Supporting students in seeing sequence convergence in Taylor series convergence

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Virtual manipulatives designed to increase student understanding of the concepts of approximation by Taylor polynomials and convergence of Taylor series were used in calculus courses at multiple institutions. 225 students responded to tasks requiring graphing Taylor polynomials, graphing Taylor series, and describing relationships between different notions of convergence. We detail significant differences observed between students who used virtual manipulatives and those that did not. We propose that the use of these virtual manipulatives promotes an understanding of Taylor series supporting an understanding consistent with the formal definition of pointwise convergence.

Keywords: Taylor Series, Virtual Manipulatives, Calculus, Cooperative Learning, Approximation

As one of the more challenging concepts in calculus, Taylor series coordinates ideas of approximations of functions with the evaluation of limits of sequences and series. Previous studies of student understanding of Taylor series indicate that many students lack a developed, working model of this concept that is sufficient to afford meaningful progress on Taylor series tasks (Kung & Speer, 2010; Martin, 2013). Tasks typically proposed to students and the ways in which they engage in such tasks may not be adequate to promote the conceiving of and relating relevant quantities so as to coordinate notions of Taylor series convergence with sequence convergence. One approach for helping students coordinate these ideas is through the use of computer software. Specifically, we aim to understand if a Virtual Manipulative (VM) might aid students in their understanding of Taylor series convergence. In particular, we ask:

1. Do students in classrooms implementing VMs respond differently to prompts asking for the production of graphical representations of Taylor polynomials and Taylor series?
2. Can differences in conceptions be observed between students from VM and non-VM classes?
3. How do students in classrooms utilizing VMs respond to open-ended questions about convergence? What do they consider most relevant to the concept of convergence?

Background

For an analytic function *f*, a Taylor series is a power series of the form $\sum_{k=0}^{\infty} c_k (x-a)^k$ where $c_k = f^{(k)}(a)/k!$ for each *k*. Almost all studies of student understanding of Taylor series document student struggles to comprehend and interpret the complicated structure inherent in Taylor series (e.g. Champney & Kuo, 2012; Kidron & Zehava, 2002; Kung & Speer, 2010; Martin, 2013; Martin & Oehrtman, 2010). Martin (2013) noted that students rarely moved beyond algebraic reasoning to offer graphical interpretations of convergence, and often failed to coordinate their notions with sequence convergence by fixing values of *x*. When looking at Taylor series graphs, Oehrtman (2009) observed that students inaccurately concluded that a Taylor polynomial and the approximated function are identical over an interval after observing a polynomial “touching” the approximated function, overlooking nonzero differences (or error or remainder) between polynomials and the function for particular values of *x*. In contrast, Martin and Oehrtman (2010) noted that students attending to structures involving “approximations,” “error,” “accuracy,” etc., where for each approximation there is an associated error and a bound on that error, can lend itself to students describing the existence of nonzero differences between approximations (Taylor polynomials) and the approximated function for particular values of *x*.

What is a suitable understanding of Taylor series convergence for the Calculus student?

Our focus has not been to bring calculus students to a formal definition of pointwise convergence using Taylor series, but to support students in algebraic and graphical reasoning consistent with the formal definitions of sequence convergence and the recognition of sequence convergence as embedded within pointwise convergence. For graphical explanations of Taylor series convergence, our students were expected to elaborate on notions of sequence convergence using vertical number lines (Figure 1) for different values of *x* while coordinating notions of “estimates,” “error,” “accuracy,” etc. which involve an unknown quantity and a known approximation. This paper investigates differences in responses to prompts between students engaging in such activity compared to students from calculus classes that tend to focus mainly on completing convergence tests using algebraic approaches.

