# ‘It’s not an English class’: Is correct grammar an important part of mathematical proof writing at the undergraduate level?

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*We studied the genre of mathematical proof writing at the undergraduate level by asking mathematicians and undergraduate students to read seven partial proofs based on student-generated work and to identify and discuss uses of mathematical language that were out of the ordinary with respect to what they considered standard mathematical proof writing. Preliminary results indicate the use of correct grammar is necessary in proof writing, but not always addressed in transition-to-proof courses.*

*Key words:* Mathematical language, Proof, Mathematicians, Undergraduate students

## Introduction

Mathematicians and mathematics educators have found undergraduate mathematics students to have difficulties when constructing (Weber, 2001), reading (Conradie & Frith, 2000), and validating (Selden & Selden, 2003) mathematical proofs. One suggested reason for these difficulties is the students’ unfamiliarity with the language of mathematical proof writing (Moore 1994). However, mathematical language at the level of advanced undergraduate proof writing is a scarcely studied topic. As a result, little is known of how mathematicians and students understand and use this technical language.

**Related Literature**

Halliday’s (1978) introduction of the notion of register (and mathematical register in particular) was groundbreaking in the study of mathematical language:

> A set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to a ‘mathematics register’, in the sense of the meanings that belong to the language of mathematics […], and that a language must express if it is being used for mathematical purposes. (p.195)

Thus, the mathematical register contains not only technical vocabulary and symbols, but also phrases and the associated syntax structures. Various mathematics educators have considered how the mathematical register plays a role in mathematics learning and classrooms. For instance, Pimm (1987) discussed how students develop the mathematical register and Schleppegrell (2007) noted students’ difficulties with differentiating between the mathematically precise and colloquial uses of words like ‘if’, ‘when’, and ‘then’.

However, much of the existing work on mathematical language focuses on K-12 mathematics and little empirical research exists on how professional mathematicians view or use the language of mathematics. Konior’s (1993) analysis of over 700 mathematical proofs revealed a common style of construction of mathematical proofs that signals the organization of the proof’s arguments. Burton and Morgan (2000) identified the roles that the author’s identity and focus played in mathematical writing in research papers. Meanwhile, a number of manuals (AMS, 1962; Halmos, 1970; Gillman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Alcock, 2013; Vivaldi, 2014) have been written describing how mathematicians and students should effectively use mathematical language. As these texts are based on the authors’ experiences rather than empirical research, the texts were used to guide the materials used for the study.

## Theoretical Perspective

This study builds upon Scarcella's (2003) conceptual framework of academic English, which was designed to study the learning of academic English. Scarcella (2003) defined academic English as a "register of English used in professional books and characterized by the specific linguistic features associated with academic disciplines" (p. 9). Scarcella argued that academic disciplines have their own sub-registers of academic English and, as such, the mathematical register can be seen as a sub-register of academic English. Thus we consider the mathematical sub-register in this study with a focus on undergraduate proof writing.

This study is also informed by Herbst and Chazan's (2003) body of work on practical rationality. Intending to study norms by evoking repairing reactions from their participants, Herbst and Chazan adapted the ethnomethodological concept of breaching experiments (Mehan & Wood, 1975). The hypothesis of the design is that when a participant of a practice is presented with a situation in which a norm of such practice is breached, he or she will attempt to repair the breach highlighting not only what the norm is, but also the role that the norm has in the practice (Herbst, 2010). Adapting this methodology, this study investigates how mathematicians view and describe conventional uses of the language of mathematical proof writing at the undergraduate level and how students understand these conventions.

## Research Questions

In this study, we am to investigate the following questions: 1) How do mathematicians view and describe common unconventional uses of mathematical language in undergraduate mathematical proof writing? 2) How do these unconventional uses affect how mathematicians evaluate student-constructed proofs? 3) How do students understand the conventions of mathematical proof writing at the undergraduate level?

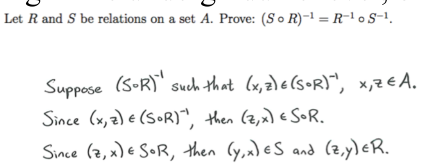
# Methods

We investigate the linguistic dimension of undergraduate proof writing by presenting participants with student-generated proofs and asking the participants to identify and describe uses of mathematical language that are out of the ordinary with respect to undergraduate proof writing. By identifying non-standard uses of mathematical language, the participants discussed their understanding of the conventions of proof writing in this context.

