On the axiomatic formalization of mathematical understanding

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This study adopts a property-based perspective to investigate the forms of abstraction, instantiation, and representation used by undergraduate topology students when acting to understand and use the concept of a continuous function as it is defined axiomatically. Based on a series of task-based interviews, profile cases are being developed to compare and contrast the distinct ways of thinking and processes of understanding observed by students undergoing this transition. A framework has been established to interpret the participants' interactions with the underlying mathematical properties of continuous functions while they reconstructed their concept images to reflect a topological (axiomatic) structure. This will provide insight into how such properties can be successfully incorporated into students' concept images and accessed; and which obstacles prevent this. Preliminary results reveal several coherent categories of participants' progression of understanding. This report will outline these profiles and seek critical feedback on the direction of the described research.

Key words: Continuous Functions, Topology, Axiomatic Formalism, Abstraction, Properties

Since Hilbert's program at the turn of the last century, modern mathematics has rested on the notion of an axiomatic system (Zach, 2015). These consist of collections of declarative statements, or axioms, whose interactions describe the properties and relationships of the primitive elements in the system. Other properties can be logically deduced from the axioms, without the need for intuition. Reasoning in this manner is considered the ideal goal for students of advanced mathematics, although it may not be natural for many at first (Freudenthal, 1991; Tall, 2013).

This research seeks to illuminate the transition that learners face when attempting to alter and embed their informal and more formal ways of understanding within axiomatic structures. By exploring the participants' transformative use of abstraction in the reconstruction of their concept images for continuous functions in axiomatic contexts, this study contributes to an emerging perspective on the construction of axiomatic mathematical understanding in general. The dual processes involved in the abstraction and instantiation of such properties should play an essential role in the development of axiomatic knowledge structures.

Background

Axiomatic mathematical understanding

Advanced mathematical thinking has been shown to be different from its earlier forms (Harel, 2000; Harel & Sowder, 2005; Sfard, 1994; Sierpinska, 1990; Tall, 2013). Students of advanced mathematics must revise their concept images for earlier ideas in ways that no longer rely on embodied metaphors and intuition about objects in the physical world (Sfard, 1994). Instead, mathematical properties gain importance as they are transformed from descriptions into definitions (Freudenthal, 1991; Tall, 2013). Eventually, the need for axiomatic understanding demands a complete reversal of the relationship between properties and mathematical objects (Freudenthal, 1991; Garcia & Piaget, 1983/1989).

The transition to axiomatic processes of understanding is fundamentally different than earlier transitions faced by mathematics students. It requires a substantial shift in the students' thinking—from descriptive activities concerning the properties of mathematical objects, to the construction of axiomatic systems and definitions based *a priori* on collections of those properties. As students are led toward increasingly abstract forms of thought that are less grounded in everyday experience, this can result in profound difficulties and misconceptions as they build their formal understanding of advanced mathematical topics (Freudenthal, 1991; Harel & Tall, 1991; Tall, 2013). Learners' abstractions, instantiations and representations of mathematical properties are therefore a vital research focus for the exploration of their transitions to axiomatic understanding. This is the primary unit of analysis in the study described here.

Cognitive structures in advanced mathematical thinking

Tall and Vinner (1981) use the term concept image to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). In this definition, the "structure" of the concept image was left largely without description; used as a holistic notion to refer to its totality, rather than as an explicit description of its organization. However, several elements of that structure have since been outlined in detail, such as:

1. Basis for categorization—Whether students categorize pre-requisite concepts through exemplar representations, prototypical abstractions, or metaphorical comparisons is a significant factor in their ability to generalize continuous functions to broader contexts (Alcock & Simpson, 2011; Lakoff, 1987).
2. Defining activities—The properties that have been abstracted into a student's personal concept definition may not coincide with the formal definition, or even the examples they consider relevant to the concept. Analyzing participants' defining activities according to the DMA framework (Zandieh & Rasmussen, 2010; Dawkins, 2012) creates a context for the interpretation of the relationship between concept image and concept definition.
3. Example space structure—Several factors involved in the structure of a student's example space have been examined by researchers in recent years including: density, connectedness, and axiological nature (Sinclair, Watson, Zazkis and Mason, 2011); and its dimensions of variation and range of permissible change (Watson & Mason, 2005). The example space will be a key structural component in this analysis of the concept image.
4. Use of metaphor and embodiment—Many students will continue to use embodied metaphors and their physical intuition to guide their understanding, rather than axioms and definitions. Whether this is intrinsic to mathematical thought (Lakoff & Nunez, 2000) or an obstacle that may be overcome (Sfard, 1994), it remains a factor in any complete study of transitions in understanding.
5. Abstraction types—Hershkowitz, et al. (2001) defined abstraction as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p. 202). As participants reconstructed and reorganized their understanding of continuous functions, certain activities related to abstraction were observed and can now be analyzed. Piaget's four types of abstraction will play an important role here (as cited in von Glasersfeld, 1995), as well as Hamton's (2005) explanation of the importance of context in abstraction and instantiation.

