Obstacles in Developing Robust Proportional Reasoning Structures: A Story of Teachers' Thinking About the Shape Task

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This paper presents some initial findings of an investigation focused on mathematics teachers' ways of thinking about proportional relationships, with an emphasis on multiplicative reasoning. Deficiencies in proportional reasoning among teachers can be serious impediments to the development of robust reasoning among their students. As such, this study focuses on how mathematics teachers reason through tasks that involve proportional reasoning by addressing the following two research questions: (1) In what ways do teachers reason through a specific task designed to elicit proportional reasoning? and (2) What difficulties do teachers encounter while reasoning through such tasks? This paper discusses the construction of a robust proportional reasoning structure in the context of a specific task and discusses one particular obstacle, which impedes the construction of such a structure.

Key words: Proportional reasoning; Multiplicative reasoning; Within-measure comparison; Across-measure comparison

Recent reform efforts to institute the Common Core State Standards for Mathematics (CCSSM, 2010) have called for an increased emphasis on multiplicative and proportional reasoning, particularly in the middle grades. According to Lesh et al. (1988), proportional reasoning is the capstone of elementary school mathematics and the cornerstone of high school mathematics. One of the most critical elements of proportionality is the ability to make sense of the multiplicative relationships among the relevant quantities. Multiplicative reasoning is rooted in the ability to reason quantitatively and make sense of contexts involving multiplicative structures. The CCSSM standards themselves call for students to be able to "describe a ratio relationship between two quantities" (CCSS.math.content.6.rp.a.1). Yet historically, the mathematics traditionally taught in K-12 has emphasized additive reasoning and ill-conceptualized procedures for multiplicative situations, rather than building productive ways of thinking about quantities, relationships among quantities, ratios, multiplicative comparisons, and proportional relationships. This paper describes an investigation conducted with middle school teachers who are participating in a large-scale professional development program designed to improve their conceptual understanding of mathematics in the middle grades. Specifically, this study describes teachers' conceptions of proportionality through the lens of a proportional reasoning structure and highlights the challenges that teachers encountered.

A Discussion of the Literature

Centrally nested in the idea of multiplicative reasoning is the ability to first conceive of the quantities that need to be compared multiplicatively. In this study, the notion of quantity is aligned with Thompson's (1993) definition of quantity: "a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (p. 165). Thompson (1994) refers to the mental operation of conceiving one quantity in relation to another as a quantitative operation. Thompson also points out that "a quantitative operation creates a structure – the created quantity in relation to the quantities operated upon to make it work" (p. 185). The mental structure created as a result of a quantitative operation ultimately supports images of other numerical operations.

Reasoning about quantities is necessary for reasoning about proportional relationships. Cramer et al. (1993) outlined several components involved in proportional reasoning: (1) understanding the multiplicative relationships that exist within proportional situations, (2) being able to differentiate proportional situations from non-proportional ones, (3) realizing the existence of and relationships between multiple solution pathways, and (4) being unaffected by the situational context or the types of numbers in the task. Kaput and West (1994) found that the context of the problem, the language of the task, the kinds of quantities involved, and the numerical values of the quantities all impact student thinking.

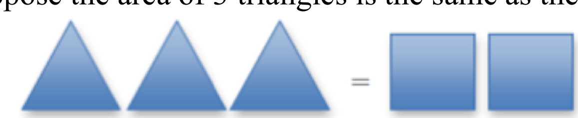
Methodology

This investigation focuses on nine middle school mathematics teachers who were recruited for this investigation based on their participation in a large-scale, two-year professional development program. Leveraging Goldin's (2000) principles, this study incorporated semi-structured, task-based interviews for investigating teachers' thinking when working through tasks involving proportional relationships. The teachers participated in five, one-hour videotaped interviews. The research team analyzed all interview sessions with the lens of characterizing teachers' thinking and reasoning as they grappled with the tasks. The design and implementation of this study was guided by the following two research questions: (1) *In what ways do teachers reason through a specific task designed to elicit proportional reasoning?* and (2) *What difficulties do teachers encounter while reasoning through such tasks?*

Creating a Robust PR Structure

In situations where two quantities are proportional, there exists an opportunity to construct a structure that can be utilized when addressing missing value proportion problems. A proportional reasoning (PR) structure is a network of multiplicative relationships that exist among the values of proportional quantities. This section of the paper presents one PR structure that is robust and founded on meaningful reasoning. Consider the Shape Task, which was used in this study:

The Shape Task: Suppose the area of 3 triangles is the same as the area of 2 squares.



