# Example construction in the transition-to-proof classrom

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**Abstract.** Accurately constructing examples and counterexamples is an important component of learning how to write proofs. This study investigates how one instructor of a transition-to-proof course taught students to construct examples, and how her students reacted to the instruction.

*Keywords:* transition-to-proof; undergraduate instruction; example construction

## Introduction

Learning to write proofs is a complicated process, and students develop a variety of beliefs about how to construct a proof (Harel & Sowder, 2007). Using examples is one possible strategy in the proof writing process. Examples can be used for several purposes when developing and proving conjectures (Alcock & Inglis, 2008; Alcock & Weber, 2010; Lockwood, Ellis, & Knuth, 2013).

The term example can have many different meanings in mathematics (Watson & Mason, 2002). Within this study, the term example is limited to a mathematical object which satisfies specific characteristics and illustrates a definition, concept or statement (Moore, 1994). This definition excludes sample proofs, e.g., demonstrations of the direct proof technique or proving by induction. Alcock and Weber (2010) claim that this definition of example is "probably the most common intended meaning of the term when it is used by mathematicians and mathematics educators in the context of proof-oriented mathematics" (p. 2).

**Research questions.** In this study, the following questions are addressed: 1. In what ways do students construct examples effectively and ineffectively on tasks in their transition-to-proof course? 2. How did the instructor teach example construction? 3. What connections are found between the students' construction of examples and the instruction given?

## Literature and Theory

Considerable literature is available on the proving abilities of students and mathematicians, and the use examples on such tasks. However, in the interest of space, much of this literature has been omitted. The review below focuses on the literature concerning the construction of examples, and the role of example in teaching advanced mathematics courses.

**Example construction.** Antonini (2006) sought to answer how examples are constructed by conducting clinical interviews with seven mathematicians. From these interviews three distinct techniques emerged: *trial and error*, *transformation*, and *analysis*. *Trial and error* is characterized by constructing objects, and then testing whether the object has the desired characteristics. *Transformation* is characterized by taking a known object which has some of the necessary characteristics, and then performing adjustments until the object has all the required characteristics. *Analysis* is characterized by deducing additional properties the object has to have. Eventually, this list of properties reaches a point that either a known example is evoked or an algorithm for constructing an example is determined.

Antonini (2006) observed that mathematicians often follow a process of starting with *trial and error* and then using *transformation* only when *trial and error* fails. The *analysis* technique was only used when after failing to construct an example with both the *trial and error* and *transformation* techniques. Antonini (2006) notes that the *analysis* technique is appropriate when there is a possibility that no example with the given properties exists, because the derivation of properties could lead to a proof by contradiction.

Behavior on one task can impact the conceptual knowledge gained from other topics. A particular instance of this occurred in a study by Iannone, Inglis, Mejia-Ramos, Simpson, and Weber (2011), where students were asked to generate examples of a particular type of function. The research team found that most students generated examples with a *trial and error* technique. Other students used a *transformation* technique where they modified known examples, or an analytic technique where the student deduced additional properties of an example. Iannone et al. (2011) theorized that the *trial and error* strategy resulted in weaker conceptual gains than the other strategies.

However, when it comes to the source of the examples used by students, Iannone et al. (2011) found that there was no significant differences between the proof productions of students who generated their own examples and those who were provided examples. This result is contrary to other literature that supports example generation as an important pedagogical tool (Dahlberg & Housman, 1997; Moore, 1994; Watson & Mason, 2002, 2005; Weber, Porter, & Housman, 2008). In fact, Iannone et al. (2011) found that the proof productions of the example reading group was slightly higher than the proof productions of the example generating group, although the difference was not significant.

**The teaching and learning of mathematics.** One of the primary goals of mathematics education is to develop and implement interventions that change mathematics teaching (Fukawa-Connelly, 2012a). At the undergraduate level, Speer, Smith, and Horvath (2010) criticized that "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (p. 99), even though poor undergraduate mathematics teaching is often cited as a reason students change majors away from science, technology, engineering, and mathematics fields (Seymour & Hewitt, 1997). In fact, Mejia-Ramos and Inglis (2009) conducted a literature of 102 mathematics education research papers concerning undergraduate students' experience reading, writing and understanding proofs, yet none of these papers described the instruction the students received. Although some studies have investigated instruction in proof writing since the publication of these critiques (e.g. Fukawa-Connelly, 2012a, 2012b; Mills, 2014), there is still a need for additional studies in this area.

Instruction can influence the choices that students make and their preferences when solving problems, including proofs. Students need strategic knowledge in order to select appropriate strategies (Weber, 2001). It is known that heuristics are difficult to teach, but that students typically do not learn them unless an attempt was made to teach them (Lester, 1994). However, some instructors do try to design the courses they teach in order to explicitly teach students strategic knowledge (Weber, 2004, 2005).