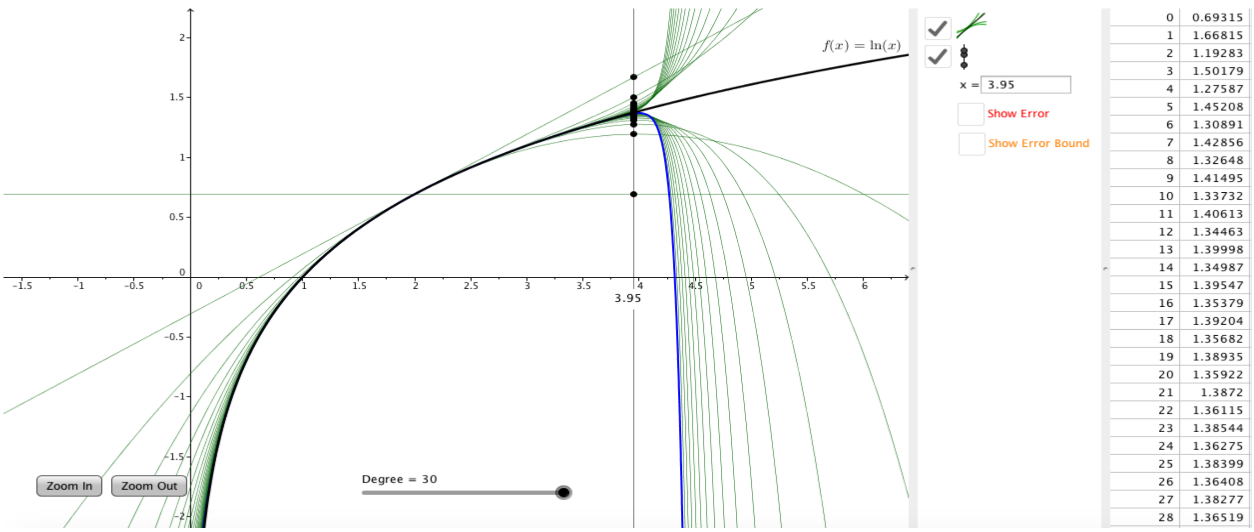

Figure 1: Screenshot of Taylor series VM

Virtual Manipulatives (VMs)

By a VM we mean an interactive computer representation of a mathematical concept. Moyer-Packingham and Westenskow (2013) note that VMs have been useful to develop certain understandings but that education research on VMs is lacking beyond 6th grade. For notions related to sequence convergence, Cory and Garofalo (2011) observed that VMs can reinforce students’ understandings of the quantitative and logical relationships captured by dynamic imagery and that these relationships can be recalled months later. Yet, when it comes to Taylor series, Kidron and Zehavi (2002) found that students can fail to correctly interpret what they are seeing in the VM if working with the VM preceded interpretation of the algebraic representation coordinated with the graphical depiction. Taylor series VMs have been helpful in supporting students with noticing general graphical trends (Habre, 2009; Kidron & Zehavi, 2002), but these VMs have also unintentionally reinforced students seeing a Taylor polynomial and the approximated function as identical over an interval after viewing a Taylor polynomial literally touching the approximated function in the VM. Even with these potential setbacks, we hypothesize that well designed graphical images (including VMs) coordinated with approximation tasks can help students come to an understanding of Taylor series convergence consistent with formal theory.

Methods

Students were recruited from calculus classes to participate in this study after receiving instruction concerning sequences, series, and Taylor series. Students were from two types of classes: calculus classes that focused on algebraic approaches to common Taylor series tasks (referred to as non-VM students), and classes that included VMs and incorporated approximation tasks (Martin & Oehrtman, 2015; Oehrtman, 2008) using laboratory-style group exercises (referred to as VM students). Instructors utilized VMs to supplement classroom instruction and group activities. Students individually interacted with the VMs to complete homework tasks. To address notions of polynomials being the same as the approximated function on an interval, zoom features were included in most VMs.

Data from this study was taken from quizzes, classwork, exams, and questionnaires that students completed after the conclusion of relevant classroom activities concerning Taylor series. In total, 139 non-VM students and 86 VM students from four institutions participated in this study. For this analysis we focused on student responses to three tasks:

Tasks			non-VM Students	VM Students
1) "Using the graph of sin(x) below, on the same axes sketch three different Taylor polynomials for sine"			✓	✓
2) "Using the graph of sin(x) below, on the same axes sketch the Taylor series for sine"			✓	✓
3) Explain how sequence convergence is related to Taylor series convergence.				
	<u>Two Question Version</u>	3a) "Briefly explain how sequence convergence is related to series convergence. Be as precise as you can." 3b) "Briefly explain how series convergence is related to Taylor series convergence. Be as precise as you can."		✓
	<u>One Question Version</u>	"List all of the ways in which Taylor series convergence is related to sequence convergence and series convergence. Make sure your explanations reference i. formulas when appropriate and ii. includes a graphical explanation that highlights sequences and/or series on your graph above. (That is, add to the graph above to appropriately highlight sequences and/or series convergence as it relates to Taylor series convergence.)"		✓

Figure 2: Tasks given to students

Responses were collected and coded (Strauss & Corbin, 1990) by a team of two undergraduate and two faculty researchers. One faculty researcher and the two undergraduates developed the coding protocol for the three tasks. The other faculty researcher coded a random sample of 20 students for each of the first two tasks, achieving over 80% reliability for each coding decision. Task 3 was coded independently by the second researcher.

Initial Results

Task 1

Task 1 was intentionally ambiguous so that students could choose to have different Taylor polynomials be centered at different points, have different degrees, or both. Responses were coded for correctness twice: once using a strict rubric and once using a more relaxed rubric. The relaxed rubric was developed since students drew even degree Taylor polynomials, while the Taylor series expansion for sin(x) centered at x=0 does not have even degree terms. These answers were counted as correct under the relaxed rubric.

Using the strict rubric, 13% of the 86 VM students answered correctly, compared to 4% of the 139 non-VM students. Using the relaxed rubric, 28% of the VM students answered correctly while 6% of the non-VM students did. This suggests that students in the VM classrooms outperformed non-VM calculus students measured by either the strict rubric, $X^2(1, N = 225) = 4.3164$, $p = 0.038$, or the relaxed rubric, $X^2(1, N = 225) = 19.592$, $p < 0.001$.

