Students' explicit, unwarranted assumptions in "proofs" of false conjectures

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Although evaluating, refining, proving, and refuting conjectures are important aspects of doing mathematics, many students have limited experiences with these activities. In this study, undergraduate students completed prove-or-disprove tasks during task-based interviews. This paper explores the explicit, unwarranted assumptions made by six students on tasks involving false statements. In each case, the student explicitly assumed an exact condition necessary for the statement in the task to be true although it was not a given hypothesis. The need for an ungiven assumption did not prompt any of these students to think the statement may be false. Through prompting from the interviewer, two students overcame their assumption and correctly solved the task and two students partially overcame it by constructing a solution of cases. However, two other students were unable to overcome their assumptions. Students making explicit, unwarranted assumptions seems to be related to their limited experience with conjectures.

Key words: Conjectures, Unwarranted Assumptions, Mathematical Proof, Task-Based Interviews

The proving process is a complex combination of creativity and rigor that encompasses a multitude of activities including analyzing and identifying patterns and relationships, generating conjectures and generalizations, and evaluating, refining, proving, and refuting mathematical conjectures (Committee on the Undergraduate Program in Mathematics (CUPM), 2004; de Villiers, 2010; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). However, many students have limited experience with the activities in the proving process that involve uncertainty and decision-making, such as exploring conjectures (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). This limited experience may inhibit students' development of "an attitude of reasonable skepticism" with respect to mathematics (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012, p. 357).

Prior research has shown that high school and undergraduate students make unwarranted assumptions in proofs (Dvora, 2012; Selden and Selden, 1987; Weiss, 2009). In these cases, the students seem to be either unaware they had made an unwarranted assumption or the assumption was based on their perception of a geometric figure and was unrecognized as unwarranted. But what leads students to knowingly make unwarranted assumptions in a non-geometric proof context? Especially when the truth value of the statement is unknown? What makes a student explicitly assume an ungiven assumption rather than consider a statement may be false? These are the questions I investigate in this paper, but my actual research questions are: Why do students explicitly make unwarranted assumptions on prove-or-disprove tasks? What types of explicit, unwarranted assumptions do students make? Under what conditions do students overcome their explicit, unwarranted assumptions?

Literature Review

"One of the most important steps in [mathematical] research is to conjecture what is the truth and then attempt to verify it by hunting down a proof" (Burger, 2007, p. xii). Suppose a mathematician believes a certain conjecture is true, but while attempting to prove it, the mathematician needs an assumption that is not a hypothesis? There seem to be three reasonable courses of action commonly practiced by mathematicians: (a) consider that the conjecture may be false and search for a counterexample, (b) add the assumption to the hypotheses and prove a weaker conjecture, or (c) assume the needed assumption and justify it later (Burger, 2007; Selden & Selden, 1987; Weiss, Herbst, & Chen, 2009). Although (a) and (c) should lead to a decision on the truth value of the conjecture, in (b), the conjecture has been weakened and there is no verification of the truth value of the original conjecture.

In order for students to experience mathematics the way mathematicians do, they need to be engaged in exploring, proving, and refuting conjectures. CUPM (2004) suggests that students majoring in the mathematical sciences "learn a variety of ways to determine the truth or falsity of conjectures…to examine special cases, to look for counterexamples," and to analyze "the effects of modifying hypotheses" (p. 45). In his article on teaching proving, Dean (1996) suggests that when students are exploring a conjecture, "if little progress is being made, the student might add an additional hypothesis and see if this leads anywhere" (p. 53). In Burger's textbook, *Extending the frontiers of mathematics: Inquiries into proof and argumentation* (2007), directions for each problem statement in the text are 'Prove and extend *or* disprove and salvage' (p. xii). Burger (2007) offers many suggestions for extending a proven conjecture or salvaging a refuted conjecture, including weakening or adding to the hypotheses, respectively. Lastly, some high school teachers believe allowing students to make an assumption with the caveat that they must return and justify it later is a valuable instructional strategy (Weiss et al., 2009).