**Theoretical framework.** This study is framed in the emergent framework developed by Cobb and Yackel (1996). This framework links the social perspectives of classroom social norms, sociomathematical norms and classroom mathematical practices to the psychological

Table 1
*The characteristics of the sampled students.*

| Name | Year | Major | GPA | Course Attempt |
|---|---|---|---|---|
| Amy | Sr. | Mathematics for Secondary Teaching | 2.50-2.99 | 3rd |
| Carl | Soph. | Mathematics for Secondary Teaching | 2.50-2.99 | 1st |
| Raul | Jr. | Applied Mathematics and Biochemistry | 3.50-4.00 | 1st |
| Mike | Sr. | Mathematics and Spanish | 3.00-3.49 | 2nd |

perspectives of beliefs about an individual's role in mathematical activity, mathematical beliefs and values, and mathematical conceptions and activity. This study was concerned with the links between the actions of the students, and the activity of the classroom community.

In addition, this study utilized grounded theory, a methodological technique developed by Glaser and Strauss (1967). Within this method, a researcher collects and organizes data by constantly organizing the data into categories or themes (Charmaz, 2006; Creswell, 2013; Glaser & Strauss, 1967; Merriam, 2009).

## Method

The case for this study is a section of a transition-to-proof course at a large university. The participants in this study are the instructor, Dr. S, and the 27 students enrolled in her course during the semester of the study.

Due to the constraints on time and resources, four students were selected for more detailed data collection during the fourth week of the semester using maximal variation sampling (Creswell, 2013). By varying the students' levels of academic success (indicated by a self-reported grade point average), mathematical preparation (indicated by self-reported grades in mathematics coursework), and specialization (pure, applied, secondary teaching, mathematics minor), the findings have increased transferability (Merriam, 2009). The characteristics of the four students included in the sample are presented in Table 1. These students were purposefully selected because they frequently spoke during class, both by asking the professor questions and presenting their own work on the blackboards.

**Data collection.** Several sources of data were used to triangulate the results (Patton, 2002; Merriam, 2009). Interviews were conducted with the four selected students, in order to observe each student's process on proof-related tasks while working independently. These interviews occurred three times during the semester: around the seventh week of the semester, the twelfth week of the semester, and the last week of the semester.

Each interview with a student had three components: a semi-structured portion addressing proof strategies and goals for the course, a task-based portion where students attempted several proof-related tasks, and a reflection on the tasks. The semi-structured portion asked the students to talk about their impressions of the course, namely what they had learned and what they thought they should be learning. The tasks for the interviews were selected from the textbook, or other studies on undergraduate proof writing (Alcock & Weber, 2010; Dahlberg & Housman, 1997; Iannone et al., 2011). The mathematical content of the questions varied over the three interviews, matching the recent content from the course. After a student completed all tasks, then the students were asked to reflect on their work. Sometimes the final reflection was omitted due to poor time management.

The classroom was observed daily to observe the examples used by the instructor during lectures and student presentations. The observations are supplemented by three interviews with the instructor. These interviews focused on the choices made during class and how those choices influenced the desired instructional goals.

## Results

**Construction of examples.** Knowing how to accurately construct examples is of crucial importance for using examples effectively. Two levels of analysis were done: 1) the accuracy of the example, and 2) the construction technique used.

Table 2
*This table summarizes the construction abilities of the students.*

| Construction | Amy | Carl | Raul | Mike | Total |
|---|---|---|---|---|---|
| Accurate Construction | 30 | 16 | 18 | 4 | 68 |
| Inaccurate or Incomplete | 6 | 3 | 3 | 2 | 14 |
| Authoritarian Source | 3 | 2 | 2 | 0 | 7 |
| Trial and Error | 17 | 15 | 11 | 4 | 47 |
| Transformation | 16 | 2 | 8 | 2 | 28 |

Three categories were used to describe the construction technique: *trial and error*, *transformation* and *authoritarian*. An *authoritarian* example is retrieved from a source, instead of being constructed by the prover. The terms *trial and error* and *transformation* were consistent with the definitions of Antonini (2006). Neither the students nor the professor discussed the *analysis* technique, so this category was not used.

The construction of examples was a difficult task for many of the students. During the first interview, Raul and Mike both made errors in constructing examples because they did not know which conditions a construction needed to satisfy to be classified as an example or a counterexample. In fact, they both identified constructions as counterexamples that did not satisfy the hypotheses of the statement.

The students generally constructed examples that were accurate, but their examples were frequently not useful. For instance, Mike was seeking a potential counterexample on a divisibility problem and chose $a = 1$ as the value for the divisor, stating that he chose this value because "1 divides everything." Mike did not realize that this choice for $a$ meant that every possible example would be true. Although other students constructed examples that were not useful for their purpose, this was the only instance in which a student stated a fact that would directly indicate the lack of usefulness.

