attempted to apply an arbitrary and unclear rule. She did not make explicit any specific relationship between the song and her solution to this problem. It seemed that she thought she should change something before computing, when in fact this was unnecessary.

## Implications

One compelling finding from this study is that children are capable of reasoning productively about integers prior to formal instruction. Research has shown that children are capable of inventing their own mental calculative strategies, when given the opportunity to do so (Carpenter, Franke, Jacobs, Fennema, & Empson, 1997). However, in order to support students' invention, teachers need knowledge of relative problem difficulty. Understanding children's ways of reasoning affords teachers models of student thinking, and therefore the ability to anticipate problem difficulty (Carpenter, Fennema, Peterson, & Carey, 1988; Fennema et al., 1996). As an example of integer problem difficulty, only 58% of seventh graders correctly solved 6 – -2 = ☐, while 75% correctly solved -5 – -3 = ☐. Procedurally, these problems look similar. If anything, -5 – -3 might appear more difficult. However, reasoning like that of James and Roland helps to explain why this problem was actually less difficult for some students.

The findings that we will present can be used instructively with preservice teachers in two ways: (1) to engage them in thinking about integers themselves, and (2) to introduce them to children's ways of reasoning, which may be very different than their own. Knowledge of children's integer reasoning is relevant to preservice elementary teachers, even if they will not be teaching about integers as such, because children's experiences in the early elementary grades will influence their preparedness for integer instruction.

## Questions

Attendees will be engaged in interpreting children's thinking and considering how the video clips could be used with preservice teachers. The specific questions will be tied to the examples of children's thinking. Questions like the following will be posed: How would you solve this problem yourself? Do you have another way of solving it? How might a second grader think about this problem? Which of these tasks would you expect to be more difficult for a child? How might preservice teachers think about this problem? How might you use this clip with preservice teachers? What would you hope they would take away from it?

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# Articulating Students' Intellectual Needs: A Case of Axiomatizing

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**Abstract**

This study uses qualitative methods to investigate how students' intellectual needs were articulated in an inquiry-based mathematics bridge course. One of the primary goals of this bridge course was to orient students toward more advanced mathematics by engaging them with an RME-inspired curriculum for learning abstract algebra (Larsen, 2004). Although intellectual need was not the initial object of analysis in this term-long teaching experiment, the teacher-researcher was curious about the nature of some of the student discussions that had taken place during the term. In a retrospective analysis of the teaching experiment, students' acts of mathematizing were examined and correlated with Harel's (2011) categories of intellectual need. A preliminary analysis of the data suggests that the act of axiomatizing a mathematical system—in this case, a group—can provide students with many opportunities to articulate and address a variety of intellectual needs.

**Keywords:** intellectual need**,** Realistic Mathematics Education, mathematizing, bridge course

## Introduction

Harel's (1998) Necessity Principle states, "Students are most likely to learn when they see a need for what we intend to teach them, where by 'need', is meant intellectual need, as opposed to social or economic need" (p. 501). The aforementioned Necessity Principle puts forth a conjecture about how students learn (Speer, Smith & Horvath, 2010) and has been used extensively by Harel as a component of a larger conceptual framework called Duality, Necessity, and Repeated-Reasoning (DNR) (Harel, 2001). More recently, Harel, (2011) has refined and expanded these intellectual needs into five inextricably-linked categories: *the need for certainty* (to establish that a statement is true), *the need for causality* (to determine why a statement is true), *the need for computation* (to quantify and calculate), *the need for communication* (to persuade others of truth and to agree on conventions), and *the need for structure* (to re-organize knowledge into a logical system). Harel has illustrated each of these categories of intellectual need using both examples from his own research, as well as documented accounts from the history of mathematics, which suggests that intellectual need permeates throughout the discipline of mathematics. However, at the same time, it leaves one to question whether all of these categories of intellectual need can be illustrated in a single context and more importantly, how students exhibit and seek to address these intellectual needs in an actual classroom setting. This research study investigates these questions by examining the role that students can play in articulating their intellectual needs within the context of axiomatizing.

## Theoretical perspective

Several theoretical perspectives—both in curriculum design and in the preliminary data analysis, influence this research study. First, the group theory curriculum (Larsen, 2004) that was used in this teaching experiment was inspired by the design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991). One of the tenets of RME is the importance of guided reinvention, in which a student is encouraged to "invent something that is new to him, but well-known to the guide" (Freudenthal, 1991, p. 48). In this particular setting, guided reinvention is the process by which students' model of an abstract group first emerges as a *model of* the symmetries of

an equilateral triangle and evolves into a *model for* the abstract dihedral group of order six (Larsen, 2004). A core component of Larsen's curriculum is the mathematical activity of *axiomatizing*, which requires students not only to construct a system of rules for operating on their symmetries, but also to refine these rules to a minimal list of axioms that could be used independently of the objects from which they were abstracted (Larsen, 2009). Therefore, the researcher in this study considered students' axiomatizing as a mathematical activity analogous to that of *symbolizing* (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), and sought to examine some of the intellectual needs that the students articulated when developing their axiomatic system. Consequently, the DNR conceptual framework was used as an indispensable tool for analyzing the data. Specifically, the categories of intellectual need were used to code instances in which students referred to knowledge that they would need to construct to resolve a problematic situation (Harel, 2011).

**Background and research methodology**

Over the course of nine weeks, the teacher-researcher and his students progressed through a subset of an RME-inspired curriculum (Larsen, 2004) for re-inventing the concept of group. Extensive written and video data were collected from this teaching experiment, which occurred in an elective mathematics bridge course at a medium-sized, suburban community college. The teacher-researcher was a full-time community college instructor with more than ten years of experience teaching courses ranging from arithmetic through integral calculus. The participants were nine community college students (five female and four male) whose ages ranged from 17 to 35 years. Four of the students were math majors, two were engineering majors, one was a music major, and two students had not yet declared a major. The students' mathematical experience varied greatly: four had taken courses through differential equations, one had completed calculus III, three had completed calculus I, and one student had only completed college algebra. None of the students had taken a junior-level collegiate math course, but two students were familiar with some group theory concepts from taking bridge course the previous year. A retrospective analysis (Cobb & Whitenack, 1996; Stylianides, 2005) was conducted on the classroom video data collected from this term-long teaching experiment. In the initial pass of the data, the researcher identified instances in which students may have had opportunities to address intellectual needs—specifically, where they were confronted with a problematic situation that was unsolvable by their current knowledge (Harel, 2011). The majority of these problems came directly from the instructional prompts that were part of Larsen's curriculum, but other problematic situations originated either from the teacher or from students in the class. In a second pass of selected classroom episodes, students' acts of axiomatizing were analyzed and correlated (when possible) with Harel's existing categories of intellectual need.