Despite the recommendations of CUPM (2004), many students have limited experiences exploring and refuting conjectures (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). In particular, "students are rarely, if ever, presented with false mathematical statements and asked to determine whether or not they are true" (Durand-Guerrier et al., 2012, p.357), and high school "students are rarely held accountable for finding the conditions under which a claim could be true (Herbst & Brach, 2006)" (Nachlieli, Herbst & Gonzalez, 2009, p. 432).

High school and undergraduate students' limited experiences may partially account for the difficulties they have studying conjectures. Students struggle with (a) knowing how to begin an exploration, (b) formulating ideas and opinions about the truth of a conjecture, and (c) connecting ideas and opinions to proofs or counterexamples (Alibert, 1988). Durand-Guerrier and Arsac (2005) suggest that students' difficulties determining the truth value of conjectures may stem from their narrow collection of possible counterexamples and limited mathematical knowledge as novices. In geometric contexts, high school students often make unwarranted assumptions based on geometric figures or diagrams even though they are taught not to do so (Weiss, 2009).

Other difficulties students have may be related to the inappropriate use of the strategies used by mathematicians and suggested by educators for exploring conjectures. Some high school and undergraduate students unknowingly make unwarranted assumptions that reduce a general conjecture to a special case (Selden & Selden, 1987; Weiss, 2009). Although mathematicians examine special cases when exploring conjectures, they do so knowingly and realize the general conjecture still needs to be considered (de Villiers, 2010). Weiss et al. (2009) reported on high school teachers' reactions to a video episode of a teacher allowing a student to make an unwarranted assumption in a proof under the condition that the student returned to justify the assumption later. Some teachers expressed concern that students would distort this practice (common of mathematicians) by developing a habit of making unwarranted assumptions and failing to return to them (Weiss et al., 2009).

Method of Inquiry

The data in this paper come from a larger study that (a) examined the reasoning students use to evaluate conjectures, (b) identified systematic errors students make during the proving process, and (c) investigated cognitive unity between students' evaluation of conjectures and construction of associated proofs and counterexamples.

Participants

The participants were twelve undergraduate students from a public university in Ohio who had passed at least one proof-based mathematics course with a grade of B or better. Ten students were in their fourth year of undergraduate study, and eleven students were mathematics or secondary mathematics education majors.

Procedures

I conducted two task-based interviews with each participant which were audio-recorded and transcribed. Participants were asked to think aloud during the completion of four tasks and to clarify or expand on their thinking as necessary. Each task was provided one at a time on a separate sheet of paper. Participants were provided with a list of definitions of terms in the tasks, but no other materials were allowed. Participants used a LiveScribe Pen and paper that recorded synchronously audio and writing. After each task, I asked follow-up questions on the participants' work on the task. Upon completing all tasks, I asked each participant general questions about their approaches to and understanding of proof and disproof.

Tasks

Each task required the participants to evaluate a conjecture and prove or disprove the conjecture accordingly. The tasks involve basic properties of functions and were chosen to be accessible to the participants. In line with Alcock and Weber (2010), each task referred to general objects and their properties and should have been approachable with either semantic or syntactic reasoning. The following three tasks will be discussed in this paper:

Injective Function Task: Let $f: A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $f(a_0) = b_0$. Prove or disprove: If $f(a) = b$ and $a \neq a_0$, then $b \neq b_0$.

Monotonicity Task: Prove or disprove: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are decreasing on an interval *I*, then the composite function $f \circ g$ is increasing on *I*.

Global Maximum Task: Prove or disprove: If f is an increasing function, then there is no real number c that is a global maximum for f.

Each statement in these tasks is false. Any noninjective function is a counterexample for the Injective Function Task. A counterexample for the Monotonicity Task requires a function g with outputs that are not elements of the chosen interval *I*. Finally, any increasing function defined on a closed interval serves as a counterexample for the Global Maximum Task.

Analysis

I identified all errors made by participants in the proving process. Instances in which participants made assumptions that were not given hypotheses in the task were classified as *unwarranted assumptions*. An unwarranted assumption was further categorized as *explicit* if the participant expressed awareness of making it.

