Undergraduate students proof-reading strategies: A case study at one research institution

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Weber (2015) identified five effective proof-reading strategies that undergraduate students in proof-based courses can use to facilitate their proof comprehension. Following Weber's (2015) study, we designed a survey study to examine how undergraduate students' proof-reading strategies relate to what proficient learners of mathematics (mathematics professors and graduate students in mathematics) say undergraduates should employ when reading proofs. Our preliminary findings are: (i) Majority of the professors in our study claimed that undergraduates should use the strategies identified in Weber's (2015) study, (ii) Professors' response significantly differed from undergraduates' in only two of the five proof-reading strategies described in Weber's (2015) study (attempting to prove theorem before reading its proof and illustrating confusing assertions with examples), and finally (iii) Undergraduate students, for the most part, tend to agree with their professors' preferred proof-reading strategies.

Key words: Proof, Proof-reading strategies, Proof comprehension, Undergraduate mathematics

In upper level mathematics courses, mathematicians regularly use proofs to convey mathematics to their students. As a result, students in these courses are expected to spend sufficient time reading and writing proofs (Weber & Mejia-Ramos, 2014). Research on undergraduates' interaction with proofs suggests that undergraduates often times have difficulty with determining the validity of a proof and/or constructing a valid proof (Alcock & Weber, 2005; Inglis & Alcock, 2012, Selden & Selden, 2003; Weber, 2010). For instance, Selden and Selden (2003) argued that when reading proofs undergraduates tend to focus on surface features of a mathematical argument as opposed to its *global* feature. Participants in their study showed only limited ability to determine if a mathematical argument is valid or qualifies as a proof or not. Studies also suggest that undergraduates often do not actually gain understanding from the proofs they read (Conradie & Frith, 2000; Cowen, 1991). There is, however, very little research on how undergraduates read proofs with the intent of learning mathematics from them. In an effort to improve students' understanding of proof, Weber and Mejia-Ramos (2013) developed five proof-reading strategies that undergraduates can use to improve their proof comprehensions, which form the basis for this study.

Theory

We designed our survey study based on Weber and Mejia-Ramos' (2013) studies on effective proof-reading strategies. In a qualitative study, Weber and Mejia-Ramos (2013) observed four mathematics majors and prospective teachers read six proofs. The authors considered these students to be strong because they were successful in both their content-based mathematics courses and on the follow up proof comprehension test that the authors designed based on Mejia-Ramos et al's (2012) proof comprehension assessment model. Their analysis revealed five proof-reading strategies that the students used to facilitate their understanding of the proofs. These five strategies identified in their study are: (1) trying to prove a theorem before reading its proof, (2) comparing the assumptions and conclusions in the proof with the proof technique being used, (3) breaking a longer proof into parts or sub-proofs, (4) comparing the proof approach to the one's approach, and (5) using an example to understand a confusing inference. Weber and Mejia-Ramos (2013) followed up their qualitative study with a large-scale internet-based survey study that included mathematics majors and mathematicians from 50 large state universities in the United States. The purpose of their quantitative study was two-fold: (1) to explore whether mathematicians prefer mathematics majors to use these five proof-reading strategies and (2) to explore to what extent mathematics majors use these strategies. The main finding of their study is that the majority of mathematics major do not use these proof-reading strategies. This continues to be the case even though the majority of mathematicians believed that mathematics majors should use these strategies. This is an interesting finding since it sheds light on why undergraduate students often times gain little from proofs (e.g., Conradie & Frith, 2000; Cowen, 1991; Rowland, 2001). Our study examines whether these findings hold in on large institution.

Previous research on student comprehension of proofs

The literature on proof comprehension is relatively sparse. Some earlier studies on proof assessment indicate that mathematicians do not necessarily evaluate their students' understanding of a given proof effectively (Conradie & Frith, 2000, Weber, 2012). Conradie and Frith (2000), for instance, maintain that mathematicians' ways of testing their students' understanding of a proof usually require nothing beyond recalling the statements and its proof. The mathematicians interviewed in Weber's (2012) study also conceded this. In Weber's (2012) study mathematicians reported that they measured their students' understanding of proofs by (1) asking students to construct a proof for a similar theorem to the one that was proven in class, and/or (2) asking them to reproduce a proof; and some said they do not assess their students' understanding of a proof. Conradie and Frith (2000) maintain that students can pass simply by memorizing the statement and proof of each theorem as presented in class; this, however, as they point out, does not effectively reflect students' understanding.