**Preliminary results of the research**

At this point in the analysis, a few themes have emerged. First, the data lends credence to one of Harel's claims about the need for computation—that it is indeed a robust intellectual need. Eight of the nine students in the teaching experiment seemed to be motivated to invent rules that aided in computation and for one student in particular, the associative axiom seemed completely unnecessary because he saw no computational need for it. Secondly, the data suggests that axiomatizing is a mathematical activity that could provide students with opportunities to address a variety of intellectual needs. For example, globally there existed a constant tension between the need to create rules that made students' computations more efficient, while at the same time, keeping the list of rules as small as possible to avoid redundancy. This tension provided an opportunity to discuss the differences among mathematical terms such as definitions, axioms, theorems, and lemmas and to point out the advantages and disadvantages of lengthening or

shortening the list of rules. As the students' model progressed from a *model of* toward a *model for*, decisions about how to state certain axioms appeared to be influenced by their needs to communicate, compute, and structure. In fact, throughout the term, the students formally axiomatized five different versions of their list of rules, which provides strong evidence for the existence of the need for structure. Finally, there is evidence in the data to support re-examining Harel's initial category of the need for elegance, which he described as "what we associate with mathematical beauty, efficiency, and abstraction" (1998, p. 502). In making decisions about notational conventions and which rules to keep or discard, students' choices may be motivated not only by the existing categories of intellectual need, but also by an intellectual need that is epistemic to the discipline of mathematics—the need for elegance. One of the students in the teaching experiment seemed to be periodically motivated by this need and used a powerful metaphor to describe the need for elegance of an axiomatic system, as this excerpt illustrates:

Chris: It's like you know, you got a hammer sitting at home…you get a blue hammer. You go out and get a blue hammer, so you hammer in nails with a blue hammer instead of a red hammer. Cuz we already got the red hammer and the red hammer works just as well to solve the problems as the blue hammer…and we already have it.

Later, Chris acknowledged that the creation of a new axiom would make certain computations "faster," but he stated that such an axiom did not make the system "stronger." Sinclair (2004) adds to the importance of this need by stating, "In terms of the aesthetic dimension of mathematical judgments, the emphasis placed on the aesthetic qualities of a result implies a belief that mathematics is not just about a search for truth, but also a search for beauty and elegance" (p. 269).

**Questions to further future research**

In traditional mathematics curriculum, students are rarely given opportunities to develop their own notations, conventions, or axioms, so examining the role that students' intellectual needs plays in designing and enacting RME-inspired curriculum may be very useful for the field. In particular, Harel (2011) claims that "DNR's Necessity Principle is an analogue of the RME dictum that students must engage in mathematical activities that are real to them, for which they see a purpose" (p. 23). If that is the case, then how do other acts of mathematizing correlate with DNR's categories of intellectual need?

Another area that might be worthy of future investigation concerns the function and role of bridge courses. If one of the primary functions of bridge courses is "to ease the transition from lower division, more *computational* [emphasis added], mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus" (Selden & Selden, 1995, p. 135), then it seems reasonable that students in bridge courses should engage in mathematical activities that give them opportunities to address intellectual needs other than those necessitated by computation. Arguably, proof and the activities associated with it attend to this larger goal, so it is not surprising that much of the research on bridge courses has centered upon proof. However, in addition to proof, what other elements could or should be included in bridge courses to support student learning of more abstract mathematics?

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# Towards a Description of Symbol Sense for Statistics

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**Abstract:**

This theoretical report aligns itself with Arcavi's (1994) work and the tradition of onto-semiotic research in mathematics education (Font, Godino, & D'Amore, 2007) and is situated in the context of statistics education. This report will:

- articulate a notion of symbol sense in statistics
- explain the importance to student understanding of the development of symbol sense.

The goal of this work is to guide both research and curriculum design efforts for introductory undergraduate statistics courses. The paper begins by describing statistical analogs of Arcavi's algebraic symbol sense, then furthers this by noting the importance of reading symbols generally, reading symbols through the context of the question, and the reading of symbols related to the visualization or selection of the display. Finally, the paper briefly explores how the understanding of symbols becomes more difficult and important in the use of the Central Limit Theorem and estimation of parameters.

**Keywords**: Statistics, symbols, symbol sense, semiotics

## 0. Introduction and Motivation

While there have been investigations of students' understanding of measures of center (Mayen, Diaz, Batanero, 2009; Watier, Lamontagne, & Chartier, 2011), variation (Peters, 2011; Watson, 2009; Zieffler & Garfield, 2009), and students' preconceptions of the terms related to statistics (Kaplan, Fisher, & Rogness, 2009). A literature search of the titles, keywords and abstracts of all papers in the *Journal of Statistics Education* and the *Statistics Education Research Journal* suggest that none had a primary focus on investigating and exploring students' use and understanding of the symbolic system of statistics although one paper did draw upon the onto-semiotic tradition to describe student errors related to representations of the mean and median (Mayen, Diaz, Batanero, 2009).

The research on students' conceptual understanding of statistical concepts has, thus far, avoided discussion of the importance of representation. Yet, the onto-semiotic research tradition proposes that "Representations cannot be understood on their own. An equation or specific formula, a particular graph in a Cartesian system only acquires meaning as part of a larger *system* with established meanings and conventions" (Font, Godino, & D'Amore, 2007, p. 6). The implication is that, without considering the effects of the different representations of the concepts under study, researchers are being, to use Font, Godino and D'Amore's term, naïve in their study of students' thinking. In particular, they argue:

> In the onto-semiotic approach, the introduction of the unitary-systemic duality in the analysis of the representations enables us to reformulate the naïve vision that *there is one 'same' object with different representations*. What there is, is a complex system of

practices in which each one of the different pairs object/representation (without segregating them) makes possible a subset of the set of practices that are considered to be the meaning of the object (p. 7).

Within the realm of statistics, even when the object under consideration seems relatively simple, such as the mean, there are often multiple symbolic representations (such as $\bar{x} = \sum x / n$ ) which are interchangeably used, by statisticians without consideration of any other type of representation (graphical, verbal etc). When moving to a more complex idea such as the standard deviation of a sample mean, the individual paired relationships between object and representations become even more complex due to a layering of representations. These different possible pairs can arguably convey entirely different meanings of the same object.

Arcavi's (1994) seminal article on symbol sense in mathematics, while not explicitly situated within the tradition of onto-semiotic adopted the position that symbolic understanding and fluency was an important component in knowing and doing algebra. That is, fluency with particular types of mathematics required fluency with a broad range or presentations, including the symbolic. In particular, Arcavi claimed that students should, at minimum:

- Know how and when symbols can and should be used in order to display relationships
- Have a feeling for when to abandon symbols in favor of other approaches
- Have an ability to select a representation and, if necessary, change it
- Understand "the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one's one intuitions or with the expected outcomes of that problem" (p. 31).

It is certainly true that algebraic skills do support students' ability to do and understand statistical concepts (Lunsford & Poppin, 2011). As a result, we argue that there are reasonable analogues to Arcavi's habits and skills in the realm of probability and statistics that are important to consider.

