Mathematicians' rationale for presenting proofs: A case study of introductory abstract algebra and real analysis courses

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Proofs are essential to communicate mathematics in upper-level undergraduate courses. In an interview study with nine mathematicians, Weber (2012) describes five reasons for why mathematicians present proofs to their undergraduate students. Following Weber's (2012) study, we designed a mixed study to specifically examine what mathematicians say undergraduates should gain from the proofs they read or see during lecture in introductory abstract algebra and real analysis. Our preliminary findings suggest that: (i) A significant number of mathematicians said undergraduates should gain the skills needed to recognize various proof type and proving techniques, (ii) consistent with Weber's (2012) findings, only one mathematician said undergraduates should gain conviction from proofs, and finally (3) some mathematicians presented proof for reasons not described in Weber's (2012) study such as to help their students develop appreciation for rigor.

Key words: Proof, Purpose of proof, Proof presentation, Undergraduate mathematics

In upper-level mathematics courses, mathematicians regularly use proofs to convey mathematics to their students. As a result, mathematicians expect their students to gain some understanding from the proofs they present. A plethora of research suggest that student find the concept of proof problematic (Harel & Sowder, 1998; Inglis & Alcock, 2012; Moore, 1994; A. Selden & Selden, 2003). Research on undergraduates interaction with proofs suggests that undergraduates often times have difficulty with determining the validity of a proof and/or constructing a valid proof (Alcock & Weber, 2005; Inglis & Alcock, 2012, Selden & Selden, 2003; Weber, 2010). For instance, Selden and Selden (2003) argued that when reading proofs undergraduates tend to focus on surface features of mathematical arguments as opposed to its *global* feature. Participants in their study showed only limited ability to determine if a mathematical argument is valid or qualifies as a proof or not.

Empirical studies focusing on what mathematicians expect their upper-level undergraduates to gain from proofs are rare. In a semi-structured interview with nine mathematicians, Weber (2012) argued that most mathematicians present proofs mainly to facilitate their students' understanding of mathematical concepts and/or illustrate some proving techniques. Yopp (2011) also reports that in advanced undergraduate mathematics courses, mathematicians mainly present proofs to show their students how to prove theorems.

The extent to which students actually learn mathematical concepts from seeing proofs remains an open research problem. However, one can infer from existing research that undergraduates actually do not gain mathematical understanding from proofs (Conradie & Frith, 2000). Weber (2012) also evidenced that mathematicians rarely present proofs to convince their students that a theorem or a proposition is true; this is in contrast to the primary role of proof in mathematics scholarship (Hersh, 1993). Alternatively, Hersh (1993) maintains that in mathematics classroom, the primary goal of presenting proofs should be to provide an explanation for why a theorem is true. Interestingly, some participants in Weber's (2012) study expressed doubt if proof is indeed an effective way to convey mathematics to all their students. Our study contributes to the growing body of literature on the purpose of proof in undergraduate mathematics instruction by examining the following research question: What roles do proofs play in the teaching of introductory abstract algebra and/or real analysis courses? In what follows, we discuss the theoretical framework guiding this study.

Theoretical framework and literature review

The three most important roles of proofs discussed in the proof literature are: (1) *conviction* or *verification*, (2) *explanation, and* (3*) illustrating proving techniques*. *Convincing* is the idea that a proof demonstrates that a theorem is true. Although undergraduates and surprisingly mathematicians (e.g., Weber, Mejia-Ramos, & Inglis, 2014) are sometimes convinced without proof, De Villiers (1990) writes that "the well-known limitations of intuition and quasi-empirical methods" underscore the vitality of proof as a useful means of *verification* (p.19). Convincing is perhaps the primary goal of any proof. Indeed, some such as Hersh (1993) actually define proof simply as "a convincing argument, as judged by competent judges" (p. 389).

Convincing may be the primary goal of any published proof; however, there is a consensus that the functionality of a proof is not, and should not, be limited to verifying that a theorem is true (De Villiers, 1990; Hersh, 1993). The fact that we have different published proofs in peer-reviewed journals of a single known result inevitably leads us to believe that proofs are far more than a certificate of truth. Indeed, it appears that there is considerable interest in the insight that is gained from the reasoning utilized in a proof. For a mathematician, a proof—beyond convincing—also functions as an explanatory argument. To explain is to provide insight as to why a theorem is true (De Villiers, 1990; Hersh, 1993; Knuth, 2002; Thurston, 1995; Weber, 2002; Weber, 2008). Explanatory proofs are insightful precisely because they make "reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depended upon the property" (Steiner, 1978). According to De Villiers (1990), explanatory proofs provide "psychological satisfactory sense of illumination" (p.19).

Mathematicians' desire for explanatory proofs is evident in the controversy surrounding Appel and Haken's joint proof of the Four-Color theorem (Thurston, 1995). Appel and Haken's joint proof heavily depended on a computer; for that reason, renowned mathematicians such as Paul Halmos showed dissatisfaction toward the proof, as it apparently did not provide any insight for why the theorem must be true. Stressing the importance of the explanation in a proof, Hanna (2000) writes: "[a proof] becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding" (Hanna, 2000). In fact, all eight mathematicians interviewed in Weber (2008) claimed that the primary reason they read published proofs is to gain insight. In particular, in undergraduate mathematics education, Hersh (1993) argued that the primary role of proofs should be to offer insights and provide complete explanations why a given theorem is true. Harel and Sowder (2007) complement this when they say: "…mathematics as sense making means that one should not only convince oneself that the particular topic/procedure makes sense, but also that one should be able to convince others through explanation and justification of her or his conclusions" (p. 808-809). In addition, Hersh

(1993) maintains that one should consider the explanatory power of a particular proof when making the decision whether or not a proof is worth presenting in class. Hersh (1983) writes: "proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding" (p.7). Therefore, it is important that instructors make use of more explanatory proofs in their instruction when possible.