There are fewer studies on what students do when they read proofs for understanding. For example, Inglis and Alcock (2012) conducted a study that compared and contrasted beginning undergraduate students' proof-reading habits to those of research-active mathematicians. By studying their participants' eye movement while reading a proof, they concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface feature" of a mathematical proof. Based on this observation, the researchers suggest that undergraduates spend less time focusing on the logical structure of the argument; this, in turn, seems to explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature (A. Selden & Selden, 2003).

There is a growing body of literature aimed at improving undergraduates' proof comprehension. Recently, Hodds et. al (2014) put forward a pedagogical technique known as *self-explanation training* that they argued can improve students' proof comprehension by improving their engagement with the proof. Weber (2015) has also described strategies that undergraduates can use to facilitate their understanding of proof. Our study contributes to the growing body of literature in proof comprehension by examining the following research questions: (1) To what extent do professors endorse the proof-reading strategies described in Weber's (2015) study? (2)To what extent do undergraduate students use proof-reading strategies described in Weber's (2015) study?

Research methodology

The population consisted of undergraduate students who have at least taken or enrolled in a transition-to-proof course, and mathematics professors. All participants were solicited from a large public university in the United States. Because we were investigating the relationship between professors' suggestions and students' uptake, we believed that asking both groups and attempting to relate them at the professor-and university-level is useful.. We should note that although the majority of our undergraduate student participants were taking a transition-to-proof course, a significant number of them had at least two proof-based mathematics course, including, but not limited to, introductory abstract algebra and real analysis.

We replicated the survey items in Weber and Mejia-Ramos' (2013) study where they asked mathematics majors to indicate the extent to which the aforementioned proof-reading strategies are reflective of their own. For undergraduate students, one of the researchers visited all proof-based undergraduate mathematics courses offered at this institution at the time this research was taking place and asked the students to complete the survey. Nearly all undergraduate students (92) who were enrolled in at least one proof-based course completed the survey. Most undergraduates completed the survey in less than 10 minutes. Following Weber and Mejia-Ramos (2013) study, we also disseminated the survey to mathematics professors in this institution. Fifteen mathematics professors agreed to participate. The survey questionnaires for the professors were virtually identical questions; however, they were directed to reflect undergraduate students' proof-reading experience as opposed their own. For instance, to examine to what extent undergraduate students employ proof-reading strategy #1, we asked them to what extent they agreed with the following statement: When I read a theorem, I usually try to think about how I would prove the theorem before reading its proof. For professors, the item above was phrased as follows: when reading a theorem undergraduate students should usually try to think about how they would prove the theorem before reading its proof. All participants were asked to indicate their choice using a five-point Likert scale (strongly agree (5), agree (4), neutral (3), disagree (2), and strongly disagree (1)).We used to the statistical software JMP 12.1 Pro to determine if there is a statically significant difference between the two groups. We will present our findings in the next section.

Results

We organize our results based on the research questions. Recall that the main goal of this study is to explore to what extent undergraduate students in one large research institution employ proof-reading strategies that professors in that same institution find desirable. As it is evidenced in Table 1, the majority of professors claimed that undergraduate students should employ all five proof-reading strategies described in Weber (2015). In particular, a significant number of professors (85.71%) strongly agreed or agreed that undergraduate students should use examples to verify the veracity of potentially confusing assertions in a proof (strategy #5); on the other hand, only 66.3% of undergraduate students claimed to employ this strategy. In fact, using

Wilcoxon Each Pair Test we found that undergraduate students' response on strategy #5 is significantly different from professors with an alpha-level of 0.05.