The study was conducted at a large research university in the United States. Eight mathematicians and sixteen undergraduate students were interviewed (eight of the undergraduates were mathematics majors who had completed the proof-based courses required for graduation and eight were undergraduates enrolled in an introduction to proof course.) The mathematicians had 1-38 years of experience teaching undergraduate proof mathematics courses, with 1-15 years of experience teaching introduction to proof courses.

## Materials

The materials for this study include seven partial proofs that are based on student-generated work. Each of the proofs was truncated to help participants focus on the use of mathematical language and not the attempted proof's logical validity. One of the partial proofs used in the study is provided below in Figure 1a. The partial proofs were chosen from student exams given in introduction to proof classes at the same university of the study. For each one of these partial proofs, a copy was created and marked for each of the instances of what we believed to be breaches of conventional uses of the language of mathematical proof writing at the undergraduate level, one example is shown in Figure 1b.



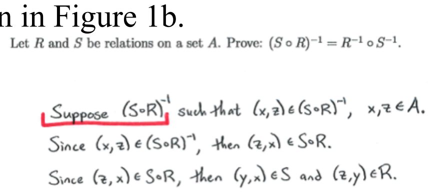



**Figure 1a.** Example of the partial proof. **Figure 1b.** Example of marked partial proof.

**Procedures**

The interview procedures for mathematicians and for undergraduate students were nearly identical. The semi-structured interviews were videotaped and lasted one to two hours. Participants were presented with the student-constructed partial proofs one at a time. They were asked to mark the partial proofs for anything that was out of the ordinary with respect to the use of the language in undergraduate mathematical proof writing. The interviews made two passes through the materials. In the first pass, participants were asked to explain why they had made each mark. Then for each mark, the participant was asked if the breach at hand was a logical issue, if it affected the validity of the proof, if it was an issue of mathematical writing, if it was definitely unconventional or a matter of personal preference, if it lowered the quality of the proof significantly, and if they (or in the students' case, if they thought a mathematician) would have deducted points based on this issue when grading the proof in an introductory proof course. These prompts were designed to elicit the participants' views on what they thought were conventional uses of mathematical language in proof writing. In particular, the prompts addressed the severity of each breach and enabled a differentiation between issues of logic and issues of mathematical writing in the analysis of the data.

In the second pass, for each of the predicted instances of unconventional use of mathematical language that had not been identified by the participant in the first pass of the data, participants were asked if they would agree that this was an issue of mathematical language. Specifically, mathematicians were asked whether or not they would agree with a colleague of theirs who had suggested these were unconventional uses of mathematical language and the undergraduate students were asked if they would agree with a classmate of theirs who believed a mathematician would think these were unconventional uses of mathematical language. If they agreed, they would be prompted to discuss the breach as in the first pass.

**Analysis**

Interview videos were transcribed and materials generated in the interviews were scanned for analysis. The interview protocol created clear episodes of discussion, each concerning a single breach of mathematical language. Thus the data is organized by these episodes and was then analyzed using open ended thematic analysis in the style of Braun and Clarke (2006). That is, we first familiarized ourselves with the data by marking for ideas and transcribing videos, generated initial codes by organizing the data into meaningful groups, searched for themes by focusing analysis at a broad level, and reviewed the themes to verify that the themes reflect he data set as a whole.

## Results

One theme that has emerged from the data is that mathematicians believe that mathematical language is a subset of the English language whereas some students believe the two are independent. This theme was brought forth by the mathematicians' attention to the need for correct grammar and complete sentences as well as some of the undergraduate students' responses indicating the rules of English do not apply in mathematical settings. In particular, this theme emerged from three categories of responses from participants discussing what they considered was non-standard mathematical language use, which are described below using interview data three different proofs: Proof A, Proof B, and Proof C.

**Mixing mathematical notation and English prose**

As shown in Figure 2, the first line of Proof A reads, "None of the sets are $\emptyset$." We expected that participants would indicate this sentence as an unconventional use of language because the mathematical symbol for the empty set, $\emptyset$, was used in a sentence that was otherwise written in English words. This was generally the case, however, two mathematicians explained further that the issue of using the symbol $\emptyset$, was an issue of grammar. For example, M8 indicated in Pass 1 through the proof that there is a problem with the part of speech of the symbol “because *empty* is an adjective and *the empty set* is a noun”. So M8’s comment highlights that when read, the statement says, “none of the sets are empty set” rather than the possible intended meaning, “none of the sets are empty”. M5 gave a similar explanation for why the use of the symbol $\emptyset$ was inappropriate in this statement. Both M5 and M8 said that they would make a note to the student suggesting that they avoid this use of language in the future.

