Student Conceptions of Integration and Accumulation

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Prior research has shown several common student conceptualizations of integration among undergraduates. This report focuses on data from a large scale written assessment of students' views of integration and accumulation to categorize student conceptualizations and report their prevalence among the undergraduate population. Analysis of these results found four categorizations for student descriptions of the definite integral: antiderivative, area, an infinite sum of one dimensional pieces, and a limit of approximations. Similarly, when asked about an accumulation function, student responses were grouped into three categorizations: those based on the process of calculating a single definite integral, those based on the result of calculating a definite integral, and those based on the relationship between changes in the input and output variables of the accumulation function.

Key words: Calculus, Integration, Accumulation

Difficulties with conceptions involving integration are documented (Bezuidenhout, J. & Olivier, A., 2000; Orton, 1983; Rasslan, S. & Tall, D., 2002; Hirst, 2002). Some of these difficulties may come from a limited conception of a rate (or derivative) as a quotient (Byerley et al., 2012).

The central conceptions of a definite integral are: procedure of antidifferentiation, area, and accumulation. These categories are consistent with those used by Jones (2015 jmb) and Hall (2010). Hall also had a "linguistic" category which he noted did not constitute actual mathematical understanding, so we exclude it here.

For conceptions of integral involving antidifferentiation, students have been found to be generally competent in executing the procedure of integration (Mahir, 2009; Orton, 1983; Grundmeier et al., 2006); however, only a small proportion of students are able to translate to the graphical representation and solve when the original problem contains a formula not elementary for integrating (e.g. [x]) (Rasslan, S. & Tall, D., 2002; Mahir, 2008).

The area can be conceived as an infinite collection of lines or limit of narrowing rectangles (or trapezoids) (Sealey 2006; Jones 2013; Czarnocha, B., Dubinsky, E., Loch, S., Prabhu, V., & Vidakovic, D., 2001).

Accumulation is an important but less widely understood interpretation (Thompson, 1994; Thompson & Silverman, 2008; Jones 2015). Tall (1992) called it cumulative growth; Thompson (1994) called it accumulation; Jones (2013) called it adding up pieces; Jones (2015) called it multiplicative-based summation. We refer to it as *rate-based accumulation*. The element of summing is captured by the word "accumulation" and the fact that each term is calculated from the rate is reflected by the phrase "rate-based." Understanding of this form is directly related to the ability to writing the integral for an application problem (Jones, 2015; Sealey, 2006).

Understanding in each of these 3 categories might be shallow or deep. For example, shallow understanding of antidifferentiation might include only procedural knowledge for basic functions (e.g. polynomial, trig, exponential). Deep understanding might include a full conception of the Fundamental Theorem of Calculus. A shallow understanding of area might include only the nominal notion itself. A deep understanding might allow reasoning about the bounds which maximize the integral given the graph of a function (e.g. Bezuidenhout, J. &

Olivier, A., 2000); it might encompass approximations through Riemann sums as well as exact answers from a definite integral. Shallow understanding of rate-based accumulation might only include a restatement of the Fundamental Theorem of Calculus. Deep knowledge might include the ability to set up integrals for applied problems (as in Sealey 2006) or the ability to represent it with the graph of the antiderivative (Tall 1991).

Our desire with this study is to understand the current landscape of students' conceptualizations of integration in multivariable calculus. Our goal is to ask the question "what are the primary conceptualizations of integration and accumulation present among students who have completed single variable calculus, and how prevalent are the various conceptualizations among this population?"

Theoretical Perspective

In this study we chose to focus on student's descriptions of integration and accumulation rather than examine their ability to give correct or incorrect responses to mathematical questions. This decision is based on our theoretical perspective which stems from Tall and Vinner's (1981) work on concept images and concept definitions. In short, our primary interest is in understanding the mental images, processes and connections that a student brings to mind when considering the topics of integration and accumulation, that is, their *concept image* of integration and accumulation. It is important to keep in mind that students involved in our study may possess elements of their concept image that were never uncovered by their responses, for this reason we say that we are studying their *evoked concept images* in response to the questions posed to them.

Methodology

The data for this report come from a large scale study involving <fill in> multivariable calculus students at <fill in> universities. The. Students in the study were asked to complete a collection of open ended written responses on various topics in introductory calculus. The current report will focus on student responses the following two questions featuring the concept of integration in single variable calculus.

> Question #1: Suppose the result of $\int_a^b q(x)dx$ is a real number, k. Explain what k means and how it was measured. Sketch any images you have in mind in the space below.
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> Question #2: Suppose that a function G is defined: $G = \int_a^t f(x)dx$. Is G a function of t or a function of x? Justify your response.

Student responses were then analyzed using an open coding scheme which preliminary results have indicated four categories for responses to question #1 and three categories for responses to question #2.

Preliminary Results

Following are the emerging categories found during our preliminary analysis of student responses to questions #1 and #2 above regarding the concept of integral in single variable calculus.

Emerging Categories from responses to question #1di

Integral as representing an antiderivative

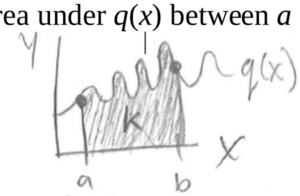
Students within this group primarily respond in terms of symbolic representations of functions and describe k as the computed result from manipulating those representations. Student responses typically omit sketched images or present symbolic representations as the 'image' accompanying the description.