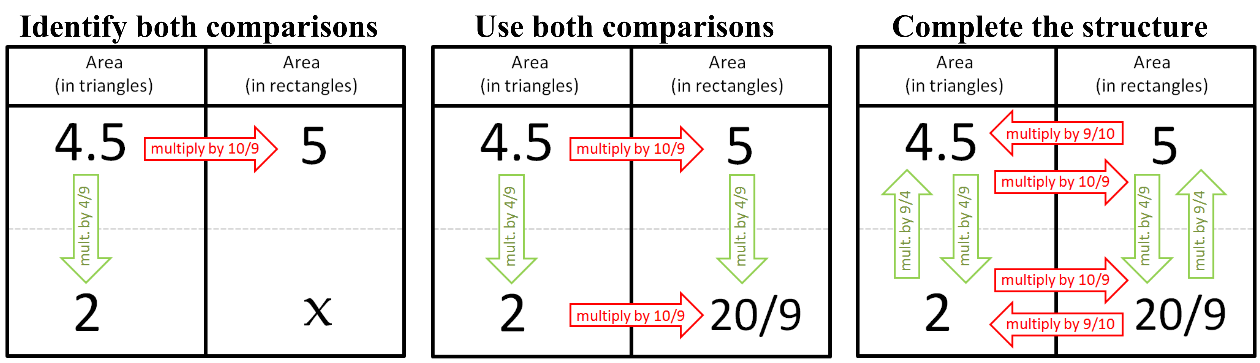
Also, suppose the area of 3 squares is the same as the area of 5 rectangles.

What is the area of 2 triangles, measured in the rectangle areas? Explain your reasoning.

In the Shape Task, it is important to recognize that the value of an area, measured in triangles, is proportional to the value of the same area measured in rectangles. Most of the teachers in the study were able to deduce through various methods that the area of 4.5 triangles is the same as the area of five rectangles. However, it was not trivial for many teachers to subsequently determine the amount of rectangles that is equivalent to two triangles, and a few were not able to overcome this challenge. We present data in this paper that highlights this one particular obstacle.

For the Shape Task, the constant of proportionality is 10/9 (found by computing 5÷4.5); which results from a multiplicative comparison of the two area measurements, five rectangles to 4.5 triangles. Kaput & West (1994) refer to this comparison of 10/9 (or its reciprocal of 9/10) as an *across-measure comparison* because it is a multiplicative comparison of two distinct ways to measure one quantity (i.e. area in triangles versus area in rectangles). We interpret the across-measure comparison of 10/9 as 10/9 rectangles for every one triangle, just as we interpret 9/10 as 9/10 of a triangle for every one rectangle.

Another approach is to construct a scale factor within the same measure (e.g. scaling one area measured in triangles, to a new area measured in triangles). By multiplicatively comparing two triangles to 4.5 triangles, the scale factor of 4/9 (found by computing 2÷4.5) can be constructed and then applied to the second measure (area in rectangles) to maintain the proportional relationship. Kaput & West (1994) call the comparison of 4/9 (or its reciprocal of 9/4) a *within-measure comparison* because it is a multiplicative comparison of two values within the same measure space, each value expressed using the same unit. We interpret the within-measure comparison of 4/9 as representing that the area of 2 triangles is 4/9 times as large as the area of 4.5 triangles. A robust PR structure includes both ways of reasoning – across-measure and within-measure – as well as the associated reverse operations. The construction of a robust structure is depicted in the figure below.



The ability to construct a PR structure as described above depends on the refinement of other ways of thinking about mathematics. For example, one should be able to use division to evaluate multiplicative comparisons as instinctively as one might use subtraction to evaluate additive comparisons. Our data indicates that utilizing division to evaluate multiplicative comparisons is not trivial for some middle school mathematics teachers. Unless one is able to meaningfully determine the across- and within-measure comparisons, one will not be able to construct the PR structure described.

Where is the relationship of cross-products: $(4.5)\left(\frac{20}{9}\right) = (2)(5)$?

The PR structure that we describe deliberately omits the cross-product relationship because it is not a necessary component of a robust proportional reasoning structure. Research has shown that students and teachers who leverage the procedure of cross-multiplying as a strategy for solving proportional tasks often lack the conceptual knowledge to explain why this strategy works (Cramer et al., 1993). An important goal in mathematics is to help students develop the ability to make sense of their world and to reason through problems. As supported by NCTM's Principles to Action (2014), procedural fluency should emerge from conceptual understanding. When trying to reason why cross-multiplication is effective, it can be challenging to explain the meaning of the cross-products. In the Shape Task, we have difficulty making sense of the product of an area (measured in triangles) and an area (measured in rectangles). Consequently, we claim that techniques involving cross-multiplication (or other procedures) should only be introduced after a solid foundation of proportional reasoning is constructed.