While these constructs have been considered in isolation, there has been less work in seeking relationships between these separate elements of the concept image structure. The main contribution of the described research will be to explore a number of distinct cases of interaction between these five elements, forging the way toward a more complete understanding of how the concept image is structured and re-structured as the participants' transition to axiomatic formalism proceeds.

Significance

This study contributes to the theoretical knowledge about advanced mathematical understanding by: 1) providing insight into students' transitions to axiomatic content, especially in the important context of continuous functions; and 2) exploring the effect of shifting the focus of mathematical learning research onto learners' mental representations of mathematical properties rather than mental or mathematical objects.

The continuous function concept is of great importance, not only as a window into the transition of students' understanding toward axiomatic settings, but also in its own right. The long historical formulation of this idea led to the development of the important field of topology (Moore, 1995), and is central to the exploration of topological invariants through its role in the definition of homeomorphism, the defining criteria for the preservation of topological properties (Munkres, 1975/2000).

By studying the transition to an axiomatic system in the context of continuous functions, this research spotlights the participants' abstraction activities relating to important mathematical properties. Not only is continuity itself a property of the reified notion of a function (Dubinsky,1991; Sfard, 1994), but it is a complex of relationships between sub-properties, such as sets that are open or closed, sequences that converge, and images/preimages of a given function. These interactions generate the property of continuity at higher levels of abstraction, and are therefore worthy of investigation. The advanced study of properties and their relationships will be needed to effectively model students' transitions to axiomatic formalism in undergraduate and graduate mathematics classrooms.

There are a number of research perspectives about how students acquire mathematical understanding (Arnon, et al., 2014; Sfard, 1994; Tall, 2013) through the cognitive representations of actions, objects, or symbols. Alternatively, Slavit (1997) demonstrated a "property-oriented view" (p. 263) of students' understanding of functions, blending with Sfard's (1994) operational-structural perspective to "discuss how a student can reify the notion of function as a mathematical object that possesses or does not possess various functional properties" (Slavit, 1997, p. 263). This investigation aims to elaborate this perspective greatly, establishing a scheme to describe the structure of participants' concept images for continuous functions in terms of mathematical properties and students' mental actions upon them. The cases constituted in this study will enable future research on targeted instructional techniques to accommodate diverse profiles of student learning in axiomatic contexts.

Methods

This qualitative, case-oriented research was first informed by several cycles of grounded theory-building that occurred over three semesters at a large university in the southwest United States. The initial studies aided in the development of a categorization scheme for possible factors and obstacles involved in the development of an axiomatic understanding of continuous functions. These categories served as the basis for the constitution of archetypal cases, which came to be organized around learners' uses of abstraction and instantiation of mathematical properties in axiomatic contexts.

Research design

Grounded theory framework for preliminary data generation

The evolution of the theoretical background for this study relied on several iterations of applied grounded theory methodology, which provided enough initial data to extract meaningful dimensions for further study. Three semesters of preliminary interviews served as the ultimate basis for discovering categories in the emerging theoretical model, although those categories were also informed by a pre-existing theoretical framework derived from the research literature. This research guided the formation of interview tasks and questions, designed to elicit specific, observable acts of understanding. These tasks and questions were then modified based on participants' responses and the researcher's own reflective insights. Categories were formulated through this iterative process, which were then modified via reflexive feedback and sharpened into relevant dimensions for study.

Case analyses

The next phase of research used these theoretical categories to develop cases of particular ways of thinking and processes of understanding students use in combination to develop axiomatic understanding. Although it is not claimed that these cases are generalizable, they contribute to an understanding of the interaction among various types of abstraction and instantiation students might use at this stage in their mathematical development. This will be an essential first step to conducting further research in this area.