Proofs, beyond convincing and explaining, can function as tools to communicate techniques or ways of reasoning that can later be used to tackle other problems. Thurston (1995) argued that mathematicians sometimes use proofs to communicate a developed body of common knowledge or new techniques in the case of truly novel proofs. For example, mathematicians interviewed in Weber's (2010) study stated that when reading a proof, they would hope to learn new techniques that might eventually help them prove conjectures or problems they have been thinking about in their research.

De Villiers (1990) proposes even more roles of proofs: proofs as a means of discovery and proofs as a means of systematization. He argues that proofs are our only tool "in the systematization of various known results into deductive system of axioms, definitions and theorems" (p.20). Take, for example, the proof of the intermediate value theorem for continuous functions; he asserts that the primary function of this proof is basically a systematization of continuous functions. Systematization, among other things, provides global perspective, simplifies mathematical theories, and enables us to identify inconsistencies, circular reasoning, and hidden assumptions (De Villiers, 1990). In addition, a proof enables us to explore, generalize, analyze, and discover mathematical ideas (De Villiers, 1990). For example, the invention of non-Euclidean geometries would have been completely unthinkable without our capacities of deductive reasoning and proof, since these ideas are unintuitive.

Research methodology

Fifteen mathematicians agreed to participate in our study. All participants were solicited from a large public university in the United States. The mathematicians come from a wide range of research interests including, but not limited to, analysis, algebra and topology. The lead author provided the mathematicians a written task asking them to briefly describe what they would hope an undergraduate student enrolled in introductory abstract algebra and/or real analysis would gain from reading or seeing proofs during lecture. Fourteen of the 15 mathematicians who agreed to complete the written task have at least seven years of teaching experience in tertiary institution. While a significant number of the participants taught at least two proof-based mathematics courses, four mathematicians said they have not taught any proof-based course at this institution.

We also conducted task-based interviews with three mathematicians (an algebraist, and analyst and a topologist). Two of the mathematicians who agreed to be interviewed did not complete the written task. The two algebraists and the one analyst that we interviewed have at least ten years of experience teaching introductory abstract algebra and real analysis respectively. During the interview we asked the mathematicians the following questions:

- Why would you present the proof of Lagrange's theorem?
- In general, what would you say is the purpose(s) of presenting proofs during lecture in undergraduate mathematics courses such real analysis (or abstract algebra if the interviewee is algebraist?

- Is there a proof that you would consider a 'must see' in your introductory real analysis (or abstract algebra if the interviewee is algebraist) (adopted from Weber's (2012) study)

Results and discussion

We present our preliminary results as follows. Recall that the main goal of this study is to explore what mathematicians hope their undergraduate students gain from the proofs they present in upper-level undergraduate courses such as abstract algebra and/or real analysis. Two researchers independently coded participants written response based on categories presented in Weber's (2012 study. As it is evidenced in Table 1, the majority of mathematicians (60%) said that they would hope undergraduates develop proficiency in recognizing proof type. This includes, but is not limited to, identifying whether the proof is a direct proof, a proof by contradiction, a proof by cases, or a proof by mathematical induction. We find this surprising because we were expecting that mathematicians would only say this for undergraduates in intro-to –proof courses. Consistent with Weber's (2012) study, we found that (1) a significant number of our participants (46.67%) said they would hope undergraduates would learn new proving techniques from seeing proofs during lecture, and (2) only one mathematician described conviction as an important role of proof for undergraduates. The following interview excerpt indicates that mathematicians present proofs to illustrate some proving techniques.

I: Is there a proof that you consider a must-see in your abstract algebra course?

P: A proof that I consider a must-see um there are a number of types of proofs that I think that they should see um for instance um when some either the uniqueness of the zero element, the uniqueness of inverses of elements, something to that effect. I think it's a must-see. Um um what other things? Uh um either the idea that a kernel of an image of a homomorphism is a subgroup

I: Why would you think that is a must-see or is important for them to see?

P: to see? Well because many constructions or many ideas that we use to study groups are based on the study of homomorphisms between groups.

Additionally, some mathematicians said they would hope that undergraduates would develop proficiency in logical inferences from seeing proofs presented in upper-level undergraduate mathematics courses. Also, a small percentage of mathematicians (13.33%) said they presented proofs so that students can appreciative the rigor that goes into writing proofs. We find this interesting because it has not been evidenced, to our knowledge, in any empirical study.

Table 1 Mathematicians' reasons for presenting proofs in upper level mathematics courses

Reason	Percentage of participants
To help students recognize proof type	60%
To illustrate proving techniques	46.67%
To develop proficiency in logical inferences	33.33%
To illustrate why a theorem is true	33.33%
To help students recognize proof type	20%
To develop appreciation for rigor	13.33%
To communicate mathematical ideas	13.33%
To establish that a theorem is true	6.67%

Discussion questions and implications for further research

We believe that our preliminary study contributes to the scarce literature on the role proofs play in undergraduate mathematics education. We plan to analyze our interview transcripts to examine if there are additional reasons why mathematicians present proofs in upper-level mathematics courses. During our presentation, we would like to get some feedback on the following questions.

(1) What methodological suggestions might you offer us to examine any non-mathematical benefits, assuming that there are some, that one can acquire from reading or seeing a proof during lecture, and to what extent do we care?

(2) Are there good reasons to believe that mathematicians present proofs in different classes for different reasons? How can we explore that?

In summary, we believe that our study provides further evidence for the claim that convincing should not be the primary goal of presenting proofs in mathematics instruction. Finally, we hope that further research such as interviewing more mathematicians can provide insight into additional roles that proof can play in undergraduate mathematics education.

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