Taylor polynomials beyond 0^{th} order must be tangent to the function being approximated. Of the VM students, 33 (38%) drew Taylor polynomials that were both tangent to the function being approximated, while 11 (8%) of the non-VM students did. Of the VM students, 28 (33%) drew Taylor polynomials not only tangent to the function being approximated, but also to each other, while 9 (6%) of the non-VM students did. This indicates a larger proportion of VM students are drawing Taylor polynomials tangent to the function being approximated than non-VM students, $X^2(1, N = 225) = 31.331$, $p < 0.001$.

Since students could answer the question using either Taylor polynomials of different degree or with a different center, we considered how students approached the problem. The number of students who answered correctly using different centers was quite small: 6 students, 5 in VM sections, 1 in non-VM sections. 62 students in VM sections (72%) answered the question by drawing polynomials of different degree, while 26 students in non-VM sections (19%) answered the question by drawing polynomials of different degree, indicating students in VM classrooms were more likely to solve the problem correctly using different degree Taylor polynomials than non-VM students, $X^2(1, N = 225) = 63.589$, $p < 0.001$.

Task 2

The ideal answer for Task 2 would be to sketch over the graph of sine. Possible misconceptions can introducing error either throughout the entire graph or part of the graph, or considering the Taylor series to be a collection of polynomials, either finite or infinite. As before, grading was done using a strict rubric and a relaxed rubric. Students were considered correct under the strict rubric if they traced over the graph. A correct answer under the relaxed rubric allowed for careless tracing.

Using the strict rubric, 41% of the 86 VM students answered correctly, while 11% of the 139 non-VM students did. Using the relaxed rubric, 44% of the VM students answered correctly, while 16% of the non-VM students did. VM students therefore outperformed non-VM students using both the strict rubric, $X^2(1, N = 226) = 23.595$, $p < 0.001$, and the relaxed rubric, $X^2(1, N = 226) = 18.29$, $p < 0.001$.

Another misconception of interest was drawing a collection of functions as a Taylor series. Students in the VM sections were more likely to report a collection of polynomials as a Taylor series than those in non-VM sections, whether including all students, $X^2(1, N = 226) = 12.34$, $p < 0.001$, or removing students who left the problem blank, $X^2(1, N = 148) = 5.23$, $p = 0.02$.

Also of interest are the students who introduce intentional non-zero error in the graph of the Taylor series on either the entire domain (negative infinity to infinity) or on some part of the domain. There was no statistically significant difference between the proportion of VM and non-VM students who drew graphs with error over the entire domain, $X^2(1, N = 226) = 1.877$, $p = 0.17$, including when students with blank responses were removed, $X^2(1, N = 148) = 0.35126$, $p = 0.553$. Similarly, there was no difference between the groups when comparing students who introduced error for only part of the domain, $X^2(1, N = 226) = 0$, $p = 1$, and still no significant difference when blank answers were removed, $X^2(1, N = 148) = 2.1018$, $p = 0.147$.

Task 3

The numbers of students describing key ideas related to Taylor series in task 3 are reported below.

Table 1: Number of students referencing concepts is task 3

	Highlight particular *x*	Mention error/error bound for fixed *x*	Mention vertical number line	Describe partial sums	Approximation language
Two question version (N=60)	4 (7%)	0 (0%)	1 (2%)	3 (5%)	3 (5%)
One question version (N=15)	10 (67%)	0 (0%)	1 (7%)	3 (20%)	0 (0%)
	Concept of error	Describe error as decreasing	Describe error bound	Describe sequence of terms	Describe interval of convergence
Two question version (N=60)	6 (10%)	2 (3%)	1 (2%)	6 (10%)	17 (28%)
One question version (N=15)	2 (13%)	2 (13%)	0 (0%)	1 (7%)	4 (27%)

Conclusion & Questions

Evidence suggests that these VMs encouraged students to have an understanding of Taylor series that may eventually support the formalization of a pointwise convergence definition. Students in classes using VMs were more likely to draw Taylor polynomials correctly, as well as to draw the Taylor polynomials tangent to the function being approximated. Students in VM classrooms, while more likely to correctly draw a Taylor series graph of sin(*x*), were also more likely to draw a collection of Taylor polynomials than students in non-VM classrooms. This may be an artifact of the VMs themselves, in which new Taylor polynomials are introduced while previous Taylor polynomials remain. Responses to the third task suggest that connecting many of the key concepts of approximation, error, and error bound to issues of convergence may not be in the fore of the VM students' minds. Despite the relatively low numbers and losses in some categories, we saw gains in highlighting a particular *x* to be especially promising as it is one of the key concepts of pointwise convergence of Taylor series. Combined with results from Task 1, VM students had improved understanding of general shapes and trends of Taylor series convergence, but more support may be necessary to promote further unpacking of the relevant quantities and move closer to a notion of pointwise convergence.

Currently, interviews are being conducted with students who have completed classes featuring VMs to further describe what students are observing when viewing a Taylor series VM.

We invite discussion about the following questions:

1) Analyzing the ways in which students interact with VMs compared to static images.
2) Is there some other way of bringing out the pointwise conception naturally?

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