Investigating a mathematics graduate student's construction of a hypothetical learning trajectory

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This study reports results of how a teacher's mathematical meanings and instructional planning decisions transformed while participating in and then generating a hypothetical learning trajectory on angles, angle measure and the radius as a unit of measurement. Using a teaching experiment methodology, an initial clinical interview was designed to reveal the teacher's meanings for angles and angle measure and to gain information about the teacher's instructional planning decisions. The teacher participated in a researcher generated HLT designed to promote the construction of productive meanings for angles and angle measure and then constructed her own HLT for her students. The initial interview revealed that the teacher had several unproductive meanings for angles and angle measure that caused the teacher perturbations while participating in the tasks of the researcher generated HLT. This participation allowed her to construct different meanings for angles and angle measure which changed her instructional planning decisions.

Key words: Hypothetical Learning Trajectories, Trigonometry, Graduate Teaching Assistant Education, Mathematical Meanings

Students and teachers often have difficulty reasoning about topics related to trigonometric functions (Moore, 2010; Thompson, Carlson, & Silverman, 2007; Weber, 2005). Moore (2010) described several reasons that students may have difficulty reasoning about trigonometric functions including the approach that current curricular materials take when introducing the sine and cosine functions. Many teachers introduce trigonometric functions in both right triangle contexts and unit circle contexts, though they rarely make connections between the two. This approach hinders students' ability to develop coherent meanings for these functions. This has led researchers to start working on how students reason quantitatively and covariationally about trigonometric functions (Moore, 2010, 2012, 2014; Moore & LaForest, 2014). For students to develop coherent meanings for trigonometric functions, they must first develop meanings for angles, angle measure, and the radius as a unit of measurement. Moore (2009) investigated students' meanings for these concepts.

Teachers should strive to have their students build coherent mathematical meanings (Thompson, 2013). Simon (1995) shared three episodes from teaching that paint the picture of a teacher guided by his conceptual goals for his students' learning. A teacher's consideration of this learning goal, the learning activities, and the thinking and learning in which students might potentially engage in make up a hypothetical learning trajectory (HLT). The term refers to a teacher's prediction of the path by which learning may occur and characterizes expected tendencies of student learning. It is hypothetical in the sense that the actual learning trajectory of an individual is not knowable in advance. A teacher's HLT for her students has three parts: the teacher's goal for students' learning, the mathematical tasks used to promote student learning, and hypotheses about the process of the students' learning (Simon & Tzur, 2004). Simon and Tzur (2004) propose that having a teacher generate a HLT is a way for a teacher to teach based on her anticipation of how students might come to learn a particular concept, knowledge of what her students' current understandings are, tasks that she can use to promote learning of the concept, and her own understandings of the goal of the lesson. The generation of a HLT requires a teacher to think about what meanings she needed to know in order to build the proposed meanings. I hypothesize that the act of generating a

HLT serves as an impetus for getting a teacher to focus on mathematical meanings and to leverage student thinking when designing instructional interventions.

Methodology and Research Questions

The primary goal of this study was to explore how a teacher's mathematical meanings and instructional planning decisions change while participating in and then creating a hypothetical learning trajectory on angle, angle measure, and the radius as a unit of measurement. The study was conducted using a teaching experiment methodology (Steffe & Thompson, 2000). The subject is a graduate student in applied mathematics who was teaching *Pathways Precalculus* (Carlson, Oehrtman, & Moore, 2014) at the university level. I will refer to the subject as Lily. All sessions were videotaped and all written work produced was scanned and used for analysis. An initial clinical interview was conducted to build a model of the teacher's meanings for angle, angle measure, and the radius as a unit of measurement and to gain information about the teacher's instructional planning decisions for angle, angle measure, and the radius as a unit of measurement. The teacher then participated in two exploratory teaching sessions that were designed to resemble a hypothetical learning trajectory for a student's meanings for angles, angle measure, and the radius as a unit of measurement. During each session, I gave the teacher tasks that I designed to reveal and push the boundaries of the teacher's mathematical meanings. These tasks were designed before the initial interview and then modified to reflect the insights I gained from working with the teacher. The last part of the intervention was to have the teacher create a hypothetical learning trajectory for her students. The teacher was given a template for a HLT that was adapted from a Lesson Logic Form (Thompson, 2008). The HLT Lily created provided insight on how her meanings for angles and angle measure had changed as well as what meanings she wished her students to construct in class. The two research questions were "in what ways and to what extent does a teacher participating in and then generating a hypothetical learning trajectory on angles and angle measure affect the teacher's mathematical meanings for angles and angle measure?" and "in what ways and to what extent does a teacher participating in and then generating a hypothetical learning trajectory on angles and angle measure affect the teacher's instructional planning and decisions?"

