Mathematicians' ideas when formulating proof in real analysis

Melissa Troudt
University of Northern Colorado

This report presents some findings from a study that investigated the ideas professional mathematicians find useful in developing mathematical proofs in real analysis. This research sought to describe the ideas the mathematicians developed that they deemed useful in moving their arguments toward a final proof, the context surrounding the development of these ideas in terms of Dewey's theory of inquiry, and the evolving structure of the personal argument utilizing Toulmin's argumentation scheme. Three research mathematicians completed tasks in real analysis thinking aloud in interview and at-home settings and their work was captured via video and Livescribe technology. The results of open iterative coding as well as the application of Dewey's and Toulmin's frameworks were three categories of ideas that emerged through the mathematicians' purposeful recognition of problems to be solved and their reflective and evaluative actions to solve them.

Key words: proof construction, Toulmin argumentation scheme, inquiry, real analysis, mathematicians

Writings of mathematicians and mathematics education researchers note that the mathematical proving process involves a formulation of ideas; specifically, for mathematicians, there is a reflection, reorganization of ideas and reasoning that "fill in the gaps" so a proof will emerge (Twomey Fosnot & Jacob, 2009). Byers (2007) described an idea as the answer to the question "what's really going on here?", and Raman, Sandefur, Birky, Campbell, and Somers (2009) observed three critical moments in the proving process in which there were opportunities for a proof to move forward. Tall and colleagues (2012) gave a description of proof for professional mathematics that "involves thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures formulate definitions as necessary and to build a valid argument" (p. 15). Rav (1999) stated that the term "proof" can describe the written product used to "*display the mathematical* machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p. 13, italics in original); however the process of constructing proof involves informal and formal arguments to find methods to attack the problem as well as incomplete proof sketches (Aberdein, 2009). Despite these writings, little research describes the context around the formulation of ideas that a professional mathematician finds useful and how these ideas influence the development of the mathematical argument. This study focused on describing mathematicians' development of these ideas when constructing proofs in real analysis made evident in changes in the structure of the argument (Toulmin, 1958/2003) utilizing Dewey's (1938) theory of inquiry to describe the problem situation.

Research Questions

Part of a larger project, this report focuses on the findings for the research questions: What ideas move the argument forward as a professional mathematician's personal argument evolves? What problem situation is the mathematician currently entered into solving when s/he articulates and attains an idea that moves the personal argument forward?

Theoretical Perspective

This research conceived of the mathematical proving process as an evolving personal argument. The *personal argument* is a subset of one's total cognitive structure associated with the proof situation (described as a *statement image* by Selden and Selden (1995)) that the individual deems relevant to making progress in proving the statement. The personal argument is graded in that some aspects of the statement image may be central and others may lie on the periphery. The personal argument evolves or moves forward when an individual develops an idea that s/he sees as useful in making progress in proving the statement. The focus of this study was to describe the ideas incorporated and the inquirential context surrounding that development.

Toulmin's (1958/2003) argumentation model provided a means of describing structurally the evolution of the personal argument as the individual incorporated new ideas. The framework notes the content of the statements given in the argument (either explicitly or not) as well as the purposes that those statements serve. The framework classifies statements of an argument in six different categories. The claim (C) is the statement or conclusion to be asserted. The data (D) are the foundations on which the argument is based. The warrant (W) is the justification of the link between the grounds and the claim. Backing (B) presents further evidence that the warrant appropriately justifies that the data supports the claim. The modal qualifiers (Q) are statements that indicate the degree of certainty that the arguer believes that the warrant justifies the claims. The rebuttals (R) are statements that present the circumstances under which the claim would not hold.

New ideas result from periods of ambiguity or when engaged in non-routine problem solving (Byers, 2007; Lithner, 2008). John Dewey (1938) posited in his theory of inquiry that new knowledge or ideas are developed when one is engaged in active, productive inquiry into a problem. An individual engaged in the cyclical process of inquiry reflects on problem situations, selects and applies tools to the situations, and evaluates the effectiveness of the tools (Hickman, 1990). Dewey's framework provided for understanding the context surrounding the emergence of new ideas from the participant's point of view.

