Unraveling, synthesizing and reweaving: Approaches to constructing general statements.

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Abstract

Learning progressions for the development of the ability to look for and make use of mathematical structure would benefit from understanding how students in mathematics-focused majors might construct such structures in the form of general statements. The author recruited ten university students to interviews focused on tasks that asked for the reconstruction of a general statement to accommodate a broader domain. Through comparative analysis of responses, four major categories of approaches to such tasks were identified. This preliminary report describes in brief those four categories.

Keywords: Undergraduate mathematics, mathematical practices, structure, generality, general statements.

Rationale

One goal of mathematics education at all levels is to promote the development of proficiency in mathematical thinking. In recent years, the Common Core Standards (Council of Chief State School Officers [CCSSO] & National Governors Association Center for Best Practices [NGA], 2010) have become a well-known framework for describing the mathematical practices that students should develop during their K-12 education.

The Standards for Mathematical Practices are meant to "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (p. 6). Included among these Standards is the practice to "look for and make use of structure" (p. 8). Mason, Stephens, and Watson (2009) define mathematical structure as "the identification of general properties which are instantiated in particular situations as relationships between elements" of some kind of collection. For the purpose of this preliminary report, a statement that describes such a structure will be referred to as a *general statement.* The construction of general statements is an essential component of mathematical activity, without which the knowledge of individuals and of the discipline cannot grow (cf. Mason, Drury, & Bills, 2007).

According to NGA and CCSSO (2010), the Standards were constructed on "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). A learning progression for the ability to create general statements should include research-based descriptions of the various ways that individuals might construct general statements as their formal education in mathematics increases. Research efforts have led to insights into the ways that elementary, middle, and secondary students construct general statements through patterning and generalizing (e.g., Becker & Rivera, 2004, 2005, 2006, 2007; Ellis, 2007; English & Warren, 1995; Fuii & Stephens, 2001; Garcia-Cruz & Martinón, 1997; Jurow, 2004; Lannin, 2005; Lannin, Barker, & Townsend, 2006), yet much less is known about the approaches that students, engaged in formal postsecondary study of mathematics, use to construct general statements

This preliminary report focuses on findings from data collected as part of a research study designed to investigate the following question:

What are the characteristics of approaches that postsecondary students in math-focused majors use when constructing general statements?

Theoretical Framework

Examples and generality

Watson and Mason (2005) suggest that example construction is an important aspect of mathematical activity. Among other possible uses, examples may serve as “placeholders used instead of general definitions and theorems” (p. 3) or as “representatives of classes used as raw materials for inductive mathematical reasoning” (p. 3). The generality that one encodes in the examples that are produced may have an influence on the process of developing a general claim. For example, as Mason and Pimm (1984) noted, the numeral 6 can be used to represent a specific value, or as a representative example of an even number, or even as a generic representation of any element of the even numbers. The claims that one makes about a specific inscription may or may not be general claims about a class of objects, depending on the generality that the inscription is meant to represent.

In addition to the generality that one encodes in an example (or attributes to an example), the symbols used to represent an example can influence the process of developing a general claim about a collection. Lannin, Barker, & Townsend (2006) hypothesized that individuals are more likely to develop numerical patterns involving recursive relationships when elements of a collection are represented in such a way that one can perceive one figure as an intact subfigure of another, such as in the arrays shown in Figure 1, and that learners are more likely to work toward patterns that relate ordinal position and numerical values when presented with figures that are not so easily perceived as embedded one-within-another (see Figure 2).