Conceptual Analysis of Angles and Angle Measure

In order to produce a hypothetical learning trajectory for angles and angle measure, I needed to identify what meanings would comprise a propitious way of understanding of angles and angle measure. From these ways of understanding I identified six learning goals for students and then used prior research on students' meanings for angle measure (Moore, 2009) to design tasks that a teacher's use of would promote his/her students' construction of these desired understandings. Some of the tasks were adapted from the *Pathways Precalculus* curriculum (Carlson et al., 2014) as well as dissertation studies conducted by Moore (2010) and Tallman (2015).

An angle is a geometric object that consists of two rays that meet at a common endpoint, often called the vertex of the angle. A measurable attribute of an angle is its "openness." When a circle is centered at the vertex of the angle, one can quantify the measure of openness by measuring the length of the subtended arc in comparison to the length of either the radius or circumference of the circle, or, more generally, any unit of length that is proportional to the circle's radius or circumference. An angle can be measured by quantifying what percentage of the circle's circumference the subtended arc length is or by measuring the subtended arc length in units of the radius length. The six learning goals I identified for the researcher-generated HLT are students will understand:

1. …that an angle is an object that consists of two rays that share a common vertex.

2. …that the measurable attribute of interest of an angle is its “openness.”
3. …the “openness” of an angle in terms of the length of the subtended arc of the circle centered at the vertex of the angle.
4. …that any particular angle subtends the same fraction of the circumference of all circles centered at the vertex of the angle.
5. …that the unit of measure of this subtended arc length must be proportional to the circumference of the circle centered at the vertex of the angle so that the size of the circle does not matter.
6. …that angles measured in radians are measured by measuring the subtended arc length in units of the length of the radius of the circle centered at the vertex of the circle and that angles measured in degrees are measured by measuring the subtended arc length in units of $1/360$ of the circumference of the circle centered at the vertex of the angle.

With these learning goals in mind, I selected and/or designed seven tasks that could be used to promote the construction of these ways of understanding. In combination with the learning goals, these made up the researcher generated HLT that was used during the study.

Results

The initial clinical interview began with Lily creating a lesson plan for angles and angle measure. Lily’s lesson plan began with asking her students “What is an angle?” Lily’s answer to the question was that “an angle is an object that can be measured” and also that she “would love for them to relate it to a circle.” Next Lily planned to look at one picture and ask her students “How many angles can we measure in this picture?” Lily’s intended answer to this question revealed that Lily’s meaning for angle and angle measure was potentially different from the meanings I outlined in the researcher-generated HLT. When asked what she wanted her students to understand about angle measure, Lily drew a picture similar to the following picture (colors added to ease discussion of what she was referencing):

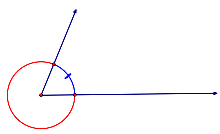

Figure 1: Lily’s initial image of two angles.

Initially Lily drew the part of the picture that is in blue, identifying that the blue arc and tick mark she had drawn indicated that students should be thinking about the interior space between the two rays. Then Lily added the red arc and said that she also wanted students to recognize that “this” was another angle. This revealed that Lily did not view the object of an angle as two rays that met at a common endpoint, but that some other aspect was also present in her scheme for angles. When asked how we measure an angle, Lily stated that we could measure an angle by comparing the subtended arc to the circumference or radius. Lily then wanted her students to imagine that every angle can be drawn inside a circle and then said that she would return to asking her students what an angle was. Finally, she would conclude her lesson by asking, “How can we measure the angle?” and brought up that she expected students to mention protractors, SOHCAHTOA, radians, and degrees. Then she would ask students what a radian was and what a degree was.

Following this, I proceeded to ask questions that I had designed to reveal more about Lily’s meanings for angles and angle measure. The first question I posed was “What is an angle?” Lily’s answer revealed that the word angle invoked a lot of meanings for her and that she had not made a distinction between an angle and an angle’s measure. Her mental image of an angle included rotations and a circle. She stated “we have this notion of going around a circle, which is where I naturally think about angles now.” She mentioned that something being 360 degrees was the same as something being 720 degrees, but did not mention what this “something” was. Lily was then presented with an image of an angle,

$\angle ABC$, and asked how many angles were pictured. Lily's answer was that there were infinitely many angles, depending on where you drew in an arc, or what you wanted to measure. This added evidence to the idea that Lily's meaning for angle consisted of more than just two rays that have a common endpoint.