The students transitioned to more advanced construction techniques late in the semester. During the first interview, Mike was the only students to utilize the *transformation* construction technique, and he only did so once. By the final interview, the students were using the *transformation* technique more frequently than *trial and error*. This interpretation of the result may be conflated with the choices the students make due to the mathematical content. Specifically, the first interview consisted entirely of number theory tasks which the students may have limited previous experience, whereas the final interview concerned real-valued functions and the students should have significant experience with these from their calculus courses. Although the students likely used the *transformation* technique due to increased experience, they also knew more examples of real-valued functions to draw upon as the starting point for the transformation process.

In particular, when asked to construct an example of a fine function on question 3 of interview 3, the first example constructed by each student was a transformation of $y = \sin x$. These students recognized that the pattern of the zeros in $y = \sin x$ could be adjusted to satisfy the conditions of a fine function. It is unlikely that the students could have constructed an example of a fine function via *trial and error* because of how difficult it would be to verify. However, it is equally difficult to imagine a students utilizing a *transformation* technique on $a|(bc)$ implied $a|b$ or $a|c$, especially for an initial example of the statement. Most students will not have a sufficient background in the formal language of divisibility to have such examples in their personal example space.

**The instruction.** Dr. S modeled example construction very rarely during the lecture. Although she presented many examples throughout the semester, she seldom talked about how these examples were constructed. Dr. S did model how to determine which properties an example or counterexample of a statement needs to satisfy, and how to go about verifying that a construction satisfies those properties. Dr. S knew that *trial and error* is the first technique used in example construction, and that the most important aspect of that is knowing which properties need to be verified. Dr. S assigned student presentations that she intended to be opportunities for the students to learn how to construct examples. She knew that the students would often fail before they succeeded at example construction, and that the best way to help the students improve would be to review their constructions attempts during their presentations.

There were two episodes from the lecture where Dr. S emphasized example construction, and the care that must take place when constructing examples. The first instance occurred shortly after defining functions. Dr. S emphasized the importance of a function being well-defined, particularly when the domain is a partition. To do this, Dr. S presented three potential functions:

$$\begin{array}{ll} f : \mathbb{Z}_3 \to \mathbb{Z}_6 & f([x]_3) = [3x + 2]_6 \\ g : \mathbb{Z}_4 \to \mathbb{Z}_2 & g([x]_4) = [3x]_2 \\ h : \mathbb{Q} \to \mathbb{Z} & h\left(\frac{a}{b}\right) = a + b \end{array}$$

The first example was generated using numbers suggested by the students, the last two were purposely chosen by Dr. S. Dr. S showed that $f$ and $h$ are not well-defined by producing counterexamples that show that two different representatives of the equivalence classes produce different outputs. For $g$, Dr. S provided the students with a proof that it was well-defined. Ultimately this episode was demonstrating what it means to be well-defined, but Dr. S knew that this would help the students when constructing their own functions especially in their Modern Algebra course.

Dr. S seldom lectured explicitly about constructing examples and counterexamples, because Dr. S had the expectation that the students would attempt and present many example construction questions on the board, and that these presentations would provide the opportunity to discuss example construction techniques.

Another reason that Dr. S did not lectured about example construction frequently is because she expected the students to utilize *trial and error* by randomly trying constructions and to test whether these are examples or not. Although this is not a sophisticated strategy for example construction, Dr. S believed that students at the earliest stages of proof writing “are not always ready yet” for other strategies. Dr. S wished that the students would move towards the *transformation* construction strategy by asking themselves questions such as “is the statement similar to one [I] know?” and then using that response to construct their example. During the final interview, Dr. S reiterated this by saying “I would like to move them toward more directed examples where they are intentionally trying to go certain places but I doubt that most of them are ready for that. Right now I’m happy if they try random examples to see what’s going on, as long as they don’t stop there.” Perseverance was a frequent theme when discussing proof and example constructions in the lecture.

When the students presented example construction tasks that we incorrect, Dr. S would usually ask the student who presented (or sometimes the whole class) to help her revise the construction. In one instance, Carl presented a relation on $A = 1, 2, 3$ that should have the properties of symmetry, transitivity and not reflexivity. Carl presented the relation $\{(1,2)(2,1)(1,3)(2,3)(3,2)(3,1)\}$, but this example is not transitive. Dr. S argued that if $(1,2)$ and $(2,1)$ are in the relation, then transitivity requires that $(1,1)$ and $(2,2)$ must also be included. As such, Dr. S changed the relation to $\{(1,2),(2,1),(1,1),(2,2)\}$, which is symmetric and transitive, but not reflexive because it is missing $(3,3)$. Through this discussion, Dr. S walked the students through using the *transformation* technique for example construction, since she transformed an existing example to satisfy the given criteria.

