DOES IT CONVERGE? A LOOK AT SECOND SEMESTER CALCULUS STUDENTS' STRUGGLES DETERMINING CONVERGENCE OF SERIES

David Earls and Eyob Demeke
University of New Hampshire

Despite the multitude of research that exists on student difficulty in first semester calculus courses, little is known about student difficulty determining convergence of sequences and series in second semester calculus courses. In our preliminary report, we attempt to address this gap specifically by analyzing student work from an exam question that asks students to determine the convergence of a series and follow-up semi-structured interviews. We develop a framework that can be used to help analyze the mistakes students make when determining the convergence of series. In addition, we analyze how student errors relate to prerequisites they are expected to have entering the course, and how these errors are unique to knowledge about series.

Key words: Series, Framework, Convergence, Calculus, Undergraduate Mathematics

Researchers have noted that there is a lack of research in the area of infinite series (González-Martín, Nardi, & Biza, 2011). Moreover, the research that does exist does not focus on undergraduates in second semester calculus, but rather on undergraduates in real analysis (González-Martín, Nardi, & Biza, 2011; Alcock & Simpson, 2004; Alcock & Simpson, 2005), how graduate students understand series (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012), and humanities students' difficulty with the concept of infinity when dealing with series (Sierpińska, 1987).

In this study, we begin to fill a gap in the literature by developing a framework that can be used to analyze student errors that occur while solving problems in second semester calculus courses related to sequences and series. Moreover, since researchers have argued that first semester calculus students struggle because they lack the necessary prerequisite skills such as the function concept (Ferrini-Mundy & Graham, 1991; Carlson, Madison & West, 2010; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997), we also look to see how the errors students make are related to prerequisite skills they should have acquired prior to entering their second semester calculus courses. In particular, we aim to (1) determine the errors students make when solving typical second semester calculus problems on series, (2) determine the relationship these errors have to prerequisite skills, (3) determine how the errors made are unique to series, and (4) develop a framework for analyzing student errors.

Research methodology

The targeted population for this study is undergraduate students enrolled in a second semester calculus course in a large public university in the northeastern United States. Fifty-five students in the course agreed to have their work on a sequences and series exam photographed. Thirty-four of these students also agreed to be interviewed working through problems on sequences and series similar to those seen on their exam, though only eight students responded to an e-mail to set up the interview with seven showing up for their interview.

Recall that the main research aims in this study restated from the introduction are to (1) determine what mistakes students make when they solve problems on series typically seen in a second semester calculus course, (2) determine how these mistakes relate to prerequisite skills students are expected to have prior to entering a second semester calculus course, (3) determine what knowledge of series aside from prerequisite knowledge students need to avoid the mistakes seen in (1), and (4) develop a framework for analyzing student mistakes determining the convergence of sequences and series.

This preliminary report focuses on student responses to one question on their exam, a problem that focused on student knowledge of comparison tests (or integral test) to determine the convergence or divergence of a series:

Determine whether the following series converges or diverges. Be explicit about any test you use to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

To address the aims in our study, we went through two rounds of coding. Since we could not find any theoretical work in this area, we opted for open coding. In the first round, we wrote a description of the type of error we saw. In the second round of coding, we came up with categories to fit our descriptions into. The categories, abbreviations, and an explanation of the categories are given below in table 1:

Table 1: Categories, Abbreviations, and Explanation Table

Categories, Abbreviations, and a Brief Explanation with an Example

Category	**Abbreviation**	**Explanation**
No Mistakes	NM	A completely correct answer
Notational Error	NE	A notational error. For example, a student says $\frac{1}{n}$ diverges without including the series symbol.
Algebra of Series	AS	Student splits up a series when one diverges. For example, he may write $\sum_{n=1}^{\infty} \frac{1}{n} + \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$
Algebra	A	An algebraic error. A student might, for example, "plug in" infinity, or incorrectly simplify a rational expression by "cancelling" through a sum
Function Choice	FC	Wrong function choice when using a comparison test. For example, a student might try to make a comparison with $\frac{1}{n^2}$.
Unchecked Assumptions	UA	Student failed to check that the function satisfied the assumptions in the integral test.
Algebra error leading to Incorrect Test Choice	AITC	Student reaches a false conclusion (usually in the ratio test) because of an algebraic mistake. This mistake typically was cancelling through a sum.
Incorrect Test Choice	ITC	Student chooses an incorrect test, such as an nth term test, or a geometric test.