The choice of case-oriented research for this purpose was justified by the complexity of the phenomena being investigated. Whereas a variable-oriented approach presupposes a homogeneous population from which to select a randomized sample, case studies seek to draw out differences in the population and to explore complex relationships between conditions and outcomes. Through in-depth investigations of cases, Ragin (2004) explains that qualitative researchers can often account for "causal heterogeneity" and "conjunctural causation" (p. 135); providing models for phenomena with multiple factors that the analytic tools of variable-oriented researchers cannot manage.

Participants and Selection Criteria

Five participants from an undergraduate topology class of approximately 30 students were selected for profiling. They were chosen for their theorizing capacity based on their answers to a prerequisite knowledge assessment and brief interviews. Criteria for selection were divided into four categories relating to their understanding of the prerequisite concepts: 1) categorization schemes and types of abstract representations, 2) personal concept definitions and their alignment with the formal definition, 3) example space structure and coherence, and 4) use of metaphor, visualization and multiple representations.

Procedures

This investigation is a multiple-case analysis consisting of five distinct cases of participants' cognitive transformations as they reconstructed their concept images for continuous functions to reflect an axiomatic structure. The cases were chosen based on classroom observations, a preliminary assessment, and a brief interview with each individual in the sample pool. The theoretical criteria for the constitution of these cases came from the research literature and insights that have emerged from preliminary study data as described above, as well as a textbook and curricular analysis.

Textbook analysis

Twelve topology-related textbooks, used widely in introductory topology courses across the U.S., were analyzed in the preliminary data collection process. In particular, one textbook (Croom, 1989) was chosen by the participating professor as the course textbook for the semester of the study. The goal was to discern the intended learning that authors expect students to follow while transitioning to an axiomatic understanding of continuity. These sequences represent classical categorization schemes for the central notion of continuous functions and several pre-requisite and co-requisite concepts such as: functions, open and closed sets, sequences, and limits. Such schemes are the goal state for the structure of students' concept images and not representative of the natural categorization schemes that most students will adopt at first.

The approaches that were studied varied widely with respect to these topics, affected in some cases by the need to construct the concepts from prior knowledge, and in others by the authors' willingness to present an abstract definition without explicit motivation. Codes for each of the analysis categories reached saturation, with themes becoming redundant among the twelve textbooks. Nevertheless, these codes represent a large variety of potential didactical approaches to the wider subject of continuous functions. Different blends of the above approaches might be chosen by the professor, with more or less emphasis on examples, prototypical abstractions, categorization rules, or metaphors. Variations in the presentation of the content may influence students' approaches to understanding the topics, presenting possible future avenues for research.

Task-based interviews and artifact analysis

Since a learner's enactment of understanding is fluid and context dependent (Duffin & Simpson, 2000; Sierpinska, 1994) qualitative, task-based interviews were deemed the most appropriate manner of eliciting appropriate actions and capturing the evolving state of her or his cognitive structure. However, there are challenges involved in eliciting a full, reasoned solution or proof in a time-limited setting. Participants may demonstrate some of their reasoning processes in this way, but they cannot necessarily demonstrate their ability to formally produce a proof, or work through complex threads of logical reasoning. For this reason, classwork (e.g. quizzes, exams) and homework was also analyzed in order to gain insight into the participants' full range of mathematizing abilities.

Analysis was centered on participants' in-class work and the results of three rounds of task-based interviews held throughout the semester. These hour-and-a-half long interviews were focused on these three broad topics: 1) the description and use of open/closed sets, sequences, and real-valued continuous functions; 2) the description of continuous functions in abstract contexts; and 3) the use of continuous functions in abstract contexts. The interview questions were designed to elicit the participants' personal concept definitions and elements of their concept images for these topics, such as salient metaphors, the example space structure and the basis for their categorization schemes. Students were then tasked with reconciling their definitions to divergent elements of their concept images and/or the formal definitions for these topics. Further tasks were designed to provoke acts of abstraction from the participants as they tried to enact their understanding in proof and problem-solving contexts.

Questions for Audience

1. Might it be possible to find different cases of student thinking in different classroom contexts (e.g. metric space courses, geometrically-oriented introductions, or more abstractly presented material)?
2. To what extent are a student's uses of abstraction and instantiation related to each other? In other words, could we hope to predict how a student uses a mathematical concept by the process they used to define it?

3. In what other ways might the transition to axiomatic formalism reflect or contrast with earlier transitions?

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