One undergraduate participant S2 made a statement arguing that words and symbols can be used interchangeably since “the symbol for the empty set is just as rigid as saying empty”. From this quote, it appears that S2 is (at least implicitly) aware of the difference between the noun and the adjective forms, however, disregards the issue. With the exception of S2, no other student mentioned the grammatical issue of using the mathematical symbol.

Let $\mathcal{P}$ be the following collection of the subsets of the integers $\mathbb{Z}$:
$$\mathcal{P} = \{A_0, A_{+e}, A_{+o}, A_{-e}, A_{-o}\}, \text{ with } A_0 = \{0\},\ A_{+e} = \{2,4,6,8,...\},\ A_{+o} = \{1,3,5,7,...\},\ A_{-e} = \{-2,-4,-6,-8,...\},\ A_{-o} = \{-1,-3-5,-7,...\}$$
Prove that $\mathcal{P}$ is a partition of $\mathbb{Z}$.

None of the sets are $\emptyset$.
All are pairwise disjoint, since the positive sets share nothing with the negative sets and the evens share nothing with the odds, and $\{0\}$ shares nothing with the rest.

**Figure 2.** Proof A.

Let $A$, $B$, and $C$ be sets. Let $f : A \to B$ and $g : B \to C$.
Prove: If $f$ and $g \circ f$ are bijections, then $g$ is one-to-one.

let A,B,C be sets and $f: A \to B$ and $g: B \to C$
$\forall a,b \in B$, let $g(a)=g(b)$
Need to show $a=b$.
Because $f: A \xrightarrow[\text{onto}]{1-1} B$, there $\exists x \in A$ s.t. $f(x)=a$ and $f(y)=b$

**Figure 3.** Proof B.

**Punctuation and capitalization**

In Proof B (as shown in Figure 3), there is a lack of punctuation and capitalized letters to indicate the ending and beginning of sentences. Mathematician M7 pointed this out during Pass 1 through the proof, saying: “the expression and the punctuation are not good” and “we can’t allow writing like that”. In Pass 2, the remaining mathematicians agreed that lacking punctuation and capitalization is definitely unconventional of mathematical proof writing. However, mathematicians M3, M4, and M5 each also agreed that they would not address this issue in their introduction to proof classes. For example, M4 explained:

> I look for understanding of the construction of the mathematical arguments. So I’m not sure you can require that deep understanding at the same time pushing them to be correct with punctuation and so on. […] And I consider that my task is to teach them reasoning, rather than to use punctuation.

Although all eight mathematicians in the study did agree that proofs should be presented in complete sentences, including appropriate punctuation and capitalization of letters, not all believed they should discuss this in class. Only M7 indicated that he would deduct points from his students’ work for missing punctuation and lacking capitalized letters. Meanwhile, M4, M6, and M8 indicated that they would mark the punctuation and capitalized letters when grading, without deducting points, to illustrate to their students that one should use complete sentences in proofs.

None of the 15 undergraduate participants discussed the lack of punctuation or capitalization in Pass 1 through the proof. In fact, during Pass 2, 12 of the 15 undergraduate participants disagreed with the suggestion that this is an issue of mathematical proof writing. When asked why not, S2 explained “well, in my experience in my classes, some of my proofs were not full sentences with punctuation and capitalization and there was never really an issue about it.” This suggests that students may not learn the conventions of mathematical writing by simply observing mathematicians write proofs in class and that students are not made aware of issues with their proof writing until points are deducted.

Others were even surprised that issues of English would be important in a math class, for example, S4 exclaimed "Oh my god, this is a mathematics major, not a linguistic major right? I think it's fine!" and S8 noted this is not an issue because "it's not an English class." This is not to say, however, that none of the undergraduates believed that mathematical language is a subset of academic English; three undergraduate participants did believe that capitalization and punctuation belonged in mathematical proof writing; for instance, S3 explained, "a proof is like a math essay of sorts and it should still be like grammatically correct".

The above suggests that the mathematicians in this study agreed that full sentences should be used when writing proofs. On the other hand, some of the responses from mathematicians and students indicated their beliefs that proper English does not play a role in proof writing at the introduction to proof level. With only one mathematician deducting points for lacking capitalization and punctuation, it is unsurprising that students do not see the necessity of proper grammar in proof writing.