Related Literature

This research followed the lead of other researchers who have conceived of the proof construction process as a particular type of problem solving (i.e. Savic, 2012; 2013; Weber, 2005). Selden and Selden (1995; 2013) maintained that there is a close relationship between problem solving and proof, and that two kinds of problem solving could occur in proof construction: solving the mathematical problems and converting an informal solution into a formal mathematical product. Building upon extensive work in understanding the problem solving process and investigating the problem solving processes of twelve mathematicians, Carlson and Bloom (2005) developed a Multidimensional Problem Solving framework providing a description of the cyclical progression through the phases of problem solving (orientation, planning, executing, and checking), cycling, and problem-solving attributes. Savic (2013) found that the four phases of Carlson and Bloom's framework could be used to code and describe most portions of the proving process. However, he found some differences including the mathematician cycling back to orienting after a period of incubation and one participant not completing the full cycle; Savic hypothesized additional problem solving phases could be added.

Some research has been conducted and documented the existence of and provided initial descriptions of the types of ideas that this study sought to describe. Raman (2003) characterized three types of ideas involved in the production of a proof: heuristic ideas (ideas based on informal understandings linked to private aspects of proof), procedural ideas (ideas based on logic and formal manipulations), and key ideas (heuristic ideas that can be mapped to formal proofs). In later work Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009) identified the potential for three critical moments when constructing proof (1) attaining a *key idea* (later termed conceptual insight; Sandefur, Mason, Stylianides, & Watson, 2012) that gives a sense of why the statement is true; (2) gaining a *technical handle* for communicating a key idea, and (3) the culmination of the argument into a standard form. The potential for a key idea to exist apart from a technical handle exists when a prover is engaged in some informal mathematical reasoning. Although they did not describe them as ideas, Ingils, Mejia-Ramos, & Simpson (2007) found mathematics graduate students used warrants based on both formal mathematical deductions (*deductive warrants*) and non-deductive reasoning including inductive reasoning (*inductive warrants)*and intuitive observations or experiments with some kind of mental structure (*structural-intuitive warrants*). Noting these ideas' existence is interesting but calls for further research into descriptions of how these ideas are developed and what kinds of ideas are deemed important when formal or informal reasoning is utilized.

Methods

Three professional mathematicians with faculty appointments at four-year universities who specialized in researching or in teaching courses in real analysis served as the participants for this study. Each participant worked on a task or tasks in a "think-aloud" interview setting, continued to work on the tasks on their own, turned in their at-home work captured via Livescribe technology, participated in a follow-up interview replaying the video and Livescribe capture of their previous work, and repeated this process with new tasks in the next interview. Each participant worked on three to four tasks in total.

Data analysis proceeded in two phases. In the preliminary analysis of the participants' work on the tasks, I noted moments where participants articulated insights, observations, or hypotheses, and these acted as markers in the transcripts. I hypothesized Toulmin models of the participants' personal argument as well as the inquirential context while these ideas were formulated (perceived problem, contributing actions and tools, and anticipated outcomes of applying the tools) prior to and following these markers. These hypotheses informed the questions asked at the follow-up interview. In the primary analysis, the follow-up interviews provided information to complete and modify the initial analyses. For each task, I wrote stories of the participant's complete work on the task sectioned by the ideas in order to capture the evolution of the argument. I conducted open iterative coding of each idea, the problem situation encountered, the tools that influenced the generation or articulation of the idea, and the anticipated outcome of said tools. Most analysis was inductive; however, I borrowed language from the literature when elements fit the descriptions given by other authors. I analyzed across the ideas of each participant and across participants along the common tasks to look for emerging themes and patterns. I report some findings regarding the types of ideas formulated and the problems encountered when ideas were articulated.

Results

In presenting these results, I first give an overview of the characteristics of the ideas that moved the argument forward and then brief descriptions of each idea type and sub-type. I describe the problems that participants were entered into solving when they developed these ideas and finally illustrate these themes through one participant's, Dr. C's, work on a task.

The ideas that moved the argument forward either were accompanied by a structural shift in the personal argument captured by a Toulmin diagram, provided a means for the participant to communicate their personal argument in a logical manner, gave a participant a sense that his way of thinking was fitting, or were explicitly referred to by the participant as a useful insight. While pictures, examples, or individual actions were not included as ideas, the insights extracted from performing and reflecting upon these tools or a collection of tools were included. Ideas were coded in terms of the work they did for the participant. In total, I identified fifteen sub-type ideas grouped into three categories: ideas that focus and configure, ideas that connect and justify, and monitoring ideas (see Table 1). An action or evaluation of that action from one particular moment could solve multiple problems or give rise to multiple feelings. Therefore, multiple idea-types at times characterized a single moment. For example, an insight that provided a *deductive warrant* could also give the prover a sense of *I can write a proof.* Note that three of the idea sub-types that connect and justify are meant to keep in the spirit of the descriptions given by Inglis et al. (2007).