**1. Research Aims**

This theoretical report aligns itself with Arcavi's (1994) work and the tradition of onto-semiotic research in mathematics education (Font, Godino, & D'Amore, 2007) and is situated in the context of statistics education. This report will:

- articulate a notion of symbol sense in statistics
- explain the importance to student understanding of the development of symbol sense.

The goal of this work is to guide both research and curriculum design efforts for introductory undergraduate statistics courses.

**2. Theoretical Perspective**

Symbolic representations are regarded as particularly critical due to Hewitt's (1999, 2001a, 2001b) distinction between arbitrary and necessary elements of the mathematical system. Hewitt notes that names, symbols and other aspects of a representation system are culturally agreed upon conventions and, while many may feel sensible, when a member of a community of practice has an understanding of the culture, "names and labels can feel arbitrary for students, in the sense that there does not appear to be any reason why something *has* to be called that particular name. Indeed, there is no reason why something has to be given a particular name" (1999, p. 3). Hewitt continues by differentiating between those aspects of a concept used by a community of practice which can only be learned by being told and then memorizing, which he labels arbitrary, and those which can be learned or understood through exploration and practice,

which he labels necessary. Additionally he notes that for students to become proficient at communicating with established members of the community of practice, they must both memorize the arbitrary elements and correctly associate them with appropriate understandings of the necessary elements.

Eco (1976) gave the term *semiotic function* to describe the dependence between a text and its components and between the components. The semiotic function relates the antecedent (that which is being signified) and the consequent sign (or that which symbolizes the antecedent) (Noth, 1995). When considering the statistical community and the representation system in use within that community, have defined a complex web of semiotic functions and shared concepts that "take into account the essentially relational nature of mathematics and generalize the notion of representation: the role of representation is not totally undertaken by language (oral, written, gestures, …)" (Font, Godino, & D'Amore, 2007, p. 4). Throughout this paper, we recognize the inherent arbitrary nature of much of the symbolic system of statistics and draw on the notion of *semiotic function* as a means of linking a particular representation with the relevant concept. In doing so, we articulate specific linkages that students should be developing and describe some of the difficulties and potential pitfalls of the symbolic system.

**3. A Notion of Statistical Symbol Sense.**

Many of the habits and skills that Arcavi (1994) described have a natural analog in statistics. Most important of these is knowing how and when symbols can and should be used. In mathematics a symbol typically represents an unknown or is defined to represent a single mathematical concept; however, in statistics symbols often carry multiple layers of meaning. For example, both $\bar{x}$ and μ are well defined as an arithmetic mean; however, each has a second layer definition defining what type of data set the arithmetic mean comes from; $\bar{x}$ is the mean of a set of sample data and μ is the mean of a set of population data. This additional layer of information is crucial in displaying relationships, that is, should be encoded in the semiotic function linking the representation (symbol) and the concept of mean, and should be a part of a student's statistical skill set at the end of a course.

Arcavi also recommends knowing when to abandon symbols in favor of other approaches. This has a non-mathematical application to statistics. While statistical procedure revolves around the relationship between symbols and their relationship to a sample and a population the practical use of statistics is much less technical. In many instances statistics is the tool used to explain or reason about something in a different discipline such as psychology or biology; disciplines that are not necessarily rooted in mathematics. It is important to be able to abandon descriptive symbols in favor of concise statements such that a hypothesis or a conclusion can be interpreted without understanding what a symbol represents. A student should not only be able to abandon formal symbol representation, but be able to "translate" symbolic statements into something easily understood by all.

Finally, Arcavi states that a constant check of symbol meanings during problem solving is needed. In statistics, the multi-layered meaning of symbols makes this important. Additionally, there are general mathematical symbols that are mathematical operators; however, in statistics it is a general rule that a Greek symbol represents a population summary and a Roman symbol represents a sample summary, but there are times when Greek and Roman symbols are nearly indistinguishable such as with *Nu*. A student might see $N = 25$, and not understand why one is to use capital N for a population and lower-case N for a sample while a statistician might be surprised that the student does not recognize *Nu*! Thus, from the different perspectives, a symbol might be completely reasonable or seemingly arbitrary. This continues

with inclusion of symbols such as "∑" as operators, rather than conveying information about a population, will sometimes confuse students and makes these general rules less clear than intended.

**3.1 An expansion of Arcavi's list.**

The following section will briefly outline a few ideas that might be understood as forming part of a statistical symbol sense. It is important that students have a clear understanding of relevant terms and be able to correctly associate each term with the most appropriate symbol. Beyond that, students should:

- Understand, in the context of a given problem, which symbols represent constants (even if unknowable) and which represent values that can vary.
- Understand that symbols which are constant for a given problem can also be understood as varying across problem contexts.
- Possess a feeling for when symbols should be used to display relationships and when visual representations better convey appropriate information.
- Demonstrate an ability to read symbolic expressions for meaning, both in the context of the problem, while also connecting them to their abstracted.
- Consistently check the meaning of the symbols against the problem and with their own intuition.
- Possess an understanding of the difference between different symbols that represent the same basic concept (such as a sample mean versus a population mean).

To illustrate these, we will use the standard error of a sample mean. In explaining how this case illustrates aspects of a statistical symbol sense, we will concentrate on two of the bullet points above; the need to understand constants and variables, as well as the ability to read expressions for meaning.

Because the standard error of a sample mean requires the creation of a sampling distribution, it would be helpful if students had a dynamic image in their heads of samples being created from the original population, each sample being of size *n*. Then, for each sample, the sample mean is computed and the distribution is created. This distribution also has a fixed mean and standard deviation. The mean of the sampling distribution is at the same value as the mean of the original distribution, that is, a subtle point that is too often glossed over. The mean of all possible sample means is the same as the mean of the original population. The standard deviation of the sample means is measuring the spread of the sample means of size *n* from their mean. That is, this formula is meant as a measure of how spread out a population of sample means is. In order to make sense of this formula, it requires the students to have constructed a mental landscape with the ability to operate on at least two levels of abstraction; one is relatively low and is the original distribution, while the second is relatively high and asks students to contemplate the distribution of all possible sample means of size *n* where the individual samples are drawn from the original distribution. Let's now describe some of the reasons that understanding this formula may be problematic for students, and, how a statistical symbol sense would help.

**3.2 Reading of symbols.**

When students confront the equation $\sigma_{\bar{x}} = \sigma/\sqrt{n}$, one of their first realizations should be the formula mixes notation for populations and samples (σ and *n*, respectively). As a result, students need to have a decision rule that allows them to understand what is being described; is this formula describing a sample? A population? In fact, this formula is describing an entirely *new* distribution, one that is distinct from the original population, demands consideration of a

sample of size $n$, and is based upon the old distribution. In order to realize this the students need to recognize that when elements related to a population and sample are mixed, the students need to realize that the new symbol must be describing a sampling distribution.