(continued)

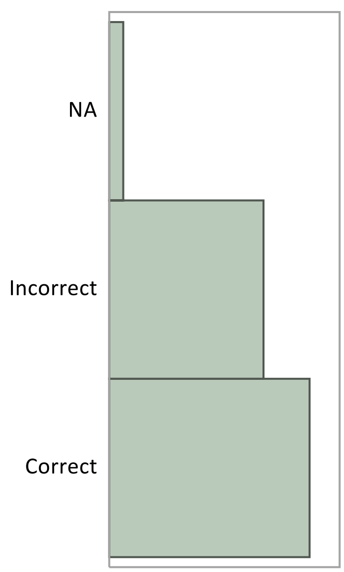
Table 1: Categories, Abbreviations, and Explanation Table (continued)

Category	Abbreviation	Explanation
Wrong Conclusion in Test #1	WCT1	Student uses a test other than the integral test or a comparison test, and reaches an incorrect conclusion from that test. For example, a student uses the ratio test and says that a value of 1 means the series converges.
Wrong Conclusion in Test #2	WCT2	The student correctly chooses a comparison test or the integral test, but reaches an incorrect conclusion using that test. For example, a student says the series converges because it is larger than the series $\frac{1}{n}$.

Preliminary results and discussion questions

In what follows we present the preliminary results of student responses to the question on the exam stated above. Figure 1 below shows that most students, about 54.5%, answered the question correctly. By a correct answer, we mean an answer that would have received full credit or only lost a point or two on the examination in the judgement of the authors of this paper, both of whom have experience teaching second semester calculus.

Figure 1: Correct/Incorrect

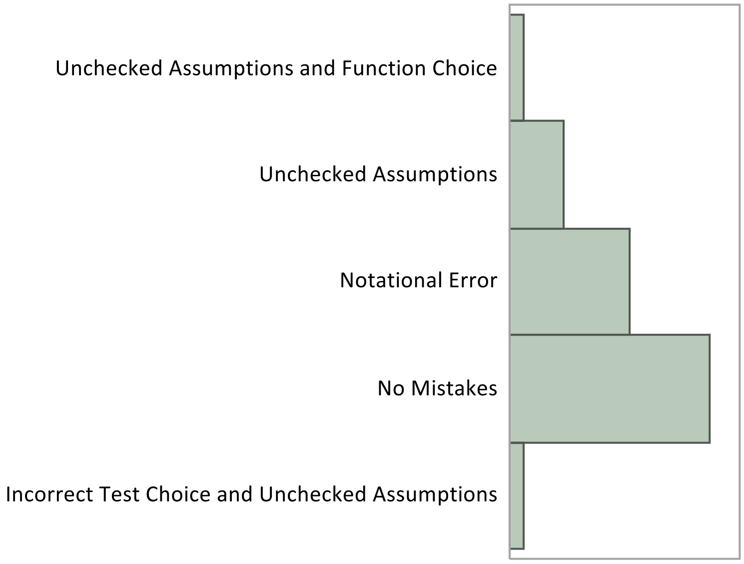


Frequencies

Level	Count	Prob
Correct	30	0.54545
Incorrect	23	0.41818
NA	2	0.03636
Total	55	1.00000

Half of the students that got the problem correct made no errors whatsoever, and another 30% only made notational errors. Figure 2 shows the types of errors made by students that answered the question correctly. Note that multiple errors were possible on the same problem.