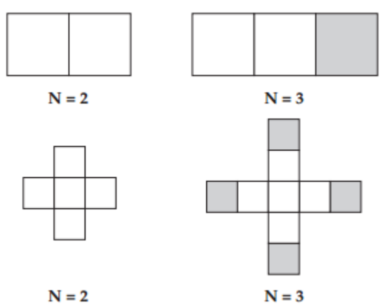

Figure 1. Recursively oriented patterns (Lannin et al., 2006, p. 22)

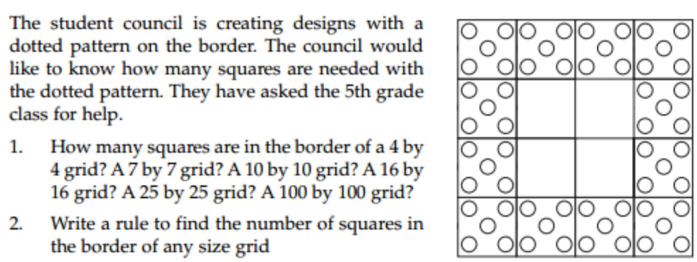

Figure 2. The Border Problem (Lannin et al., 2006, p. 18)

Relationships

A general statement is, in its presentation, nothing more than a claim that one is making about elements in a collection. Behind a general statement, however, are the structures and relationships that one understands and that undergird the statement itself. The relationships to which one attends when examining examples and building relationships can influence and even characterize the resulting general statement. Stacey’s (1989) illustration of students’ approaches to linear generalizing tasks indicated that some learners identify relationships between examples and use those relationships to transform one example into another. For example, a student who is given the images shown in Figure 1 and asked to predict the number of rectangles and squares in each set for N=4 might identify an additive relationship and predict that the number of rectangles will be one more than for N=3 and that the number of squares will be four more, thereby transforming the total of 3 rectangles and 9 squares for N=3 into totals of 4 rectangles and 13 squares for N=4. Alternatively, some respondents will focus on relationships between the index value and the number of elements, noting that, for example, the number of rectangles is 2 for N=2 and 3 for N=3 and hypothesizing that the number of rectangles will always equal the index value. In the case of patterning activities such as those used by Stacey (1989) and Lannin and colleagues (2006), the type of relationship that the participant finds salient can impact the development of either a recursive relationship or a functional relationship.

Methods

Ten students from a large mid-Atlantic university were recruited as participants. All were pursuing degrees in math-focused majors: Six were pursuing degrees in secondary mathematics education, and four were pursuing degrees in mathematics. Each participant was enrolled in mathematics coursework intended for students in their fourth year of study, and each had completed at least one mathematics course at that level prior to participating. Participation consisted of three task-based interviews, each lasting approximately one hour and consisting of one or more tasks designed to engage the participant in the construction of a general statement. Recordings were used to capture participants' statements and to provide a video record of the participants' written work and nonverbal gestural communication. Each interview was transcribed and each transcript was parsed into responses that began at the introduction of a task prompt and ended at the introduction of a subsequent task prompt or at the end of the recording.

This preliminary report is based on participants' responses to tasks that provided a general statement (we will refer to this as the *anchor statement*) and that asked the participant to reconstruct the claims made in the anchor statement as claims that would be true for a superset containing the original domain (we will refer to the superset as the *target domain* and to the requested set of claims as the *target claim*). Specifically, participants responded to one or more of the following task prompts:

Reconstructing products (RP). Consider the following statement: Any four consecutive whole numbers is divisible by 12. Can you rewrite the statement so that it is true for products of three or four consecutive whole numbers?

Reconstructing Unit Ball (RB). Every point (x ,0) on the interior of the interval [-1 , 1] has the property that $|x| < 1$. Can you rewrite the statement so that it is true for all points on the unit circle and its interior?

Reconstructing Sums (RS). Consider the statement that the sum of the first n counting numbers is $n(n + 1)/2$. Can you find a way to rewrite this statement so that it is true for any sequence of n consecutive integers?

Consistent with the theoretical framing presented here, participants' responses to these tasks were analyzed and categorized by comparing the ways that the participants exemplified the anchor domain and target domain, to the presence of evidence that illuminated the generality encoded in the examples that participants created, to the relationships (if any) that the participants analyzed while responding, and to the relationships that participants constructed while responding.

Findings

The comparative analysis of responses yielded five qualitatively distinct approaches to the tasks presented in the methods section of this preliminary report. Rough descriptions of each approach are presented in Table 1, and illustrative examples will be shared here, as space permits.