The students should also look at the equation and read in terms of how the standard deviation of the sample mean compares with the original standard deviation. Students should ask themselves what division by the square root of $n$ does, especially as $n$ varies. Students should ask, what happens when $n$ is 1? Students should understand that this would recapitulate the original distribution, both because each 'sample' would be exactly one individual (meaning that each individual in the population is then in exactly one sample) and because the symbols show that the square root of 1 is 1, and then the standard deviation of the sample means is the same as the standard deviation of the population because of division by 1. Then, the students should be able to explain how the value of the standard deviation of the sample means will change as the sample size increases by nothing that sigma is a constant and, then, division by an increasing value will cause a corresponding decrease in the final result. The students should imagine the distribution (the graphical representative) collapsing about the mean in a dynamic way.

*Insert Diagram 1a: A normal distribution and the distribution of sample means from samples of size 2, 10 and 100.*

*Insert Diagram 1b: A normal distribution and the distribution of sample means (n = 2, 10 and 100) scaled towards the parent distribution*

**3.3 Reading symbols for meaning related to the problem**.

A student must be able to answer "What can vary?" and "What's constant, even if unknown?" to fully understand a problem. In the context of the formula above, students should be asking themselves these questions. Yet, the answers require a non-trivial ability to negotiate between contextualized and generalized understandings. At the most general, both $\sigma$ and $n$ can be understood as varying, the formula is applicable to all distributions, and, therefore, any sigma. But, in most situations that the students encounter, they should be thinking in terms of a specific underlying distribution, which means that $\sigma$ is fixed; although, it may be unknown (which the students should be able to discern). Yet, we want the students to understand that once the population, and thereby $\sigma$ is fixed, that by changing sample sizes they create a large number of different sampling distributions. That requires students to understand the sample size $n$ as able to vary and we should teach them to think this way.

To liken this to an element of algebra, when students consider quadratic functions, they should understand that $f(x) = ax^2$ gives rise to a quadratic, and, that for a particular instance, $a$ is fixed, but we also want them to understand that $a$ can vary and what that variation does to the function. Yet, they also need to be able to proceed into further contextualized problems where $n$ has also been fixed and they, then, need to be able to picture the shape of the distribution and describe what effect $n$ has on the shape of the distribution. Students might do this by drawing an appropriate picture of the distribution with ranges variation, as described by differences from the mean, marked.

The example of the standard error of a sample mean is an example of a concept that, when understood, makes understanding expected results straightforward. It is this concept of what is expected that is a building block of statistical inference. Students often dive into

inference without conceptual understanding of what "should" happen under the premises provided. The ability to read expressions for meaning is a skill we should expect of statistics students. If a student has information about σ, then that student should have the ability to infer what outcomes for the sample mean are most common, and how they vary. This skill, directly leads to the concept of "unlikely events" and a student can then infer what is likely versus what is unlikely by only understanding what the premise of the problem.

**3.4 On visualization and selection of the display.**

One of the challenges for students in understanding the sampling distribution is making sense of what individuals represent. They typically begin a statistics class by exploring data where an individual is a single measurement from one member of the population under study. This might be a heartbeat, count of siblings, or Likert scale rating, but, each number could be understood as describing one individual and often a person. That is, a single thing that could be visualized. When students start to consider a sampling distribution, the individual members of the population are now samples, and the measurement of each individual that we are considering is a mean. That is, we have asked the students to operate on, as an individual, this concept that was originally introduced as a collection of individuals.

When we talk about visualizations of distributions, we might want students to visualize the individuals in the original distribution being selected into the sample. Then, they need to see the sample mean becoming an individual in the sampling distribution. Let us look at a diagram that might depict these ideas.

*Insert Diagram 2a: A normal distribution with a sample of 13 plotted and the meanof that sample identified.*

*Insert Diagram 2b: The distribution of all sample means (of size 13) from a normal distribution with the sample mean of the 13 points from Diagram 2a shown.*

**4.0 Pointing towards more advanced statistical concepts**

Finally, we will to undergird this discussion, with a few extensions, outlined here and to be discussed in more detail in the presentation and subsequent papers. We first note the complications in understanding that result from estimated constants, and, we'll discuss the role of the standard error in the Central Limit Theorem (CLT) and how the coordinated understandings that we have described above are essential to understanding the CLT. Oversimplifying a bit, the CLT is a weak convergence theorem that states the conditions under which a sampling distribution approaches normality. It is one of the most important results in probability and lies at the heart of much of the inferential statistics taught in an introductory course. For the purposes of an introductory statistics course, Moore offers the following statement, "as we take more and more observations at random from *any* population, the distribution of the mean of these observations eventually gets close to a normal distribution. (There are some technical qualifications to this big fact, but in practice we can ignore them.)" (2001, p. 488). The normal distribution of sample means that is being defined has the standard error as one of its two parameters.

There are instances when a sampling distribution for the sample mean is desired, but the parent distribution is unknown. In this situation one is able to apply the central limit theorem and obtain an estimated sampling distribution. Without access to information from the parent distribution, one must estimate the constants based on information gathered in the sample. These

estimated constants are denoted by a "hat" and are able to vary. Understanding when a sampling distribution is reported verses when an estimated sampling distribution is reported and the differences between them is an important skill that is confusing due to similar looking parameters as both are denoted with marks above the symbol.

A further generalization is the standard error's importance to a student's understanding of the Central Limit Theorem (CLT). A primary goal in an introductory statistics course is to convey an understanding of the CLT. The relationship between a distribution's variance and the standard error that defines the distribution of its sample means is a fundamental component in inference and CLT. A student must be able to ascertain if the true standard error is being used or an estimate and be able to understand and convey how that affects any inferential conclusions.

**5.0 Summary**

The list of behaviors and understandings (including semiotic functions) proposed above is knowingly incomplete. It is meant as a beginning description of the significant difficulties that students face in coming to know statistics. We believe it helpful as a first step for researchers in statistics education in that it can set the direction for future research. It reminds us that understanding is multi-faceted and that symbol reading, recognition, and use is intimately tied to students' conceptual development. For instructors, we believe that this description can raise awareness of the issues, emphasizes the difficulties for students, and argues for more targeted teaching and explicit descriptions of the codes carried by the symbols (perhaps explanations of why a particular symbol was chosen to represent a particular concept). Finally, we note the overall inadequacy of merely cataloguing and argue that significantly more work is needed in this field to further explore the types of needs that learners have, the means by which people develop appropriate (and inappropriate) semiotic functions and symbol sense, and the development of instructional sequences that support students' learning.