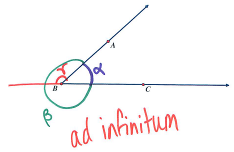

Figure 2: Lily's image of an infinite number of angles.

I then asked Lily what it meant to measure an angle. Lily stated, "we're looking at maybe a proportional relationship of what is cut off if we were to imagine the entire circle there. It's the relationship between this arc here and the entire circle." I then asked her to clarify what was proportional and she responded, "we're looking at the proportion of this arc to this radius." This revealed that Lily is able to think about angle measures as a ratio of the subtended arc length to the circumference and the ratio of the subtended arc length to the radius length, but that she was using these two ratios interchangeably. She was initially discussing the subtended arc length as a proportion of the circumference, but then drew a picture and defined an equation that found the proportion of the subtended arc to the radius. When asked what it means for two angles to have the same measure, Lily referred back to the ratio of subtended arc length to the length of the radius and stated that "two angles have the same measure if and only if s-one over r-one is equal to s-two over r-two where s-one and r-one are from angle one and s-two and r-two are from angle two." S-one and s-two stood for the subtended arc length and r-one and r-two stood for the radius length of each angle. When asked further questions about measuring an angle in degrees or in radians, her lack of distinction between the two ratios she had identified caused her problems when writing equations that described what it meant for an angle to have a measure of one degree or of one radian. In summary, the initial clinical interview revealed that Lily's definition on an angle was conflated with her process for measuring the angle. Lily did not make the distinction between the object of an angle and the measureable attribute of openness. Lily also had a strong conception of the measure of an angle being related to the portion of the circle subtended, though she used ratios of the subtended arc length to the radius and circumference interchangeably, and not always correctly. Identifying these meanings led to the researcher's modification of some of the tasks and questions to try and address what the researcher viewed as unproductive meanings that Lily had.

The next two sessions involved Lily working through 7 tasks with the researcher. I designed the first task to help Lily distinguish between an angle as an object and the measure of an angle as a quantity. I presented Lily with a Geometer's Sketchpad (Jackiw, 2011) file that had an angle pictured and asked her to describe the picture. She was then able to drag a point located on one of the rays of the angle, which changed the openness of the angle and traced out an arc of the circle the point was located on in red. Even though the full circle was not drawn, Lily imagined that she could think about continuing to trace out the subtended arc as a way to create a circle that would be related to the subtended arc, and described that she could measure the amount of openness between the two rays by creating a relationship between the portion of the circle the arc subtended. Her initial description involved creating a ratio between the subtended arc length and the circumference but then Lily described that we could measure an angle by relating the arc length to the radius length. Throughout the sessions, Lily identified two consistent ratios that can be used to measure an angle:

$\frac{\text{length of the subtended arc}}{\text{length of the circumference}}$ and $\frac{\text{length of the subtended arc}}{\text{length of the radius}}$. However, Lily did not distinguish between these two ratios, often citing one in an explanation, but actually using the other in her work.

In the second task, Lily used these ratios to again talk about how she could measure an angle. Lily was given that for a particular angle, the length of the subtended arc was 11.48 cm and the length of the circumference was 26.04 cm and asked if she would know what the length of the subtended arc would be if the circumference changed to 16.8 cm. She demonstrated fluency of using consistent ratios to find the missing subtended arc length. I then asked her if she was measuring the same angle in her picture. Lily's response was that no, she was not measuring the same angle but was instead measuring two angles that have the same measure. I took the opportunity to further probe Lily's definition of an angle (Excerpt 1). This showed a shift in Lily's definition of an angle from her initial clinical interview.

Excerpt 1

Interviewer: So why would there be different angles?

Lily: So again, we talked about that an angle is an object. So these are two different objects.

I: An object that consists of?

Lily: That consists of two rays meeting at a common point.

I: How many rays meeting at a common point have you drawn?

Lily: Well actually, I guess I'm thinking of line segments. If I were to think of it as rays, where rays go on forever, then they would have the same rays and so they would be the same.

I: So does changing the size of the circle you're looking at change the original object of the angle?

Lily: I'm going to go with no, because if you're thinking about rays, they go on forever.