Results

Six of the twelve students in this study made an explicit, unwarranted assumption. Each of these students did so on exactly one task. In each case, the assumption the student made was exactly what was needed to make the statement in the task true, but was not a given hypothesis. Additionally, in the face of the needed assumption, no student considered the possibility that the statement may be false without prompting from the interviewer. In this section, I describe these students' explicit, unwarranted assumptions and the extent to which they overcame them. First, I discuss Edward and Jalynn, each of who overcame their explicit, unwarranted assumptions and correctly solved the associated tasks. Next, I present Evan and Inigo, who partially overcame their explicit, unwarranted assumptions by constructing task solutions involving cases. Lastly, I discuss Aurelia and Jay who failed to overcome their explicit, unwarranted assumptions and incorrectly solved the associated tasks.

Edward and Jalynn

Edward and Jalynn each made an explicit, unwarranted assumption while attempting to prove the statements in the Monotonicity and Injective Function Tasks, respectively. With prompting from the interviewer, they eventually realized that their assumptions were problematic and correctly decided the statements were false.

Edward

Edward decided that the statement in the Monotonicity Task was true and constructed a proof for it. Within his proof, Edward made the explicit, unwarranted assumption that the range of the function g was in the interval *I*. Upon completing his proof, he noted, "I'll say it's increasing on *I*. Although I didn't do a good job at all of proving where *I* is or working with where *I* is." I asked him how concerned he was about that, and he said:

> If they are both decreasing on an interval *I*, that doesn't necessarily mean the intervals overlap…Because we would need the range of g to be in I…we would need the domain of f to be the same decreasing interval as the range of *g,* and we'd need the domain of *g* to be decreasing. So, and I didn't prove that connection. I should have.

I inquired, "Does that invalidate your proof?" He responded, "Yes. I would not necessarily believe this proof because I didn't match up the range to the domain."

I pressed further regarding this assumption in his proof, and he indicated that it was a necessary but unwarranted assumption: "If I make that assumption,…it does work…But without making that assumption, I don't think it holds….I don't think that's an assumption I can legitimately make." Upon making sense of why the assumption was necessary for the statement to be true, Edward finally decided that the statement was false. He concluded:

> Without this [the assumption], f…could be increasing or decreasing on *I*. I mean, depending on where the range of g is mapped onto the domain of f and what, whether it's increasing or decreasing at that interval…'cause the interval…doesn't necessarily line up at f and g. That makes this statement false.

Thus, through interviewer prompting and analysis of the necessity of his assumption, Edward realized that he could not justify his assumption and the statement was false.

Jalynn

Jalynn knew that the Injective Function Task was related to the concept of one-to-one, but was confused by the notation $f: A \to B$, wondering whether it only indicated the domain and range of the function or if it also implied that the function was onto or one-to-one. After she began her proof, she realized she needed the assumption that the function f was one-to-one and said, "I can assume that it's one-to-one….There would just be a condition for it then." With this explicit, unwarranted assumption, Jalynn constructed a proof for the statement.

After she completed her proof, I asked Jalynn if she thought that the assumption that f was one-to-one was a necessary condition for her proof. She said that she was unsure because she was still confused about whether the notation indicated that the function was one-to-one. So, I asked her what she thought if we just assumed that the notation only indicated the domain and range of the function, and she replied "[that] probably would change it, but, I'm just trying to think of an example." She wrote $f(x) = x^2$, and showed that $f(3) = f(-3) = 9$. She indicated that this function was not one-to-one and that f being one-to-one was a necessary condition for this statement to be true.

Finally, I asked Jalynn to clarify whether she thought the statement was true or false, and she replied "it's true if it's one-to-one and it's false if. Overall it would be false in any case, just like how here [referring to her counterexample $f(x) = x^2$]…I guess it just asks for the general case." Like Edward, through prompting to consider the necessity of her explicit, unwarranted assumption, Jalynn analyzed it in the context of an example and realized she needed to consider the general case in which the statement was false.

Inigo and Evan

Inigo and Evan each made an explicit, unwarranted assumption while proving the statements in the Injective Function and Global Maximum Tasks, respectively. They were able to partially overcome these assumptions by constructing cases–one with and one without the assumption–for their solutions to the tasks. However, neither student realized that only one of the cases applied to the given task.