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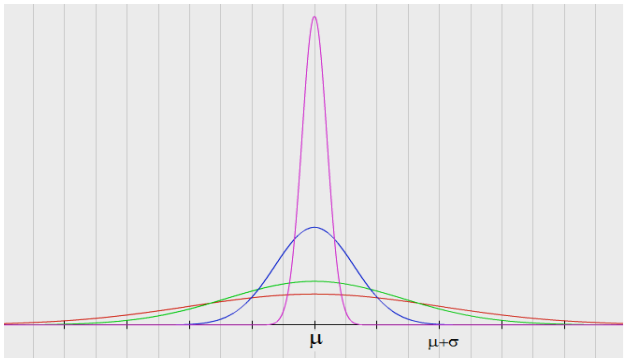

Diagram 1a: A normal distribution and the distribution of sample means from samples of size 2, 10 and 100.

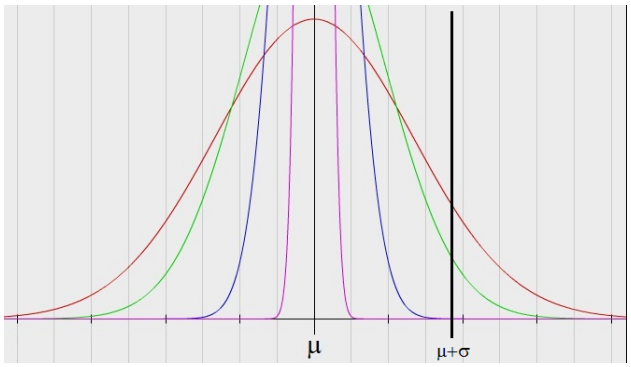


Diagram 1b: A normal distribution and the distribution of sample means (n = 2, 10 and 100) scaled towards the parent distribution

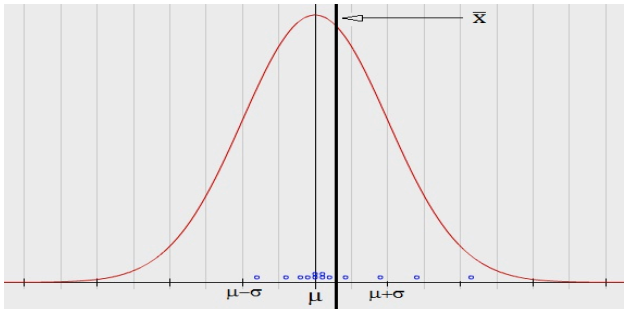

Diagram 2a: A normal distribution with a sample of 13 plotted and the mean of that sample identified.

Diagram 2b: The distribution of all sample means (of size 13) from a normal distribution with the sample mean of the 13 points from Diagram 2a shown.

**Promoting students' object-based reasoning with infinite sets**

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Abstract: Recent research about the thorny Tennis Ball Problem has revealed that students respond in different ways depending on the type of properties they generalize from the finite steps to the envisioned final state of the infinite process. These generalizations, in turn, depend on the features of the finite steps their attention is directed toward. Undergraduate students who attend to the labeling of objects, rather than simply counting the objects, are using object-based reasoning, which is crucial to their ability to understand Cantorian set theory. We propose a sequence of tasks centered around the Tennis Ball Problem that our research has shown to help students build object-based reasoning.

Keywords: infinite process, encapsulation, transfer

*The Tennis Ball Problem and student reasoning about infinite sets*

In the last decade, the Tennis Ball Problem has been used to reveal how students reason about infinite processes (Dubinsky, Weller, McDonald, & Brown 2005; Dubinsky, Weller, Stenger, & Vidakovic 2008; Ely 2007, 2011; Mamolo & Zazkis 2008; Radu 2009; Radu & Weber 2011; Weller, Brown, Dubinsky, McDonald, & Stenger 2004). Although this problem appeared at least as early as Littlewood (1953), we believe its appearance in education research was largely due to Falk (1994). One variant of the problem is this:

Suppose you are given an infinite set of numbered tennis balls (1, 2, 3,...) and two bins of unlimited capacity, labeled A and B. At step 1 you place balls 1 and 2 in bin A and then move ball 1 to bin B. At step 2 you place balls 3 and 4 in bin A and then move ball 2 to bin B. At step 3 you place balls 5 and 6 in bin A and then move ball 3 to bin B. This process is continued in this manner ad infinitum. Now assume that all steps have been completed. What are the contents of the two bins at this point?

Undergraduate and graduate students typically produce three kinds of solutions. The most common is (i) the infinitely-many-balls answer: Bin A contains infinitely many balls, or Bin A contains "half of infinity" (e.g., Ely 2007, 2011; Mamolo & Zazkis 2009; Radu 2009; Radu & Weber 2011). A typical reason given is that after every step there is one more ball in each bin than there was after the previous step, so after infinitely many steps each bin will hold infinitely many balls. Another response that is less common is (ii) the empty-bin answer: Bin A contains no balls. The reason here is that on step 1 ball 1 is moved from Bin A to Bin B, on step 2 ball 2 is moved from Bin A to Bin B, and so on, so that after infinitely many steps all of the balls have

been moved out of Bin A into Bin B. A third type of response is (iii) that you can't answer the question because "you're never done moving the balls." Many of the students who produce this third answer are also willing to produce one of the other two answers as well; sometimes it helps if they hear a "time-sensitive" version of the problem in which the steps are performed at 1 minute till noon, ½ minute till noon, ¼ minute till noon, etc. (step $k$ occurs at $1/2^k$ minutes till noon), and then they are asked what the bins hold at noon (Radu 2009; Ely 2011). In addition, one could choose to ask what the "limit state" is rather than asking about the process being completed.

Some researchers consider the empty-bin answer to be unambiguously the correct one (Dubinsky, et al. 2005; Dubinsky et al. 2008; Mamolo & Zazkis 2008; Weller, et al. 2004), although research in philosophy indicates that this conclusion is far from obvious (Allis & Koetsier 1991, 1995; van Bendegem 1994). Our purpose is not to take a stance on this issue, but rather to discuss (a) what student responses to this problem indicate about student reasoning and (b) how this problem, and other carefully-designed problems involving infinite processes, can be used to promote what Radu & Weber (2011) call "object-based reasoning" (OBJ) among upper-level mathematics students. As we explain later, our interest in helping students engage in object-based reasoning in the context of infinite processes stems from our belief that this type of reasoning is important in Cantorian set theory, particularly because it promotes the understanding of correspondences between infinite sets.