No distinct pattern involving the types of problems and tools that contributed to the generation of certain ideas. There was a discernable pattern of a participant proposing or articulating an idea or tool, testing the usefulness of the proposed idea or tool or the prior ideas against the consequences of the new idea, and then articulating a new idea or evaluation. This process involved the passing through, perhaps multiple times, the inquirential cycle of reflecting, acting, and evaluating against the ideas' abilities to solve a perceived problem. The participants transitioned through the following four phases of problems to tackle or tasks to complete in order to finish the construction of the proof.

1. Understanding the statement and/or determining truth
2. Determining a warrant of some kind
3. Validating, generalizing, or articulating those warrants
4. Writing the argument formally

At times the participants proceeded linearly through the four phases; however, there were instances where participants needed to cycle back to a previous phase when a proposed idea or tool was not fitting or if no tool could be found to solve the current problem (see Figure 1). Aside from these major problems to solve, the participating mathematicians also tackled problems parallel to or embedded within these problems such as dealing with a found problem with a tool. Writing the argument formally typically was not problematic for the professional mathematician once they had developed a deductive warrant.

To illustrate these themes, consider Dr. C's work on the task: *Let f be a function on the real numbers where for every x and y in the real numbers, f(x + y)= f(x)+ f(y). Prove or disprove that f is continuous on the real numbers if and only if it is continuous at 0.*

Upon his initial reading of the problem, Dr. C declared that he believed the statement was true for the rational numbers but not generally true for the real numbers.

> Dr. C: I was thinking about the well-known fact that the only continuous linear functions in the reals to the reals are those of the form y equals mx for some fixed m. And one shows that those are continuous on the rationals fairly easy - linear functions are continuous on the rationals pretty easily by doing some induction.

This idea was in response to the problem of determining the truth of the statement and was coded as a truth proposal, informing the statement image, and a structural-intuitive warrant since he was basing his conjecture on a connection between the additive property and linearity and his conceptual knowledge.

Table 1
Descriptions of ideas that moved the argument forward sub-types

Idea sub-type	Description
Ideas that focus and configure	Ideas that gave a sense of what was relevant, what claims to connect to the statement, fitting strategies to achieve connections, and how to structure and articulate the argument
Informing statement image	Ideas that broadened or narrowed the conception of the situation.
Task type	Assessments about what tools or ways of approaching developing connections between the conditions and the claim would be fitting
Truth proposal	Participant-generated conjectures about the validity of a given claim based on a warrant of any type
Identifying necessary conditions	A sense that "The statement can't possibly be true unless this condition is fulfilled"
Envisioned proof path	A proposal of a series of arguments that will lead to a solution that may be missing connections
Logical structure & representation system of proof	Decisions regarding structuring and communicating the formal argument
Ideas that connect and justify	Warrants and backing, the means of connecting data with claims
Deductive warrant*[1]	Reasoning based on generalizable logical statements
Inductive warrant*	Reasoning based on specific examples
Structural-intuitive warrant*	Reasoning based on a feeling that is informed by structure or experience
Syntactic connection	Symbolic manipulations deemed useful to connect given evidence to a claim that may not be supportable by deductive reasoning or attend to the mathematical objects that the symbols represent
Proposed backing	Proposed support for previously identified non-deductive warrants or vague senses of what would underlie a possible warrant
Ideas that monitor the argument evolution	Ideas or feelings about the mathematicians' progress
Truth conviction	Personal belief as to why a statement must be true
"I can write a proof"	A feeling of formulating the connections necessary to communicate the argument in a final proof
Unfruitful line of inquiry	An idea that persuaded the participant that the tools or actions pursued or considered were not optimal for achieving the set goal
Support for line of inquiry	A sense that one's actions were fitting

[1] The asterisks indicate that the titles of idea sub-types of deductive warrant, inductive warrant, and structural-intuitive warrant borrow from the descriptions of reasoning given by Inglis et al. (2007).