**Non-statement**

Proof C (shown in Figure 1a) included the following phrase that was ungrammatical and meaningless: "Suppose $(R \circ S)^{-1}$ s.t. $(x, z) \in (R \circ S)^{-1}$". As an imperative phrase with a transitive verb, English grammar dictates the need for both a direct object and an object complement to be a complete sentence. That is, the sentence must suppose the direct object in relation to another object or a property about the direct object. While the mathematicians did not give this exact grammatical explanation, they did note the incompleteness of the sentence.

In Pass 1, seven of eight mathematicians discussed that the proof's first line is not a complete sentence and has no meaning. M8 explained, "the way that I would parse this sentence is, suppose $(S \circ R)^{-1}$. That's in itself a part and again it has no verb. Suppose $(S \circ R)^{-1}$?" M5 similarly noted "Students sometimes say 'let a set' which doesn't mean anything. This is just a nonsense thing to say, suppose this set." Thus, the statement does not suppose a property of the relation, is not a complete sentence, and conveys no meaning. Moreover, seven of eight mathematicians indicated they would deduct points for a nonsensical and incomplete statement. The eighth indicated they would make a note to the student to show the student that the statement was incomplete, but would not deduct points.

Meanwhile some undergraduate participants saw an issue with the statement and attempted to rectify it by completing the sentence, but were unable to articulate what was wrong in the first place. For instance, N3 explained, "I would say 'Suppose $(S \circ R)^{-1}$ is a relation such that $(x, z)$' is in this relation". On the contrary, some of the undergraduates found no problem with the incomplete sentence; for example, N4 saw no difference between saying 'Suppose $(S \circ R)^{-1}$' and 'Suppose there is a relation $(S \circ R)^{-1}$'. This suggests that some students do not view mathematical language as a sub-register of academic English and do not see the importance of using compete sentences in mathematical proof writing.

## Discussion

As this qualitative study considers only a small sample of mathematicians and undergraduate students, the findings are simply suggestive of how mathematicians and undergraduates view the need for proper grammar in undergraduate proof writing. Based on the above, we see for the most part that mathematicians in the study believed that grammar and the parts of speech of mathematical words should be attended to when writing mathematical proofs. This need for complete sentences and attention to grammar is supported by the mathematical writing guides written by mathematicians (Gilman, 1987; Krantz, 1997; Higham, 1998; Houston, 2009; Vivaldi, 2014), who indicate that correct grammar and complete sentences should be used in proof writing. Meanwhile, the results suggest that mathematicians may not be attending to these issues in introductory proof courses.

**Questions for the audience**

- How might one instruct mathematical grammar to undergraduate mathematics students?
- How can we motivate students to use correct grammar in proof writing if they believe it is unnecessary?

## References


Alcock, L. (2013). *How to Study as a Mathematics Major*. Oxford University Press.

AMS (1962). Manual for authors of mathematical papers. *Bulletin of the American Mathematical Society*, 68(5), 429-444.

Braun, V. and Clarke, V. (2006) Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77-101.

Conradie J., & Frith, J. (2000). Comprehension tests in Mathematics. *Educational Studies in Mathematics, 42*(3), 225-235.

Gillman, L. (1987). *Writing mathematics well: a manual for authors*. Washington, DC: Mathematical Association of America.

Halliday, M. A. K. (1978). Language as social semiotic. London: Edward Arnold.

Halmos, P. R. (1970). How to write mathematics. *L'Enseignement Mathématique*, *16*.

Herbst, P. (2010). Practical rationality and the justification for actions in mathematics teaching. In the *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education*, 46-54.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes- The case of engaging students in proving. *For the learning of mathematics*, 2-14.

Higham, N. J. (1998). *Handbook of writing for the mathematical sciences*. Siam.

Houston, K. (2009). *How to think like a mathematician: a companion to undergraduate mathematics*. Cambridge University Press.

Konior, J. (1993). Research into the construction of mathematical texts. *Educational Studies in Mathematics*, 24(3), 251-256.

Krantz, S. G. (1997). *A primer of mathematical writing: Being a disquisition on having your ideas recorded, typeset, published, read and appreciated*. American Mathematical Society.

Mehan, H. and Wood, H. (1975). *The reality of ethnomethodology*. New York: John Wiley & Sons, Inc.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge & Kegan Paul.

Scarcella, R (2003). Academic English: A conceptual framework. University of California Linguistic Minority Institute. Retrieved from http://escholarship.org/uc/item/6pd082d4

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem?. *Journal for research in mathematics education*, 4-36.

Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, *23*(2), 139-159.

Vivaldi, F. (2014). *Introduction to Mathematical Writing*. London: The University of London.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119.