Additionally, a large percentage of professors (73.33%) either strongly agreed or agreed that when reading a theorem, undergraduate students should attempt to prove the theorem before reading its proof (strategy #1), however, only 60.87% of undergraduate students claimed to have used this proof-reading strategy. Indeed, a Wilcoxon Each Pair Test revealed that undergraduate students' response statistically significantly differed from professors with an alpha-level of 0.05. This finding is consistent with that presented in Weber and Mejia-Ramos (2013) study where the majority of mathematicians (88%) agreed that mathematics majors should try to prove a theorem before reading its proof. Weber and Mejia-Ramos (2013) also argued that the majority of mathematics majors in their study do not use these strategies. In our study, we have no evidence to support this claim; on the contrary, our study revealed that the majority of undergraduate students did in fact claim to use these strategies. We will present a plausible explanation for this discrepancy in the next section.

Table 1 Percentage of participants who strongly agree or agree on the survey items (see Appendix 1)

Strategy	Professors	Undergraduates
1. Attempt to prove theorem before reading its proof	73.33%	60.87%
2. Consider proof frameworks	86.67%	88.04%
3. Compare proof method with one's own approach	60%	63.04%
4. Break proof into parts	66.67%	69.57%
5. Illustrate assertion with example	85.71%	66.3%

Table 2 *p-values* in Wilcoxon Each Pair Test (based on Wilcoxon rank scores, also called Mann-Whitney test) using the statistical software JMP 12.1 pro

Participants	Strategy #1	Strategy #2	Strategy #3	Strategy #4	Strategy #5
Professors vs. Undergraduates	**0.0303**	0.3663	0.645	0.7095	**0.0471**

Discussion and implications for further research

In this paper we argued that statistically significant difference between undergraduate students and professors existed only in the two of the five proof-reading strategies (strategies #1 and #4), suggesting that undergraduate students mostly claimed to employ desirable proof-reading strategies. We have also argued that undergraduate students' proof-reading strategy, for the most part, tend to agree with what their professors say undergraduates should do when reading proofs.

The level of agreement between undergraduates and professors on strategy #3 (breaking a longer proof into parts or sub-proofs) is encouraging. It is encouraging because they are using a reading strategy that is identified in the literature as effective for proof comprehension (Weber, 2015, Weber & Mejia-Ramos, 2013). At the same time, we are surprised by this result because Weber and Mejia-Ramos (2013) in their survey study found that only 38% of mathematics major claimed to have employed it. We believe there are several plausible explanations for this discrepancy. First, while our survey questions were identical to theirs, the choices our participants had were slightly different. In their study, participants were given two choices and asked to indicate if they agree or disagree; in contrast, in our study, participants were asked to indicate their choice on a five point Likert scale. Second, their Internet based survey included participants from 50 large institutions in the United States; on the other hand, our pool of participants comes from a single institution. Thus, it could be the case that mathematicians in this institution explicitly discuss these proof-reading strategies with their students. Finally, our undergraduate participants were different from theirs in the sense that our participants were not only mathematics majors, our study incorporated participants majoring in computer science, and secondary mathematics education. We plan to conduct further analysis of our data to examine if our preliminary results hold for mathematics majors only, prior to this we would like to use our presentation to receive feedback regarding the inconsistency of our result to that of Weber and Mejia-Ramos' (2013). In particular, we would like to focus on the following discussion questions: (1)To what extent do you agree or disagree with our potential explanation for inconsistency? (2) What further analysis of our survey data might explain the inconsistency?

In summary, our study provides further evidence that the strategies described in Weber (2015) are indeed effective in facilitating proof comprehension. We have also argued, contrary to Weber and Mejia-Ramos (2013) study, undergraduate report to use these effective proof-reading strategies. As a result, we believe this study is a welcome addition to the paucity of the literature in proof comprehension. Finally, we hope that further research such as interviewing these mathematicians might provide insight into the surprising level of agreement between them and their students.

Appendix 1. Survey Items (slightly modified from Weber and Mejia-Ramos (2013) study)

Strategy #1: When I read a theorem, I usually try to think about how I would prove the theorem before reading its proof.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #2: When I read a proof of a theorem, I consider what is being assumed, what is being concluded, and what proof technique is being used.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #3: When I read a new assertion in a proof that I find confusing, I sometimes check whether that assertion is true with specific example.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #4: When I read a long proof, I try to break it into parts or sub-proofs.

Strongly agree Agree Neutral Disagree Strongly disagree

Strategy #5: When I read a proof, I compare how the methods used in the proof compares to the methods I would use to prove the theorem.

Strongly agree Agree Neutral Disagree Strongly disagree

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