Researchers who use APOS (action, process, object, schema) theory to interpret student responses on this problem focus on how students who produce the infinitely-many-balls answer are unable to treat the infinite process as being encapsulated into a single object (Dubinsky, et al. 2005; Dubinsky et al. 2008; Mamolo & Zazkis 2008; Weller, et al. 2004). On the other hand, we have found students who can encapsulate the infinite process but who defend either the infinitely-many-balls answer or the empty-bin answer or both, based on which properties of the finite states they choose to generalize when envisioning a final state. Furthermore, these generalizations depend on the properties of the infinite process their attention is directed toward. For instance, students who were asked "which balls" were in the bin (instead of "how many") were much more likely to generate and be able to explain the empty bin solution, even if they personally preferred the infinitely-many-balls solution more. When students provide the infinitely-many-balls solution, they generalize properties of count or cardinality from the finite states—they count the number in the bin at each step and generalize that this count is always growing. They ignore the labels on the balls entirely, and often claim that the problem would be exactly the same as the "odd-even" version (see below). On the other hand, when students provide the empty-bin answer, they instead attend to and generalize to the final state a pattern in the labeling of the objects in the finite states (Ely 2011). These properties that are generalized from the finite states to an envisioned final state have been termed infinite projections (Ely 2011).

By attending to the labeling on the balls, students demonstrate object-based reasoning (Radu & Weber 2011), rather than reasoning by counting or cardinality (or "rate", Mamolo & Zazkis 2009). In this context the term "object-based" is not meant to be in contrast with "process-based;" it has nothing to do with whether a student views the infinite process or its result as an object or a process. Rather, object-based reasoning (OBJ) indicates that the student primarily

attends to, and generalizes properties of, objects. The student focuses on objects and where they end up rather than on sets and a trend in their sizes. By focusing on the objects first, students who use this kind of reasoning attend to the way the objects are labeled, not just how many there are at finite states.

We argue that the ability to use OBJ is important in advanced mathematical thinking, and it is crucial to Cantorian set theory. In order to extend the notion of "size" from finite sets to infinite ones in Cantor's way, it is not the count of the objects, but rather the way that they are indexed or labeled, that is important to attend to. This is counterintuitive—when we count a finite set the labeling, the way that we temporarily assign names to the objects in the set ("one", "two", "three", …) is unimportant. The last name we say is what is important. When ascertaining the size of an infinite set, the idea of the last number loses all importance but the way that we index the set becomes crucial. The set's size is determined by what kind of set suffices for indexing it.

For example, a problem that might appear in an upper-level mathematics course is to suppose $Q=\{x_1, x_2, ..., x_n, \ldots\}$, and let $B_n = \{x_1, x_2, ..., x_n\}$ and $A_n = Q - B_n$. How many elements are in each $A_n$? What is the intersection of all the $A_n$s? With object-based reasoning, the student is able to fix a given element and look at what happens to the element as $n$ increases, rather than to consider only the sizes of the sets and what happens to those sizes. In fact, the explicit notion of the limit of a sequence of sets can be found in some courses, where an upper limit set (which contains all elements that are contained in infinitely many sets in the sequence), and a lower limit set (which contains all elements that are eventually in the sets of the sequence, and a notion of the convergence of a sequence of sets precisely if its lower and upper limit coincide (e.g., Hausdorff 1957). Such a situation requires OBJ, because the limit set contains each elements that ends up in all of the $A_n$s from some point onward.

It is OBJ, particularly the attention to and generalization of labeling rather than count, that is indicated by a student's ability to understand and justify the empty-bin answer to the Tennis Ball Problem. It is for this reason that we want students to be able to envision and to explore the implications of the empty-bin solution, not because we believe that this solution is uniquely and unambiguously correct. For this reason, we devised a sequence of activities with infinite processes that focus on developing students' object-based reasoning. Based on how undergraduate students' thinking developed with these activities in a teaching experiment, we propose a sequence of activities that could be used for developing object-based reasoning for mathematics majors (Radu 2009; Radu & Weber 2011).

Based on our research with these problems, one way students' object-based reasoning was promoted was when they were asked to investigate features of an envisioned "final state," even if they themselves were not willing to commit to the answer they were exploring the implications of (Ely 2011). Because it is ambiguous how to mathematically model the Tennis Ball Problem context using a sequence of sets with a specified metric for convergence, it is possible for the discourse to devolve into a debate about this mathematization process, which, while potentially worthwhile from a broader mathematical point of view, is unproductive for developing students' object-based reasoning. By instead bringing focus to how the properties of the finite states are generalized or extended to the envisioned final state, particularly to the property of labeling, rather than counting, the instructor can help foster the development of object-based reasoning.

The sequence of problems is in keeping with Wagner's theory of transfer in pieces (2006). According to this framework, transfer of knowledge is a complex process during which an initially topical set of principles is constantly refined to *account for* (and not ignore) the new contexts of the problems encountered as one progresses through a sequence of problems with a common mathematical core. Thus, the acquisition of abstract knowledge can be seen as a *consequence* of transfer and not a required initial component for it to happen. In our own work with students, we found students did not abstract general principles from one of these problem contexts and then apply them to another. Rather, as they worked through a class of related problems, cross-references between prior and current tasks were made based on perceived structural commonalities among the tasks, which often resulted in changes in students' reasoning on one or more of the tasks involved in the comparison, and thus in the refinement and expansion of topical principles (Radu 2009; Radu & Weber 2011). Below we present a proposed sequence of tasks designed to help students envision object-based reasoning, accompanied by the rationale for each task.

*A sequence of tasks that support object-based reasoning*

1. The Tennis Ball Problem (described at the beginning of this paper)

This can serve as an informal assessment of how the students react to an infinite process problem that challenges them to envision a limit (final) state, and what infinite projections they focus on (if any).

2. The Odd-Even Tennis Ball Problem

This problem is a variation of the first problem: at step $n$, balls $2n$ and $2n$-1 are placed in bin A, then ball $2n$-1 is moved from bin A to bin B. It can be used for two purposes:
i) with student(s) who cannot envision *any* limit state to the original Tennis Ball Problem. Since in the odd-even problem each individual ball is affected (moved) by exactly one step, students will likely have no difficulty in envisioning a limit state where bin A contains all even-numbered balls and bin B all odd-numbered balls.
ii) with students who could envision only an "infinitely-many-balls" limit state for bin A for the original Tennis Ball Problem. In the context of the Odd-Even version, once the student envisions the odd-even limit state, the facilitator can ask questions about specific balls (e.g., why is ball 5 in bin B?), thus helping the student reflect on the action of the steps of the process on a particular ball and how that affects the position of that particular ball with respect to the limit state. Finally, students can discuss the difference between this problem and the original Tennis Ball Problem. In our experience, students who are reasoning according to count rather than using object-based reasoning will consider the two problems to be the same, but that in the Odd-Even version one is more certain about which balls remain in the bin.

3. The Vector Problem. Let $v = (1, 0, 0, 0,...)$. You are going to "edit" this vector step by step.

- Step 1: v = (0, 1, 2, 0, 0, ...)
- Step 2: v = (0, 0, 1, 2, 3, 0, 0, ...)
- Step 3: v = (0, 0, 0, 1, 2, 3, 4, 0, 0 ...)

……..

This process is continued ad infinitum. Now assume ALL steps have been completed.

Describe v at this point.