Discussion of Findings

In this study, initial data have revealed that the difficulties teachers encountered while solving the Shape Task were consistent with past research findings about student thinking (Kaput & West, 1994; Thompson, 1994). This study contributes to the field by investigating obstacles that *teachers* encounter while reasoning about situations that involve proportional relationships. The following is a discussion of the data from two teachers who grappled with the Shape Task, each of whom demonstrated difficulty in answering the mathematical question: *What do I need to multiply this by to get that?*

The Case of Ellie:

Within the first couple of minutes of engaging with the task, Ellie deduced that 4.5 triangles was equivalent to five rectangles. She recognized the need to scale 4.5 triangles to two triangles, but she was unable to determine the scale factor by which to do so.

> Ellie: I have to divide nine halves by, to get to two, I have to divide it by two ninths? No, that's going to give me one…What I was trying to do was, okay, I have to get down to two triangles (points at the 4.5 triangles drawn on the page)…

Ellie's inability to determine how to scale 4.5 triangles to two triangles led her to abandon a sensible way of thinking – a way of thinking that is essential to the construction of the PR structure set forth in this paper. During another attempt to answer the question, she again encountered difficulty when trying to scale 1.5 triangles to two triangles.

> Ellie: So then I've confused myself again.
> Interviewer: How have you confused yourself? What are you thinking?
> Ellie: …How do I get to two from one and a half? What do I have to multiply by? And I could not, for the life of me, think of what that would be. But it would have to be (long pause) four thirds? Does that work?

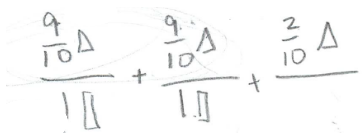
Although successful in determining how to scale 1.5 triangles to two triangles, the cognitive load was heavy and she ultimately relied on algebraic methods – writing down and then solving the algebraic equation $\frac{3}{2}x = 2$.

The Case of Anne:

Like Ellie, Anne quickly deduced that 4.5 triangles were equivalent to 5 rectangles. Anne unitized this relationship to 0.9 triangles per one rectangle, but struggled to leverage this information productively.

> Anne: So two full triangles would be…Oh now for some reason I'm getting stuck and I know all I have to do is enlarge it. What do I do? Okay, um, to get to two full triangles…

Unable to multiplicatively scale 0.9 triangle to two triangles, Anne relied on additive reasoning to combine the amounts of triangles (see written work below).



This initial approach was eventually abandoned since Anne could not determine how many rectangles were equivalent to 2/10 of a triangle. After moving on with other tasks in the interview, Anne returned to the Shape Task and successfully completed the task using a modified strategy of scaling three triangles to two triangles, which did not seem to pose a challenge.

> Anne: But I want two triangles. I have three triangles. So I'm gonna multiply this, I'm just gonna multiply this whole thing by 2/3, will let me say two triangles.

Anne's initial challenge with the task could be indicative of issues pertaining to scaling with fractional numbers – scaling from a whole number to another whole number is less cognitively demanding than scaling from a fraction to a whole number. According to Cramer et al. (1993), Anne does not have a robust ability to reason proportionally because the numbers in the task affected her reasoning.

Conclusion and Discussion Questions

The initial data reveal that several teachers struggled to evaluate multiplicative comparisons, which is a severe hindrance to the construction of the robust PR structure that we have described. Also, the data reveal that teachers in this study have inconsistent – and sometimes incoherent – ways of thinking about the quantities and their proportional relationships. At times, the teachers in the study lose track of the quantities they are relating together and they experience difficulty in describing how the quantities are related. Other instances have revealed that teachers rely on additive reasoning in order to cope with an inability to compare two quantities multiplicatively. A PR structure that sensibly relies on multiplicative comparisons may have provided the teachers with a conceptual understanding of proportionality to further facilitate their thinking and mitigate their struggles. As part of the presentation, the following questions will be posed and discussed to further the direction of this research investigation: (1) Have other PD researchers encountered similar obstacles to proportional reasoning and, if so, how have they addressed them? (2) How can we develop video coding frameworks to investigate proportional reasoning? (3) Are there researchers who already have such coding frameworks that we could adopt?

Acknowledgements

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