Inigo

Inigo assumed the statement in the Injective Function Task was true. While constructing his proof, in order to claim $f(a) = f(a_0)$ implies $a = a_0$, Inigo said he needed to assume f was one-to-one. He did so, making an explicit, unwarranted assumption, and completed his proof. He then said, "I know there's a flaw in some logic there because of this [underlining his assumption that f is one-to-one], but I'm finished." Inigo was content to stop with an invalid proof, but I was not willing to let it stand. I asked him if he could tell me why he thought it was wrong, and he said "I am assuming that this is one-to-one. And it's not necessarily one-to-one….And I know you can't actually make that assumption here". Inigo then realized that $f(x) = x^2$ served as a counterexample and said "So when it's one-to-one, that holds [indicating his proof]; and then when it's not, there [underlining his counterexample]….I broke this into cases." Thus, through prompting, Inigo only partially overcame his explicit, unwarranted assumption, deciding that a complete solution to the task included two cases. He did not realize that only the case without the assumption applied to the statement in the given task.

Evan

On the Global Maximum Task, Evan thought mistakenly that the given statement said the function *did* have a global maximum rather than saying it did *not* have a global maximum. Thus, Evan decided the statement was false and constructed a proof by contradiction to disprove the statement (proving the function did not have a global maximum). However, this proof included the implicit, unwarranted assumption that the domain of the function f was $\mathbb{R}$.

Because Evan had misread the statement, I confirmed with him that he thought the statement was false and asked him to reread the statement to ensure he was saying what he wanted to say. Upon looking back at his disproof, he realized he made the assumption that the domain of f was $\mathbb{R}$ and said he would "add a disclaimer" to his proof. He included his assumption in his disproof which made it an explicit, unwarranted assumption. Additionally, Evan wrote a second case in which the domain was a closed interval and proved the statement was true in that case. Like Inigo, Evan concluded he had two cases, but did not realize only one actually solved the given task.

Aurelia and Jay

Aurelia and Jay each made an explicit, unwarranted assumption while proving the Global Maximum and Injective Function Tasks, respectively. Both students failed to overcome these assumptions and incorrectly solved the tasks.

Aurelia

Aurelia struggled to determine the truth value of the Global Maximum Task. Upon first reading the statement, Aurelia said "So, I'm assuming that means if f is increasing throughout the whole entire function? So, this is obviously not true if you have…[a] function that stops at a certain point." However, she questioned whether a function could have a restricted domain. She drew a graph of $f(x) = x^2$ restricted to $[0,2]$ and asked herself "is that considered a function?" She was uncertain whether it was a function, but decided to assume that it was not a function because she thought I was "not trying to trick [her]". Thus, she made the explicit, unwarranted assumption that a function cannot have a restricted domain. This allowed her to assume that the function in the task was defined on $\mathbb{R}$, and she used this assumption to incorrectly "prove" the false statement.

Jay

Jay assumed the statement in the Injective Function Task was true and constructed a proof in which he made the explicit, unwarranted assumption that the function f was one-to-one. After he completed his proof, I asked him what the key step was in his proof, and he said "Well, just, for me, the idea since $a \neq a_0$, then, I, sort of, made a jump and assumed that $f(a)$ then is not equal to $f(a_0)$." I inquired about making this "jump", and he replied "That'll only be true if the function was one-to-one, but from just the given information, I don't know exactly if it is one-to-one." I continued attempting to draw information out of him about his use of one-to-one despite being uncertain whether f was one-to-one, but I was unable to get him to reconsider his assumption. He repeated that his proof would work if he knew the function was one-to-one, but he never indicated decisively whether he knew this. Despite my pressing, Jay was unable to overcome his explicit, unwarranted assumption and was satisfied with his "proof" for this false statement.