In a teaching experiment with four math majors, each student easily employed OBJ in the context of this problem. Furthermore, the discussion of the Vector problem evoked spontaneous references to the Tennis Ball Problem. In students who had previously produced only rate/cardinality reasoning to the Tennis Ball Problem, these back references to this problem resulted in students envisioning what OBJ reasoning may mean in that context for the first time. In cases where the student had envisioned both OBJ and rate/cardinality reasoning when working on the Tennis Ball problem in a prior session, work on the Vector problem resulted in students' revisiting of the Tennis Ball problem and ending up preferring OBJ over rate/cardinality arguments. We argue that the vector context of this particular problem encouraged students to focus on individual positions/objects and made it less likely that they would focus on cardinality issues (given that there's no evident "growing set" in this process). For more detailed discussion of student episodes related to the Vector Problem see Radu (2009) and Radu and Weber (2011).

4. The 10-Marble Problem.

This problem is similar to the Tennis Ball Problem, except that at step $n$, marbles $10n$ - 9 through $10n$ are put in a bin, and then marble $n$ is removed from the bin. In this problem the set of marbles in the bin "grows" by 9 marbles at each step, which may make it even more counterintuitive to students to envision using OBJ and claim that the limit state is the empty set. The role of this task at this point in the sequence (after the likely OBJ-inducive Vector Problem) is to explore the students' reaction to a task whose context strongly encourages a rate/cardinality approach. For detailed accounts of student reasoning on a timed version of this problem see Mamolo & Zazkis 2008.

5. The Writer Problem. Tristram Shandy, the hero of a novel by Laurence Sterne, starts writing his biography at age 40. He writes it so conscientiously that it takes him *one week* to lay down the events of *one day*. If he is to document each day of his life and the pace at which he writes remains constant, can you envision a situation in which his autobiography can be completed?

This task offers a significant change of context, in the sense that we are no longer adding and removing objects from a bin. Additionally, the time component may cause the students to bring in a number of real-life considerations while reasoning on this task. The role of the Writer Problem in the sequence is to explore the students' reaction to change of context and influence of real-life surface features of the problem on the students' reasoning.

For a detailed discussion of how two different groups of students progressed through this sequence see Radu (2009). While there were certain differences between the paths of the two groups, what can be said about both groups is that i) there were numerous instances in which the students referenced prior tasks, and often such back references resulted in the students' refining their reasoning on one or more of the tasks involved in the comparison; and ii) The Vector problem elicited OBJ from all students involved and significantly influenced the students' reasoning on the rest of the problems, both prior and subsequent.

The task sequence can be extended to include the case of "oscillating" objects (objects that belong to infinitely many of the intermediate states while also <u>not</u> belonging to other infinitely many intermediate states), as well as processes manipulating objects in an implicit topological space (see Radu 2009 for examples of both). We believe such tasks are of interest from the point of view of transfer theories, but less so from a standard set theory perspective.

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# 1 + 1 = ⊞ : On the Polysemy of Symbols

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*This paper illustrates how mathematical symbols can have different, but related, meanings depending on the context in which they are used. In other words, it illustrates how mathematical symbols are polysemous. In particular, it explores how even basic symbols, such as '+' and '1', may carry with them meaning in 'new' contexts that is inconsistent with their use in 'familiar' contexts. This article illustrates that knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol.*

Key words: ambiguity; polysemy; symbols; addition

Polysemy is a form of lexical ambiguity. A polysemous word is one that has two or more different, but related, meanings. For example, the English word "tie" may refer to an article of clothing worn around the neck or to the action of making a Windsor knot. The ambiguity may be resolved by considering the context in which the word is used. In mathematics, a word may be polysemous if its mathematical meaning is different from its everyday, familiar meaning (Durkin and Shire, 1991), or if it has two related, but different, mathematical meanings (Zazkis, 1998).

In a mathematical discourse, symbols such as +, =, and 1, may also be considered 'words' – they have their own definitions and can be strung together to form coherent mathematical phrases, such as 1+1=2. As such, they may also be a cause of 'lexical' ambiguity. Ambiguity in mathematics is recognized as "an essential characteristic of the conceptual development of the subject" (Byers, 2007, p.77) and as a feature which "opens the door to new ideas, new insights, deeper understanding" (p.78). Gray and Tall (1994) first alerted readers to the inherent ambiguity of symbols, such as 5 + 4, which may be understood both as *processes* and *concepts*, which they termed *procepts*. They advocated for the importance of flexibly interpreting procepts, and suggested that "This ambiguous use of symbolism is at the root of powerful mathematical thinking" (Gray and Tall, 1994, p.125). A flexible interpretation of a symbol can go beyond process-concept duality to include other ambiguities relating to the diverse meanings of that symbol, which in turn may also be the source of powerful mathematical thinking and learning. In this paper I discuss cases of ambiguity connected to the context-dependent definitions of symbols, that is, the *polysemy of symbols*. In particular, I examine the polysemy of the symbol '+' as it manifests in the context of modular arithmetic and transfinite arithmetic. I also present an argument that suggests that the challenges learners face when dealing with polysemous terms are also at hand when dealing with mathematical symbols, focusing on cases where acknowledging the ambiguity in symbolism and explicitly identifying the precise, context-specific, meaning of that symbolism go hand-in-hand with understanding the ideas involved.

## A familiar meaning: The context of natural numbers

The first context in which one encounters the symbol '+' is in natural number arithmetic. The familiar phrase 1+1=2 can be considered as the sum of two cardinalities that are associated with two disjoint sets, and which yields the cardinality of the union set. (However, it is not uncommon to hear children claim, as my title suggests, that 1+1 does *not* equal 2, but instead equals a window.) For the purposes of this paper, let us consider the sum 1+2. In the familiar context of natural numbers, the meaning of 1+2 can be broken down as in Table 1:

| Symbol | Meaning in context of natural numbers |
|---|---|
| 1 | Cardinality of a set containing a single element |
| 2 | Cardinality of a set containing exactly two elements |
| 1+2 | Cardinality of the union set |
| + | Binary operation over the set of natural numbers |

Table 1: Summary of familiar meaning in $\mathbb{N}$

As a binary operation, addition, its definition and properties, depends necessarily upon the domain to which it is applied – and this fact underlies the polysemy of '+'. Building on the idea of addition as a domain-dependent binary operation, the following sections consider two other domains: (i) the set {0, 1, 2} and (ii) the class of (generalised) cardinal numbers. These domains are of interest since: (i) the extended meanings of symbols such as '$a + b$' contribute to results that are inconsistent with the 'familiar', and (ii) they are items in undergraduate mathematics courses and also pre-service teacher mathematics education. It is useful for purposes of clarity to distinguish between different definitions of the addition symbol as they apply to different domains. The symbol $+_N$ will be used to represent addition over the set of natural numbers, $+_Z$ as addition over the set of integers, $+_3$ as addition over the set {0, 1, 2} (i.e. modular arithmetic, base 3), and $+_\infty$ as addition over the class of cardinal numbers (i.e. transfinite arithmetic).