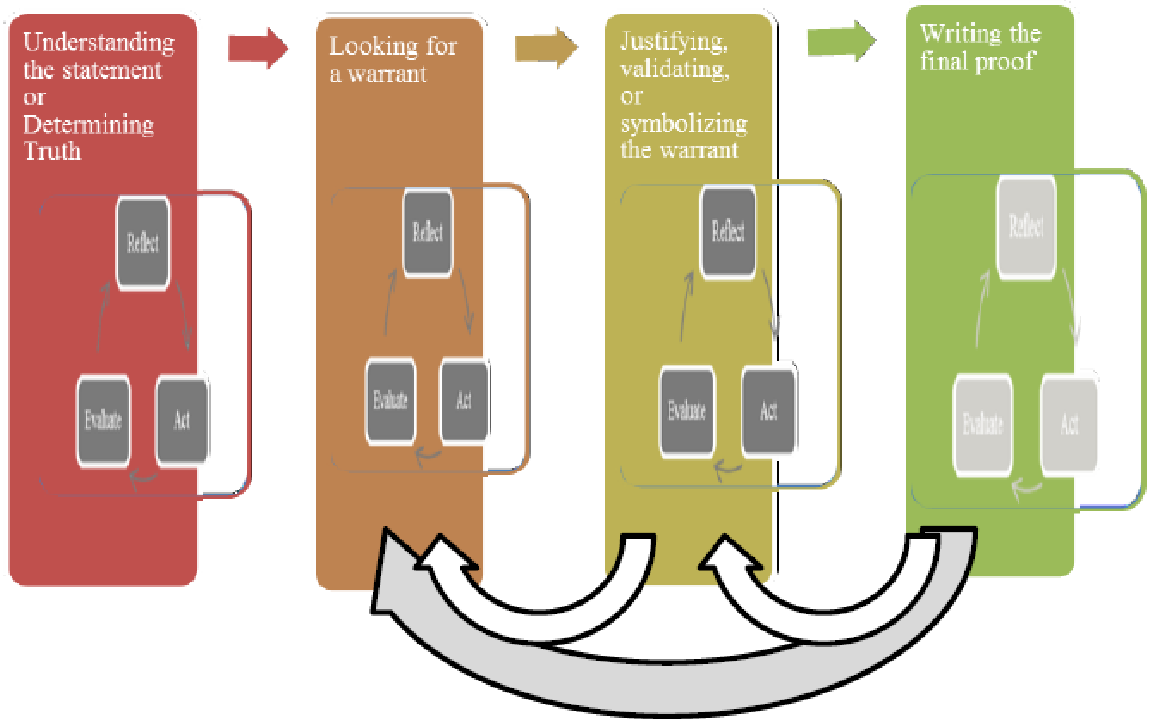

Figure 1. Illustration of the problem phases observed and the potential to cycle back.

Dr. C then set about determining a deductive warrant by proposing a counterexample function that was continuous at zero and the rational numbers but discontinuous on the reals, namely, the piecewise defined function that has an output of zero when the input is rational and the value of the input otherwise. He then tested this function and found it to not possess the additive property and concluded that the given statement might be true.

Dr. C: It turned out that didn't work. And if the easier ones didn't work, then the harder ones probably wouldn't either. Matter of fact, if the easier one didn't work, then it seemed likely that none of the harder ones would work.

I: Okay. So I was going to ask about that. So after you found that it didn't work, it didn't satisfy it. You paused for a while. Was it because you were trying to think of different examples, or were you convincing yourself that it-

Dr. C: Yeah. I was trying to convince myself that if this didn't work, then nothing would.

Dr. C recognized an *unfruitful line of inquiry*, moved back to the problem of determining the truth of the situation, and gave a new *truth proposal* based on the generated example function coupled with his knowledge of functions (an *inductive warrant*). He then moved to try to prove the statement was true (look for a deductive warrant). In exploring, he developed a string of inequalities based on instantiations of the definition of continuity and logical mathematical deductions, and he identified the *necessary condition* that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. He recalled a proof that $f(0) = 0$ and that the function was given to be continuous at zero to fulfill the condition. Dr. C symbolically evaluated that his written assertions were correct and declared a sense that he could now write the proof based on his deductive warrants. Because his work in proving the task was based on deductive warrants within the representation system of proof, the writing of the proof did not require the formulation of any new ideas.

Discussion and Conclusions

Every participant on each task identified ideas from each of the three idea categories. As was described above with Dr. C, the evolution of the personal argument was not linear in identifying focusing and configuring ideas, identifying connections and justifications, and then making monitoring decisions. The process of articulating ideas, testing the new idea or previous ideas against these new ideas, and then proposing new ideas was apparent. The process of testing ideas varied by idea-type, but the process involved active, productive inquiry in that ideas were tested against their abilities to do work in solving a perceived problem.

