Beyond procedures: Quantitative reasoning in upper-division Math Methods in Physics

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Abstract. Many upper-division physics courses have as goals that students should 'think like a physicist.' While this is not well-defined, most would agree that thinking like a physicist includes quantitative reasoning skills: considering limiting cases, dimensional analysis, and using approximations. However, there is often relatively little curricular support for these practices and many instructors do not assess them explicitly. As part of a project to investigate student mathematical learning in upper-division physics, we have developed a number of written questions testing the extent to which students in an upper-division course in Mathematical Methods in Physics can employ these skills. Although there are limitations to assessing these skills with written questions, they can provide insight to the extent to which students can apply a given skill when prompted.

Key words: Physics, Mathematics, Upper-division, Quantitative reasoning

Introduction

This work is part of a collaboration to investigate student learning and application of mathematics in the context of upper-division physics courses. Our project seeks to study student conceptual understanding in upper-division physics courses, investigate models of transfer, and to develop instructional interventions to assist student learning.

While physics education research (PER) has primarily focused on introductory-level courses, there are increasing efforts to expand into the upper division [1]. The core sophomore- and junior-level theory and laboratory courses taken by most physics majors have begun to receive the attention of researchers and curriculum developers, including electricity and magnetism [2], thermal physics [3], classical mechanics [4], quantum mechanics [5], and advanced laboratories [6]. One key course that remains under-researched (with a few exceptions [7]) is a course taught by most departments that is commonly known as "mathematical methods." Unlike similarly-named courses for prospective mathematics teachers, this highly theoretical course focuses on the mathematical techniques that students will encounter in upper-division physics courses. Such a course is typically intended to serve as a bridge between introductory level courses and the more challenging physics and mathematics students encounter in the core upper-level theory courses in the physics major (particularly electricity and magnetism, classical mechanics, and quantum mechanics).

Typically the learning goals for courses of this nature (called MM courses for this paper) focus on content goals, with a syllabus that covers a list of topics including differential and integral calculus, series, complex numbers, vectors and vector calculus, differential equations, and linear algebra. As if this daunting list of topics were not sufficient, often the MM course also has stated or implicit goals that go beyond specific physics and mathematics context. For example, students in these courses are expected to 'think like a physicist' when solving quantitative problems. However, despite its seeming importance, this phrase is not always operationally defined. Examples of skills that might be included in this term include connecting physical intuition with mathematics, checking units and performing dimensional analysis, considering limiting cases, and using approximations. While instructors value these skills, and there has been some previous discussion of them [8], their value is often left implicit and they are not often explicitly taught or assessed.

In this portion of the project, we have sought to investigate the development of mathematical understanding that goes beyond procedural skill and calculation, to probe the quantitative reasoning skills whose development is often left implicit. A key goal of our larger project is to develop a series of tasks that would be suitable for use by instructors in the MM course. For this report, we describe preliminary efforts to develop written tasks that ask students to apply such skills and to document student responses to them. For the purpose of this paper, we will refer to questions designed to assess several quantitative reasoning skills. The set of these skills is not intended to be complete, but we have identified several that appear to be relevant as starting points:

- Using dimensional / unit analysis
- Testing expressions with limiting cases
- Using approximations, e.g., with Taylor series
- Identifying errors in solutions
- Predicting the effects of problem changes on the resulting solution

During the current study we have examined several of these skills. Examples of using approximations and of predicting the effects of changes to a problem on the resulting solution have been described previously [9, 10]. For this short paper we restrict the discussion to the second: limiting cases.

This work has taken place in the context of a MM course taught at a large public comprehensive university serving a diverse student population. The course is required for physics majors and is a prerequisite for upper-division theory courses; for most students it is one of the first upper-division courses taken. The course uses the text by Boas [11] and covers a fairly standard list of topics. It meets for two 75-minute blocks per week. The course has as prerequisites three semesters of calculus, and most students have completed at least two semesters of introductory physics. The author has taught the course six times, with enrollments between 12 and 19. Data shown is from written responses to the *Evaluate the Expressions* task shown below.

Theoretical Perspective

This portion of our project is driven by practice; we are seeking to learn what is difficult for students and develop instructional interventions. The theoretical framework guiding our analysis for this portion is "identifying student difficulties" [12].