I used the next two tasks to help Lily distinguish between the need for a unit of measure that would be used to measure an angle and a unit of measure that would be used to measure the subtended arc. Lily was asked to create a protractor that would measure an angle in gips, given that any circle is eight gips. Initially Lily talked about measuring the angle and measuring the subtended arc length interchangeably, but as we discussed what we were measuring, Lily identified that units of measure for those two things should not be the same since one was a length and the other was an amount of openness. I took the opportunity to ask Lily what the difference was between something that had a measure of one radian and something that had a measure of one radius length. Lily articulated that if we are measuring using the radius, we are measuring a subtended arc length. If we are measuring in radians, we are measuring an amount of openness. Throughout subsequent tasks, Lily still used the radians and radius lengths interchangeably, though when it was brought to her attention, she could identify which one she had actually meant. Lily stated, "a radian is an angle measure that corresponds to the number of radius lengths in the arc subtended by said angle."

During the last session, I presented Lily with a template for a HLT and asked her to plan a lesson for angles and angle measure for her students. Lily identified five learning goals:

1. An angle is formed when 2 rays meet at a common vertex.
2. How do we measure angles? (Determine the openness between the rays, use circles)
3. "openness" can be the larger or smaller value.
4. We measure angles commonly in units called radians. A radian is a unit equivalent to 1 radius length of the circle in question.
5. There are 2π radians in one circle.

Lily then identified that students would need to be familiar with circles, including the circumference formula and how it relates to radius length, prior to the lesson. Lily then started designing/selecting tasks that she could use to promote students' construction of the five learning goals she identified. As she went through this process, several of her learning goals evolved as she continued to think about them. She wanted her students to understand that when they are measuring an angle, they are describing the openness between the rays that form the angle and intended to do this by starting with an example of an angle with a measure of ninety degrees because her students would be familiar with this angle. Her goal was to have students make the connection that an angle with a measure of 90 degrees also cut off one-fourth of the circumference of a circle. This led to her changing her third learning goal to "understand that the subtended arc and circumference have a consistent relationship that can be used to measure angles," and the fourth goal became "a radian is a unit of measure often used for angles that is equivalent to one radius length on the subtended arc of the drawn circle." She later defined that "an angle that subtends an arc with a length of one radius (of the circle) is said to have a measure of one radian." Lily continued to select tasks that she would use in a lesson. She identified that she wanted to spend the first day of the unit focusing on the meaning of angles and angle measure and then spend a second day practicing these meanings in application problems.

Discussion and Implications for Future Research

Several changes occurred in Lily's mathematical meanings for angles and angle measure between the initial clinical interview and her generation of a HLT for her students. Initially, Lily's definition of an angle included more aspects than two rays that meet at a common endpoint. She believed that when she drew an arc between the two rays that the arc was part of the object of the angle, which meant that if she drew a circle centered at the vertex of the angle, she viewed this as creating two angles, where each subtended arc was part of a separate angle. This contributed to her unclear distinction between an angle as an object and an angle as a quantity. I hypothesize that this is because she viewed the subtended arc as part of the angle, and therefore the subtended arc was no longer a measurable attribute of an angle, but was instead part of the angle itself, thus making an angle a measureable attribute of another object, such as a triangle. Lily also used both radians and radius lengths when talking about measuring both an angle and an arc length. Lily was initially using these two units interchangeably. Lily also showed a tendency to talk about ratios whenever she was asked to explain the meaning or process of measuring an angle. At different times, Lily mentioned two consistent ratios when measuring an angle: $\frac{\text{subtended arc length}}{\text{radius length}}$ and $\frac{\text{subtended arc length}}{\text{circumference length}}$. Lily was aware that both of these ratios were consistent and on multiple occasions did not differentiate between which ratio she intended to use. Lily fluidly switched between saying that an angle will cut off the same portion of any circle's circumference and saying that the ratio of subtended arc length to radius length would be the same for any circle.

During the next three sessions, some of these initial meanings caused Lily perturbations while working through the tasks of the researcher generated HLT and when generating her own HLT. These perturbations caused Lily to make accommodations to her schemes for angle and angle measure. In the final session, Lily defined an angle as an object that is formed when two rays meet at a common vertex. This accommodation to her meaning for angles was a result of being confronted with tasks in which she was unable to assimilate the information in front of her to one of her already existing schemes. Specifically, this accommodation was a result of realizing that in order for you to be able to use a circle of any size to measure the same angle, you had to think of these different sized angles as still representing the same original angle.