**Comparing the instruction and the students.** The students used the *trial and error* construction technique for all of the examples constructed during the first interview, with one exception. However, as the semester progressed the students used the *transformation* technique with increasing frequency. Dr. S predicted this behavior of the students. The *analysis* technique was not demonstrated by the instructor or used by the students; however, during the member checking interview, Dr. S argued that the *analysis* technique was too advanced to be useful to the students at their current level of understanding. In the first interview, Dr. S said

> It depends on the problem, but to some extent, *trial and error* is the very first step. You just try stuff. I’ve seen this even with advanced REU students, where there is a good strategy. They’re not always ready yet. I’m okay with them randomly trying at first. Now, I want them to move toward more careful construction. As they go through this, they should be looking for things that are similar and using that to give them a hint.

Dr. S recognized that as beginning students, they would not have the mathematical experience to use the more advanced *transformation* and *analysis* techniques, but she hoped they would grow to that point. During the same interview, Dr. S elaborated that although she expects the students to have some familiarity with using examples from their calculus classes, “they just never had to construct [examples] themselves before.” As such, some of the difficulties the students had with example construction were expected.

Dr. S did not vocalize an expectation of the accuracy problems exhibited by some the students during the initial interviews. Both Raul and Mike had created examples that violated the statement hypotheses. Raul did not seem to realize that failing the hypotheses was a problem. During the member checking interview, Dr. S said students often make these types of construction errors at this point in their development. She furthered this by explaining that many students present counterexamples that are not actually counterexamples, especially on the first test of the course.

Dr. S usually did not talk about the construction technique when she presented examples to the class. She designed the course so that most of the example construction tasks were assigned as student presentations, and that she would talk about example construction as she reviewed and corrected the examples in the presentations. Unfortunately, the students did not present many problems and they tended to present problems asking for proofs rather than the problems asking for examples. Consequently, Dr. S did not have the opportunity to talk about construction techniques with the expected frequency.

Overall, Dr. S had the experience to know the capabilities of the students with respect to example construction. She recognized that *trial and error* would be the primary technique at the beginning of the semester, and that many of the students would not be able to move beyond that technique in this course. However, towards the end of the semester, she introduced the *transformation* construction technique for the benefit of the students who were ready for more advanced techniques. The students in the sample were able to apply the *transformation* technique in some circumstances, and likely will be able to utilize it more frequently in their subsequent courses.

## Discussion

By the end of the semester, all of the students were selecting examples with more thought, and used the *transformation* construction technique with increased frequency. It is unclear exactly what caused this growth. Possible explanations include the students' individual development throughout the semester, the influence from the instruction, and the new content.

Previous research on undergraduate example construction showed that the students used *trial and error* techniques approximately 80% of the time (Iannone et al., 2011). This percentage is considerably higher than than the 57% *trial and error* observed in this study. It is unclear what accounts for this discrepancy, although the most likely causes are the sample and the task selection. Both studies also had small samples, this one had four participants and Iannone et al. (2011) had nine, so the individual characteristics of the participants strongly affected the percentages.

**Implications for teaching transition-to-proof courses.** One implication is that students should be explicitly taught strategies for constructing and verifying examples. One of the hardest parts of trial and error is picking the construction to test. However, by explaining how the examples in the course are constructed, it may be possible to guide the students beyond blinding picking parameters to test.

In this study, most of the students became convinced that a *prove or disprove* statement was true after constructing only one or two examples. However, when mathematicians obtain conviction from empirical evidence it is often from multiple examples or for unusual properties (Weber, 2013; Weber, Inglis, & Mejia-Ramos, 2014). Although it is unreasonable to assume that numerous examples should be constructed before trying to prove a statement, we need to teach students to consider the quality of the examples they construct, and to view the examples as a collection. For example, a statement that is true for a prime number, a perfect square, and another composite number is far more believable than a statement evaluated only with a prime number. But students need to be taught to consider examples collectively rather than individually.

**Future research.** Additional research concerns the instruction on example construction. How does instruction impact a provers ability to effectively use and construct examples? It is unclear whether or not such instruction will actually help the students learn how to construct examples effectively. Some studies suggest that instruction in problem solving frameworks alone does not help students become better problem solvers (Garofalo & Lester, 1985; Schoenfeld, 1980), so it is possible a similar phenomenon will occur here. This can only be established through additional testing and study.

Finally, it is unclear whether effective example construction will positively impact proof writing. Iannone et al. (2011) found that generating examples provided no benefits to the students as compared to receiving a list of examples. One interpretation of this is that it does not matter where the examples come from, what matters is how the examples are used and what conclusions are drawn from the examples. As such, it is possible that knowledge in using examples effectively can improve a persons ability to successfully write proofs, but additional study is needed on this topic.

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