A growing body of work in PER has examined student use of mathematics in physics. In particular, several models have been proposed to describe student use of mathematics. In each of these models, successfully executing the mathematical procedure in question is only one element of success. Redish has proposed a framework to describe student usage of mathematics in science course [13], describing stages of modeling, processing, interpreting, and evaluating. For the specific case of upper-division physics courses, Wilcox et al. have proposed the ACER framework 'to guide and structure investigations of students' difficulties with the sophisticated mathematical tools used in their physics classes.' [14] In this framework, students must activate the appropriate mathematical tool in addition to constructing a model, executing the mathematics, and then reflecting on results.

In traditional physics courses, instruction and assessment tend to focus disproportionately on what Redish describes as the processing phase, or what the ACER framework frames as executing mathematics. In a recent paper, Redish states that 'our traditional way of thinking about using math in physics classes may not give enough emphasis to the critical elements of modeling, interpreting, and evaluating' [12]. The tasks in this study reflect our attempt to investigate other aspects of mathematical thinking. For example, to probe reflection or evaluation of results, students might be asked to evaluate expressions for correctness or identify errors in solution, rather than performing procedures to generate expressions.

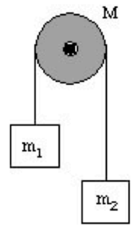
We have examined the RUME literature for corresponding studies but there appears to be relatively little focus on these sorts of skills, at least at this level of instruction. The work by Sherin describing how physics students read mathematical expressions bears upon the task described below [xx]. Thompson has examined 'quantitative reasoning' and its relationship to modeling of phenomena [1x5]. In addition, this project as a whole considers transfer broadly, as we examine how students apply their mathematical knowledge in the context of physics courses, and we have been guided in part by the work of collaborator Wagner [1x].

It is important to note there are limitations of the current study, which artificially separates these tasks from a problem solution. Qualitative studies of students in the process of solving problems can help to give insights into when and how students activate such resources and at what phase of problem solving they are employed. This project has a more modest goal, asking whether students can apply skills when their use is explicitly cued. Additional qualitative work, including interviews, is ongoing.

Methodology

For the purpose of this paper, we will focus on a sample task that is illustrative of the quantitative reasoning skills that we are describing. This task, which we refer to as the *Evaluate the Expressions* task, involves the evaluation of mathematical expressions for correctness. The problem is posed on the first day of the MM course on an ungraded quiz subsequently explored in a large group discussion while the tasks projected on a screen.

Consider the motion of two blocks connected to form an Atwood's machine. The masses of the two blocks are m1 and m2 and the mass of the pulley is M. The following expressions are proposed for the acceleration of block 1. For each, evaluate whether the expression could be correct an explain briefly:



$$a = \frac{m_2 - m_1}{m_2 + m_1} g \qquad a = \frac{m_2}{m_2 + m_1 - {}^{M}\!/_{2}} g \qquad a = \frac{m_2 - {}^{M}\!/_{2}}{m_2 + m_1 + {}^{M}\!/_{2}} g$$

FIGURE 1. A written task in which students are asked to evaluate multiple expressions for possible correctness given a physical situation.

The task describes a simple physical system (known in physics as an 'Atwood's machine') in which two massive blocks are connected by a string across a pulley (see Figure 1). Students are shown three expressions for the acceleration of one of the two blocks and asked to determine whether the expressions could be correct. (All three expressions are incorrect.) The Atwood's machine is a common instructional task and widely used in physics courses as an example of the application of Newton's laws. Most students would have solved a similar problem in their introductory mechanics course, encountering the problem first in the case of a massless pulley, solving the problem by drawing free body diagrams for the two blocks and applying Newton's second law with appropriate constraints to generate equations of motion. Later, in the section of the course focusing on rotational dynamics, the effect of torques on the pulley mass are treated explicitly.

The problem is different from many tasks that students have encountered to this point in that it asks for evaluation (per Redish) or reflection (per Wilcox) rather than a procedural computation. Students are not asked to solve the problem, which has no numerical values. The expectation is that students will use quantitative reasoning to arrive at an answer, by checking limiting cases (e.g., considering extreme values of variables; a very large pulley mass means small acceleration) or by identifying cases in which the expression becomes unphysical (e.g., in the second expression, if $M/2 = m_1+m_2$ acceleration would be infinite).