Figure 2: Distributions of Correct Responses

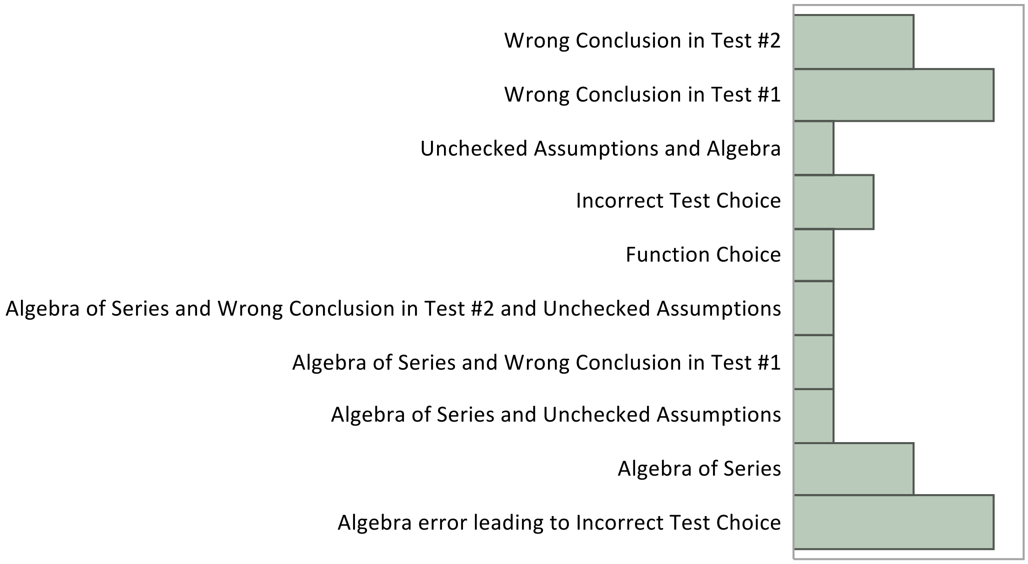


Frequencies

Level	Count	Prob
Incorrect Test Choice and Unchecked Assumptions	1	0.03333
No Mistakes	15	0.50000
Notational Error	9	0.30000
Unchecked Assumptions	4	0.13333
Unchecked Assumptions and Function Choice	1	0.03333
Total	30	1.00000

Finally, figure 3 shows the types of errors made by students that answered the problem incorrectly.

Figure 3: Distributions of Incorrect Responses



Frequencies

Level	Count	Prob
Algebra error leading to Incorrect Test Choice	5	0.21739
Algebra of Series	3	0.13043
Algebra of Series and Unchecked Assumptions	1	0.04348
Algebra of Series and Wrong Conclusion in Test #1	1	0.04348
Algebra of Series, Wrong Conclusion in Test #2, Unchecked Assumptions	1	0.04348
Function Choice	1	0.04348
Incorrect Test Choice	2	0.08696
Unchecked Assumptions and Algebra	1	0.04348
Wrong Conclusion in Test #1	5	0.21739
Wrong Conclusion in Test #2	3	0.13043
Total	23	1.00000

The preliminary data in the three figures above indicate algebraic manipulation as a prerequisite skill that causes student mistakes. While none of the students that answered the question correctly made an algebra mistake, 12 of the 23 students that answered the question incorrectly made some kind of algebra mistake.

Students also appeared to have difficulties that are somewhat unrelated to prerequisite knowledge. Nine students of the 53 failed to check the assumptions in the test they were using. For instance, they often did not check the continuity or the monotonicity of the function when using the integral test. Seven students chose the wrong test to use in this problem, and another 10 students made a mistake regarding the conclusion of their selected test.

Moving forward, we plan to continue our analysis of the other problems on the exam as well as analyze interview transcripts to get a better idea on why students might be making some of these errors. Prior to further analysis, we would like to use our presentation to receive feedback on the following questions:

(1) Which of the categories we have used might be unique to this particular problem and not appear when we look at other traditional series problems?

(2) What categories might we need to add to encompass mistakes that we might see in other problems that we did not see here, particularly for problems related to sequences?
(3) What other methodological suggestions might you offer us to examine the data further?

Implications for further research

Once we have a better understanding of the types of errors students are making and why these errors are being made, we can begin investigating teaching strategies to help students avoid these errors. In addition, by finding the most common prerequisite mistakes, we can investigate the curriculum and teaching of prior mathematics courses and help students be better prepared when entering second semester calculus courses. Finally, we can continue studying student errors and improving upon our framework.

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