Characterizing Approach: Don, RP

In his response to the RP task, Don (a pseudonym) wrote examples of products of three consecutive whole numbers as shown in Figure 3. He noted that each 3-tuple contained an even number and a 3, and hypothesized that products of 3 consecutive whole numbers might always be divisible by 6. He then tested this for 4*5*6, 5*6*7, 6*7*8, and 7*8*9. This part of Don's response consists of characterizing the collection of examples without reference to the anchor statement.

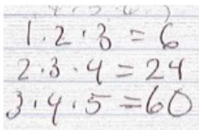

Figure 3. Don's examples of products of 3 consecutive whole numbers.

Oblique Approach with Specific Examples: Chris, RP

Chris created a set of specific examples similar to those used by Don (see Figure 4). However, instead of developing a claim inductively from examples, Chris searches for those 3-tuples that satisfy the anchor claim – in other words, those whose products are divisible by 12.

Figure 4. Chris' examples of 3-tuples in the RP task.

Unraveling and synthesizing: Jolene, RP

Jolene approached the RP task by analyzing the anchor claim. She determined that a 4-tuple would always have two even factors using a generic representation shown in Figure 5, and used the placeholder representation shown in Figure 6 to conclude that a 4-tuple would always include one number that was divisible by 3. She then used these understandings to synthesize the claim that a 3-tuple would always include one number divisible by 2 and one divisible by 3 and would, therefore, have a product that is divisible by 6.

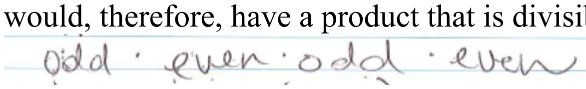

Figure 5. Jolene's general representation of 4-tuple.

Figure 6. Jolene's placeholder representation.

Unraveling and adapting: Edward, RS

Edward conceptualized an arbitrary sequence of positive consecutive integers as the difference between two sequences of counting numbers:

> Let's say we started just at 5 and . . . . I wanted to know the sum of the numbers from 5 to 10. I would do the first ten counting numbers and then I would take away the first four counting numbers.

Edward then used this relationship between the target domain and the anchor domain as a conceptual lens through which to adapt the anchor claim, writing a target claim that the sum of a sequence of integers from *k* to *n* would be computed through the expression in Figure 7.

$$\frac{n(n+1)}{2} - \frac{k(k+1)}{2}$$

Figure 7. Edward's formula for the sum from k to n.

Table 1
Approaches to Reconstruction Tasks

<table>
<tr><td></td><td></td><td colspan="4">Approaches</td></tr>
<tr><td rowspan="4">Representation of Domain</td><td></td><td colspan="2">Structural</td><td>Oblique</td><td>Empirical</td></tr>
<tr><td>Collection of Specific Examples</td><td rowspan="3">Unravel relationships between the anchor domain and anchor claim, then synthesize relationships from the target domain to a target claim.</td><td rowspan="3">Unravel relationships between the anchor domain and the target domain, then adapt the anchor claim.</td><td>Find examples that satisfy the anchor claim.</td><td>Reason inductively.</td></tr>
<tr><td>Generic example</td><td></td><td></td></tr>
<tr><td>General representation</td><td></td><td></td></tr>
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The essential differences among these approaches lies in the generality with which the elements of the domain are represented and in the relationship that grounds the response. Structural approaches seem to be amenable to the greatest variation in ways of representing the anchor and target domains, and are more well-suited to general representations than are approaches in which the target claim is populated without calling on an analysis of relationships among the anchor statement and target domain.

Implications

As one advances in mathematics education, strategies for developing general statements do not necessarily become more sophisticated. However, some individuals develop the ability to analyze and utilize mathematical structure with respect to general representations to produce new general statements. Questions for those who attend this session will be:

1. Are there particular tasks that might provoke more structural approaches?
2. What teaching strategies might help students learn to use more structural approaches?

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