The problem has been administered in three sections of MM ($N = 47$) before any instruction. Student written responses were examined and coded. Answers were coded for answer and explanation. Student responses are tentatively assigned to categories that arise iteratively according to the major themes identified in the data. For the explanations, we use open coding (drawn from grounded theory), in which the entire data corpus is examined for common trends, and all data are reexamined and grouped into the defined categories.

Results

Student responses were coded iteratively based on both the correctness of their assessment and the explanation used in support. In the final version of the rubric, twelve distinct but not exclusive codes were used, with an 'other/blank' category used when students provided no intelligible explanation. Of the twelve codes, only a few were commonly used. We provide brief examples of several of the most common codes and criteria in Table I.

While some of the codes that emerged were expected, many of them were not. For example, a few students evaluated the correctness based on whether the resulting expression would have the correct algebraic sign, although no coordinate axis is specified. Of course the written explanation may not indicate fully the underlying reasoning that students are using. For example, a few students gave explanations in which they stated that the rotational inertia of the massive pulley mass would decrease the acceleration. Others simply noted the presence or absence of the pulley mass in the expression. We cannot be sure whether the students in two groups were using similar reasoning.

TABLE 1. Sample codes for *evaluate the expression* task. The codes were not exclusive, so a student response might include both mechanism and mass difference, for example.

Code	*Description*
Solution	Attempted to solve problem directly
Variables	Noted presence or absence of variables in expression
Mechanism	Described physical mechanism for motion (forces, energy)
Mass difference	Commented on presence or absence of term describing difference in masses $\pm(m_2-m_1)$

The data indicate that this task is challenging task for students. Only one student offered a completely correct solution. Ten others identified all three solutions as incorrect but with incomplete or incorrect explanations. Many students gave no response. About 10% of students were coded as 'blank' for the first expression, and 20% for the second. Written explanations often indicated a lack of confidence in responses: "This is not correct because the mass of the pulley needs to be incorporated (although to tell the truth I am not sure how)."

The approaches used by students varied considerably, and while many did give explanations that call upon physical intuition or an attempt to parse the mathematical expression, others seemed to respond as though this task were a more typical end-of-chapter problem. About 10% of the students solved the problem directly, and a few others performed algebraic manipulations of the given expressions. A few responses appeared as though they were to a multiple-choice question, with a circle or check mark next to one expression. One student wrote that the first response "Needs the pulley!" and circled the second response, writing "This one!!!" A few students mentioned partially remembered results: "my very rusty memory only recalls subtracting from the bottom." These responses suggest an epistemological stance that is quite different that the problem intends, and students may need to have the purpose of this activity framed very explicitly.

Limiting cases: Only a handful of students explained using quantitative reasoning from the categories described above to evaluate this expression. As an example of a response that we coded as using limiting cases, one student wrote, "No, if m_2= 0 kg then this formula makes a = 0 m/s^2 which is clearly not true."

Presence / absence of variables: A very common code reflected responses in which explanations referred to the presence or absence of variables: "This could be correct because all relevant variables are used." For the first expression, most students gave explanations that referred to the absence of the mass of the pulley. A few (~5%) stated that the expression was incorrect because of the absence of the pulley mass *M,* but the more common response was to state (correctly) that the expression would be correct if the mass of the pulley is negligible.

Physical mechanism: Several student responses described a physical mechanism for the motion. A smaller group reflected an attempt to reconcile the mathematical form of the expression with this sense of physical mechanism. The most obvious examples of this were students who referred explicitly to the presence or absence of a term with the difference in masses. These responses included some in which the presence of this term was noted: "Correct [first expression]; m_2 is countering m_1 so m_1 is accelerating at a portion of g." A few students gave explanations that reflected similar reasoning but with respect to other quantities: "this [second] expression raises the value of acceleration as the mass of the pulley increases leading me to believe this is incorrect."

Discussion

This portion of the project is in initial stages, and further research is needed. Interviews focusing on these skills are in progress, including the *Evaluate the Expressions* task. We offer two tentative observations.

First, many students entering the MM course do not successfully reason quantitatively even when explicitly prompted to do so. The responses given by some students suggest that they do not recognize that the tasks shown require them to step away from solving the problem directly or remembering its answer in order to reason whether a solution might be correct. Relatively few students spontaneously examined the expressions for special cases of the variables in the problem or related to a sense of physical mechanism. Traditional instruction, focused almost entirely on procedures, does not necessarily lead students to develop other quantitative reasoning skills.