By the end of the study Lily also had a clearer distinction between a radian as a unit of measurement and a radius length as a unit of measurement. While writing her HLT she used radian when she meant radius length once, but was able to identify that she had done so and made the change. Being asked to articulate what a gip measured in task 3 had required Lily to think about the difference between measuring a subtended arc and measuring an angle.

While working on creating her HLT, Lily was still inconsistent in her use of the two ratios she had identified as staying consistent for the same angle. The first time the researcher asked Lily if she realized she had been using two different ratios was during the final session while Lily was creating her HLT. Lily identified that they were two different ratios but that she hadn't really thought about that for herself. The opportunity for Lily to reflect on this distinction was lost because the researcher did not address this inconsistency in her use of the two ratios during the tasks of the hypothetical learning trajectory.

Together, the implications of the changes that occurred and the changes that did not occur provide evidence of the importance of the initial model the researcher created of Lily's mathematical meanings after the initial clinical interview. The researcher modified the questions asked during the tasks of the researcher-generated HLT to try to specifically address the meanings identified from the first interview. The activity of completing the tasks allowed Lily to make the necessary accommodations to her schemes for angle and angle measure. The only concept that is potentially still problematic for Lily is that she is inconsistent in her use of and meaning for the ratios of subtended arc length to radius length and to circumference length. This highlights the importance of identifying your student's meanings as a starting point for constructing a HLT. These results also highlight the effectiveness of the tasks in the researcher-generated HLT on providing the activity for a teacher to make accommodations to his/her schemes invoked by the tasks in the HLT.

These accommodations to Lily's scheme showed up in the second lesson plan she created. I hypothesize that Lily recognized the usefulness of the meanings she had constructed during the tasks of the previous two sessions and wanted to help her students construct these same useful meanings. Several of Lily's learning goals were meanings that she had either not had prior to the study, or had not been able to articulate. Lily's third learning goal is a reflection of a meaning that Lily had prior to the study. This shows that Lily also recognized the importance of her prior meanings, and did not only model her HLT after the accommodations she was aware of making. I include this to highlight that teachers do not start as a blank slate. Teachers are unable to help their students construct productive meanings if they do not have these meanings for themselves. Thus any sort of hypothesized intervention for improving teaching has to take this in to account. This study provides evidence that working through the tasks in a researcher-generated HLT and then creating your own HLT is one possible way to help teachers make accommodations to their schemes and then recognize the impact that these accommodations can have on their students.

Lily's reflection on the tasks she had completed and the accommodations she had made helped her identify different learning goals for her students. The new learning goals she identified make up a more robust understanding of angles and angle measure than what her initial lesson plan contained. Lily's second lesson plan also included specific activities and tasks that she intended to do with her students and conversations she hoped to have with her students, both of which were barely contained in the first lesson plan. This suggests that providing Lily with a template that specifically asked her to identify learning goals and tasks that would help students construct those understandings helped contribute to a more detailed and robust lesson plan.

The results of this study suggest that having Lily work through tasks in a researcher generated HLT caused changes in both her schemes for angles and angle measure as well as what she identified as being important to teach her students. The study also suggested that the use of a HLT provided a way to encourage a teacher to think about student thinking as she planned her lesson. A future study involving these ideas will allow the researcher to identify what aspects of participating in the HLT caused this effect.

References

Carlson, M., Oehrtman, M., & Moore, K. C. (2014). *Pathways to Calculus: A Problem Solving Approach*: Rational Reasoning.

Jackiw, N. (2011). The Geometer's Sketchpad.

Moore, K. C. (2009). *An investigation into precalculus students' conceptions of angle measure*. Paper presented at the Twelfth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC.

Moore, K. C. (2010). *The Role of Quantitative Reasoning in Precalculus Students Learning Central Concepts of Trigonometry.* (Unpublished doctoral dissertation). Arizona State University.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, 75-92.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.

Moore, K. C., & LaForest, K. R. (2014). The circle approach to trigonometry. *Mathematics Teacher, 107*(8), 616-623.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Tallman, M. (2015). *An Examination of the Effect of a Secondary Teacher's Image of Instructional Constraints on His Enacted Subject Matter Knowledge.* (Unpublished doctoral dissertation). Arizona State University.

Thompson, P. W. (2008). Lesson Logic Form.

Thompson, P. W. (2013). In the absence of meaning... In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). New York, NY: Springer.

Thompson, P. W., Carlson, M., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

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