**An extended meaning: The context of modular arithmetic**

Consider the group $\mathbb{Z}_3$ – the set of elements {0, 1, 2} with associated operation addition modulo 3. Within group theory the meanings of symbols such as 0, 1, 2, +, and 1+2 are extended from the familiar in several ways. As an element of $\mathbb{Z}_3$, the symbol 0 is short-hand notation for the congruence class of 0 modulo 3. That is, it is taken to mean the set consisting of all the integral multiples of 3. The symbols 1 and 2 are analogously defined, and the symbol '+' is defined as addition modulo 3. As such, the familiar '1+2' now carries with it meaning quite distinct from before: just as '1' and '2' were, '1+2' is also a congruence class. Dummit and Foote (1999) define the sum of congruence classes by outlining its computation, e.g. 1+2 (modulo 3), is computed by taking *any representative integer* in the set {… -5, -2, 1, 4, 7, …} and *any representative integer* in the set {… -4, -1, 2, 5, 8,…}, and summing them in the 'usual integer way'. Thus, recalling the notation introduced in the previous section, sample computations to satisfy this definition include: $1 +_3 2 = (1 +_Z 2)$ modulo $3 = (1 +_Z 5)$ modulo $3 = (-2 +_Z -1)$ modulo 3 all of which are equal to the congruence class 0. Table 2 below summarizes the meanings of the symbols '1', '2', and '1+2', and '+' when considered within the context of $\mathbb{Z}_3$:

| Symbol | Meaning in context of $\mathbb{Z}_3$ |
|---|---|
| 1 | Congruence class of 1 modulo 3: {… -5, -2, 1, 4, 7, …} |
| 2 | Congruence class of 2 modulo 3: {… -4, -1, 2, 5, 8,…} |
| 1+2 | Congruence class of (1+2) modulo 3: {…, -3, 0, 3, …} |
| + | Binary operation over set {0, 1, 2}; addition modulo 3 |

Table 2: Summary of extended meaning in $\mathbb{Z}_3$

The process of adding congruence classes by adding their representatives is a special case of the more general group theoretic construction of a quotient and quotient group – central ideas in algebra, and ones which have been acknowledged as problematic for learners (e.g. Asiala et al., 1997; Dubinsky et al., 1994).These concepts are challenging and abstract, and are made no less accessible by opaque symbolism. As in the case with words, the extended meaning of a symbol can be interpreted as a metaphoric use of the symbol, and thus may evoke prior knowledge or

experience that is incompatible with the broadened use. In a related discussion, Pimm (1987) notes that "the required mental shifts involved [in extending meaning from everyday language to mathematics] can be extreme, and are often accompanied by great distress, particularly if pupils are unaware that the difficulties they are experiencing are not an inherent problem with the idea itself" (p.107) but instead are a consequence of inappropriately carrying over meaning. A similar situation arises as one must extend their understanding of a mathematical symbol – an important mental shift that is taken for granted when clarification of symbol polysemy remains tacit.

**An extended meaning: The context of transfinite arithmetic**

Transfinite arithmetic may be thought of as an extension of natural number arithmetic – its addends represent cardinalities of finite or infinite sets and a sum is defined as the cardinality of the union of two disjoint sets. Transfinite arithmetic poses many challenges for learners, not the least of which involves appreciating the idea of 'infinity' in terms of cardinalities of sets (i.e. the transfinite numbers $\aleph_0$, $\aleph_1$, $\aleph_2$, …). In resonance with Pimm's (1987) observation regarding negative and complex numbers, the concept of a transfinite number "involves a metaphoric broadening of the notion of number itself" (p.107). In this case, the broadening also includes accommodating properties which are unfamiliar and inconsistent with natural number arithmetic.

Consider a generic example: the sum $\aleph_0 + 1$. It is the cardinality associated with the union set $\mathbb{N} \cup \{\beta\}$, where $\beta \notin \mathbb{N}$. In this context, the addends are elements of the (generalised) class[1] of cardinals, which includes transfinite cardinals. Between the sets $\mathbb{N} \cup \{\beta\}$ and $\mathbb{N}$ there exists a bijection, which, in line with the definition (Cantor, 1915), guarantees that the two sets have the same cardinality – that is, $\aleph_0 + 1 = \aleph_0$. Table 3 summarizes the meaning of these symbols:

| Symbol | Meaning in context of transfinite arithmetic |
|---|---|
| 1 | Cardinality of the set with a single element; class element |
| $\aleph_0$ | Cardinality of $\mathbb{N}$; transfinite number; 'infinity' |
| $\aleph_0 + 1$ | Cardinality of the set $\mathbb{N} \cup \beta$; equal to $\aleph_0$ |
| + | Binary operation over the class of transfinite numbers |

Table 3: Summary of extended meaning in transfinite arithmetic

Similarly, one can show that $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbb{N}$, and that $\aleph_0 + \aleph_0 = \aleph_0$. Thus, whereas with '$+_N$' adding two numbers always results in a new (distinct) number, with '$+_\infty$' there exist non-unique sums. A consequence of non-unique sums is the existence of indeterminate differences. Explicitly, since $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbb{N}$, then $\aleph_0 - \aleph_0$ has no unique resolution. As such, the familiar notion that 'anything minus itself is zero' does not extend to transfinite subtraction. This property is part and parcel to the concept of transfinite numbers. Identifying precisely the context-specific meaning of these symbols ('$+_\infty$' and '$-_\infty$') can help solidify the concept of transfinite numbers, while also deflecting naïve conceptions of infinity as simply a 'big unknown number' by emphasizing that transfinite numbers are different from 'big numbers' since they have different properties and are operated upon (arithmetically) in different ways.

**Concluding Remarks**

This paper illustrates how even basic symbols, such as '+' and '1', may carry with them meaning that is inconsistent with their use in 'familiar' contexts. It focused on cases where acknowledging ambiguity in symbolism and explicitly identifying the precise (extended) meaning of that symbolism is necessary for understanding. While the focus was on examples of how

[1] For distinction between set and class, see Levy (1979).

distinguishing among the symbolic notation of $+_N$, $+_3$, and $+_\infty$ is fundamental to appreciating the subtle (and not-so-subtle) differences among the corresponding addends, this argument has broader application. Just as knowledge of language includes "learning a meaning of a word, learning more than one meaning, and learning how to choose the contextually supported meaning" (Mason et al., 1979, p.64), knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol. Attending to the polysemy of symbols, either as a learner, for a learner, or as a researcher, may expose confusion or inappropriate associations that could otherwise go unresolved. Research in literacy suggests that students "will choose a common meaning, violating the context, when they know one meaning very well" (Mason et al., 1979, p.63). Further research in mathematics education is needed to establish to what degree analogous observations apply as students begin to learn '+' in new contexts.

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