The four identified phases of understanding the statement or determining truth, looking for a warrant, working to validate, generalize, justify or articulate their warrant; and writing the formal proof are reminiscent of findings of other researchers. The following aspects have been identified as part of the proof construction process: understanding the statement or described objects (Alcock, 2008; Alcock & Weber, 2010; Carlson & Bloom, 2005; Savic, 2013); determining the truth of the statement (Sandefur et al., 2012); determining why the statement is true (Raman et al., 2009; Sandefur et al., 2012); translating ideas into analytic language (Alcock & Inglis, 2008; Alcock & Weber, 2010; Weber & Alcock, 2004); and justifying a previous idea (Alcock, 2008; Alcock & Weber, 2010). This research is unique in its specific efforts to identify the problems encountered as participants developed new ideas and in its use of Dewey's theory of inquiry to explain how ideas were developed and tested against these problems. The mathematicians progressed through these four phases but needed to cycle back to a previous phase when the ideas that the mathematicians had previously incorporated into the personal argument were insufficient in resolving a situation in a later phase.

The choice to conceive of the proof construction process as involving an evolving personal argument was made due to a desire to talk about all the ideas, relationships, concepts, pictures, and so on that an individual personally judges as important to providing a final proof and the relationships amongst these elements at various points in time. This conception allowed for attending to moments when ideas were generated that the prover saw as useful which broke the construction process into significant events to illustrate the story of the argument's evolution. As researching the proving process in this manner is relatively unexplored, many avenues of research are open to explore how these ideas develop, how they are tested, and the consequences their development provides for the evolution of the argument. The findings of this study were descriptive and exploratory and the fifteen idea sub-types found may or may not be salient in other studies. It is probable that varying the mathematical content area or narrowing the research questions would provide new and clarifying findings to refine the categorizations or provide insight as to how the proof construction process compares across mathematical content.

References

Aberdein, A. (2009). Mathematics and argumentation. *Foundations of Science, 14,* 1-8.

Alcock, L. (2008). Mathematicians' perspectives on the teaching and learning of proof. *CBMS Issues in Mathematics Education, 16,* 73-100.

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129

Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof construction: Purposes and effectiveness. *Investigations in Mathematics Learning, 3*(1), 1-22.

Byers, W. (2007). *How mathematicians think*. Princeton, NJ: Princeton University Press.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*, 45-75.

Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Henry Holt & Co.

Hickman, L. A. (1990). Knowing as a technological artifact. In *John Dewey's Pragmatic Technology* (pp. 17-59). Bloomington: Indiana University Press.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Lakatos, I. (1976). *Proofs and refutations: the logic of mathematical discovery*. Cambridge:

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255-276.

Raman, M. (2003). Key ideas: What are they and how can they help us understand people's views of proof?. *Educational Studies in Mathematics, 52*(3), 319-325.

Raman, M., Sandefur, J., Birky, G., Campbell, C., & Somers, K. (2009). "Is that a proof?": Using video to teach and learn how to prove at the university level. In F. L. Lin et al. (Eds.), *Proof and proving in mathematics education: ICMI Study 19 conference proceedings* (pp. 2-154 – 2-159). Taipei: National Taiwan Normal University.

Rav, Y. (1999). Why do we prove theorems?. *Philosophia Mathematica, 7*(1), 5-41.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2012). Generating and using examples in the proving process. *Educational Studies in Mathematics,* Published online, 29 December 2012, doi:10.1007/s10649-012-9459-x

Savic, M. (2012). What do mathematicians do when they reach a proving impasse?. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1-388-1-407). Portland, OR: Online at http://sigmaa.org/rume/Site/Proceedings.html.

Savic, M. (2013). An examination of proving using a problem-solving framework. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 381-493). Denver, CO: Online at http://sigmaa.maa.org/rume/Site/Proceedings.html.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*, 123-151.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematics Enthusiast, 10*(1/2), 303-334.

Tall, D. O., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive development of proof. *Proof and Proving in Mathematics Education*, 13–49.

Toulmin, S.E., (1958, 1969, 2003). *The uses of argument*. Cambridge, UK: University Press.

Twomey Fosnot, C., & Jacob, B. (2009). Young mathematicians at work: The role of contexts and models in the emergence of proof children's reasoning: Discovering the idea of mathematical proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 102-119). New York: Routledge.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior*, *24*, 351-360.
Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*, 209-234.