Second, given that physicists value the quantitative skills described, that there is a need for tasks that can be used in instruction and assessment. Redish and Kuo [12] have recently written that students "need to learn a component of physics expertise not present in math class—tying those formal mathematical tools to physical meaning….We as physics instructors must explicitly foster these components of expert physics practice to help students succeed in using math in physics." Yet the majority of problems in the course text are merely mathematical exercises that do not explicitly address these reasoning skills.

Questions for audience

Is there existing theoretical or empirical work in RUME that would complement or inform this study?

PER-influenced physics instruction has led to increased emphasis on conceptual understanding and sense-making rather than procedural tasks and routine computations. We clearly have a lot to learn from the RUME community regarding the ways that students think about mathematical sense-making; what have we overlooked due to our disciplinary filters?

What are the implications for research and practice in mathematics education?

References

[1] M. E. Loverude and B. S. Ambrose, "Editorial: Focused Collection: PER in Upper Division Physics Courses," *Physical Review Special Topics in Physics Education Research* 11, 020002 (2015).

[2] See, e.g., R. E. Pepper, S. V. Chasteen, S. J. Pollock, and K. K. Perkins, "Observations on student difficulties with mathematics in upper-division electricity and magnetism," *Physical Review Special Topics Physics Education Research* **8**, 010111 (2012).

[3] See, e.g., B.S. Ambrose, "Investigating student understanding in intermediate mechanics: Identifying the need for a tutorial approach to instruction," *American Journal of Physics*, *72*(4), 453-459 (2004).

[4] See, e.g., T. I. Smith, J. R. Thompson and D. B. Mountcastle, "Student Understanding of Taylor Series Expansions in Statistical Mechanics," *Physical Review Special Topics - Physics Education Research* 9, 020110 (2013).

[5] See, e.g., S. B. McKagan, K. K. Perkins, and C. E. Wieman, "A deeper look at student learning of quantum mechanics: the case of tunneling." *Physical Review Special Topics – Physics Education Research* 4, 020103 (2008).

[6] B. M. Zwickl, T. Hirokawa, N. Finkelstein, and H. J. Lewandowski, "Epistemology and expectations survey about experimental physics: Development and initial results," *Physical Review Special Topics – Physics Education Research* **10**, 010120 (2014)

[7] For an example of studies in a combined mathematical methods / classical mechanics course, see SEI: Science Education Initiative at University of Colorado Boulder, http://www.colorado.edu/sei/departments/physics.htm

[8] E. F. Redish, "Problem solving and the use of math in physics courses." In Conference World View on Physics Education in 2005, Delhi, August 21–26, 2005.

[9] M.E. Loverude, "Assessment to complement research-based instruction in upper-level physics courses, 2011 PERC Proceedings [Omaha, NE, August 3-4, 2011], edited by N. S. Rebello, P. V. Engelhardt, and C. Singh [AIP Conf. Proc. 1413, 51-54 (2012).

[10] M. E. Loverude, "Surprisingly, there is an actual physical application," Student understanding in Math Methods, 2013 PERC Proceedings [Portland, OR, July 17-18, 2013], edited by P. V. Engelhardt, A. D. Churukian, and D. L. Jones.

[11] M. Boas, *Mathematical Methods in the Physical Sciences,* Wiley (2006).

[12] P. R. L. Heron, "Empirical investigations of learning and teaching, Part I: examining and interpreting student thinking," *Proceedings of Enrico Fermi Summer School on Physics Education Research.* (pp. 341-350). Italian Physical Society (2003).

[13] E. Redish & E. Kuo, "Language of physics, language of math: disciplinary culture and dynamic epistemology," *Science and Education* 24, 561-590 (2015).

[14] B Wilcox et al, "Analytic framework for students' use of mathematics in upper division physics," *Physical Review Special Topics – Physics Education Research* **9**, 0201191 (2013).

[15] B. L. Sherin, "How students understand physics equations," *Cognition and Instruction* 19, 479–541 (2001).

[16] P. W. Thompson, "Quantitative reasoning and mathematical modeling," in L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming.

[17] J. F. Wagner, "Transfer in pieces," *Cognition and Instruction* **24**, 1–71 (2006).