Example response: "You take the antiderivative of $q(x)$. Once you do that, you substitute b for x and a for x and subtract. The difference is k"

Integral as representing an area

Students within this group primarily describe the integral in terms of area without reference to how the area can be computed or interpreted. Accompanying sketches typically include the graph of a function with the area underneath the function shaded; however, the sketches contain no means of dividing the area into simpler shapes.

Example response: "k is the area under $q(x)$ between a and b"

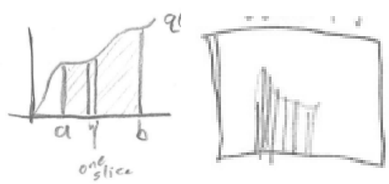


Example Sketch

Integral as representing an infinite sum of one-dimensional objects

Students within this group describe integration as a process of adding together an infinite number of infinitely small pieces, often referred to as 'lines' or 'slices.' Students within this group often describe this process as a means of measuring the area underneath the function. This category closely resembles the "collapse metaphor" of limit as described in Oehrtman (2009). Sketches accompanying these descriptions often contain either a single representative 'slice' of the function or an area composed of vertical lines.

Example response: "The area under the curve, $q(x)$, is equal to some real number, k. k was measured by taking an infinite number of slices of the area under the curve."



Example Sketches

Integral as representing a limit of approximations

Students within this group describe integration as an approximation process, usually in terms of Riemann Sums. The students' descriptions of the limiting process of these approximations can vary widely including descriptions of repeating the approximation process indefinitely, doing a single approximation at a very high level of accuracy, or creating approximations to meet a desired accuracy as described in Sealey and Oehrtman

(2007). Like the previous category, students within this group often describe the approximation process as a means of measuring the area underneath the function. Sketches accompanying these descriptions often reflect the traditional images associated with Riemann Sums.

Example response: "*k* means area under the curve. It was measured by taking segments of the curve and multiplied by the height of the function, thus creating rectangles. This process was then repeated by taking a limit and taking smaller and smaller segments each time."

Example Sketch

Emerging Categories from responses to question #2af

x and t are described by their role while performing integration

Students within this group focus on the process of computing a single value from integrating and the roles of *x* and *t* within that process. Students within this group tend to describe *x* as the "variable" involved in the process and *t* as a "parameter" or "boundary value." For this reason, students within this group respond that *x* is the variable present and argue based on the roles of either *x* or *t* rather than in terms of the function *G*. This may be due to a weak understanding of the covariational nature of functions (Carlson et al., 2002) or an unreified view of the process of integration (Sfard, 1991)

Example responses: "*G* is a function of x. It's not *t* because *t* is just a boundary." "*G* is a function of *x* because *x* is the input."

x and t are described by their role after performing symbolic integration

Students within this group tend to speak primarily in terms of the symbolic process of integrating; however, unlike the previous group of students, students within this category focus their attention on the result of integrating rather than the process of computing the integral. Students within this group respond that *t* is the variable because after using the fundamental theorem to integrate, *t* is substituted back for *x* to achieve the final answer.

Example response: "*G* is a function of *t* because *t* will replace the *x* from *f*(*x*) when integrating."

x and t are described by how changes in each variable affect the value of the function, G

Students within this group emphasize in the input-output nature of the function *G* and respond in terms of how changes in either *t* or *x* will result in changes in *G*. Students within this group respond that *G* is a function of *t* because changing the value of *t* changes the resulting value of *G*.

Example responses: "*G* is a function of *t*. by modifying *t*, one can change the interval over which *f*(*x*) is integrated." "A function of *t*, since any change in *t* would change the value of *G*."

Discussion

Preliminary results from the analysis of question #1 show that the majority of students (67%) describe the definite integral in terms of area and roughly half of the remaining students (16%) describe it in terms of an antiderivative. For question #2 the preliminary analysis shows that among the students who gave a classifiable response, the majority (54%) described *x* and *t* in terms of their roles while performing integration with most of the remaining students (34%) described *x* and *t* in terms of their role after performing symbolic integration.

It is worth noting that these preliminary results indicate that the majority of the students in our study evoked conceptualizations of integration and accumulation that fail to emphasize the underlying structure of the definite integral. Such conceptualizations could be an indication of pseudo-structural thinking (Sfard & Linchevski, 1994) among the students in our study. It is interesting that, although a majority of the students responded to question #1 in terms of the area underneath the function, very few students used area as a means to reason about the roles of *x* and *t* in question #2, opting instead to reference the symbolic process of calculating a definite integral using the fundamental theorem of calculus. This is likely due to the increased complexity found in moving from a single definite integral to treating that integral as a process in the accumulation function.

The data for this report is situated within a larger project exploring student learning in multivariable calculus. In particular, the authors are interested in exploring the implications of these results for teaching and learning in multivariable calculus. For example, we have anecdotal evidence that an emphasis on interpreting a single variable integral as an area can lead to students interpreting multivariable integrals in terms of surface area or other two dimensional quantities. For these reason, we have chosen to focus our audience questions on the implications of these results for teaching and learning in multivariable calculus.

Questions for the audience:

- How do these results impact classroom instruction in multivariable calculus?
- What affect would you expect multivariable calculus to have on student responses to a post-test?
- What types of multivariable calculus experiences would most likely influence students' evoked images of integration?

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