Discussion

Consistent with prior research with high school and undergraduate students, the students in this study seemed to lack key strategies for thinking about and identifying false statements. Some students made explicit, unwarranted assumptions rather than consider a given conjecture was false. In each case, the student completed a "proof" of a false statement that relied on and included the explicit, unwarranted assumption. Multiple students in this study, including Edward and Inigo, said they are rarely asked to consider statements in which the truth value is unknown. Inigo noted, "All throughout math classes, we're bombarded with what's true and not with what's false." It seems possible that limited exposure to conjectures may have inhibited these students' development of a healthy skepticism toward mathematics, as has been suggested in the literature (Alibert & Thomas, 1991; de Villiers, 2010; Durand-Guerrier et al., 2012). This may have led the students to do whatever it took to prove the statements rather than consider their potential falsity. This suggests students need more opportunities to engage in evaluating, refining, and refuting conjectures.

Another possible explanation for the students' behavior, as indicated by the concerns voiced by the high school teachers in Weiss et al.'s (2009) study, is that these students were misusing a common technique practiced by mathematicians. Edward was the only student who indicated he knew he should have returned to his assumption to justify it. The other students seemed content with simply adding to the hypotheses, even though some expressed concern over doing so. It is possible that these students were misusing a legitimate strategy they had seen mathematicians use, which may account for their uneasiness. However, these students also expressed a clear understanding of the logical nature of proofs during follow-up questioning, so perhaps their concern resulted from their knowing the assumptions were unwarranted, but not knowing what else to do. This would suggest again that the students' struggles were related to their limited experience with statements of unknown truth value.

Each explicit, unwarranted assumption made by the students in this study was an ungiven hypothesis that was necessary for the statement to be true. Thus, the need for each assumption should have indicated the potential falsity of each statement as well as exactly what was needed in a counterexample. On the Monotonicity Task, Edward assumed the range of the function $\boldsymbol{g}$ was in the interval I. On the Injective Function Task, Jalynn, Inigo, and Jay each assumed the given function $\boldsymbol{f}$ was one-to-one, and on the Global Maximum Task, Evan and Aurelia assumed the domain of the function $\boldsymbol{f}$ was $\mathbb{R}$. For each task, the fact that these assertions are not necessarily true is precisely why the statements are false. If students were accustomed to Burger's (2007) instructions to 'prove and extend or disprove and salvage,' the realization that they needed these assumptions to prove the statements should (a) indicate the statements are false, (b) provide the necessary conditions for a counterexample, and (c) specify an assumption to add to the hypotheses to salvage the statements. This would make the need for the assumption in a proof attempt a powerful tool in solving the task. However, it does not seem as though the students in this study were trained to recognize this power of needed assumptions.

Despite some of the students' concerns regarding their assumptions, prompting from the interviewer to reconsider their "proofs" or assumptions seemed necessary for them to overcome or partially overcome their explicit, unwarranted assumptions. However, this did not work in all cases as Aurelia and Jay were unable to overcome their assumptions. Interestingly, Inigo, Evan, and Jay each indicated during follow-up questioning that prove-or-disprove tasks are more difficult than prove tasks because if they got stuck in the middle of a proof, they would have to question whether they were trying to prove a false statement and consider looking for a counterexample. However, none of these students did this when confronted with the need for an ungiven assumption. Perhaps they did not consider or recognize this as a form of 'getting stuck.' Either way, it seems they possessed knowledge of an appropriate strategy to use the situation, but failed to use it.

The results of this study suggest a couple ideas for dealing with students making explicit, unwarranted assumptions. First and foremost, engage students in evaluating conjectures, including false conjectures. And so often. If students are rarely faced with conjectures, then it will be difficult for them to develop and use appropriate strategies for dealing with situations that are common in conjecturing contexts but not in contexts in which the truth value of a statement is known. Additionally, it seems as though students may not be inclined to question explicit, unwarranted assumptions on their own. They may need prompting from their instructors in order to recognize that needing an ungiven assumption means that they are 'stuck.' And we need to encourage students to explore this special type of being 'stuck' because of its potential power to indicate why a statement is false, what is needed for a counterexample, and what is necessary to make it true. Engaging students in evaluating conjectures and helping them recognize the potential power of a needed assumption may allow them to move ever closer toward thinking like mathematicians think.

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