The Intermediate Value Theorem as a starting point for multiple real analysis topics

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In this paper I argue that the proof of the Intermediate Value Theorem (IVT) provides a rich and approachable context for motivating many concepts central to real analysis, such as: sequence and function convergence, completeness of the real numbers, and continuity. As a part of the development of a local instructional theory, an RME-based design experiment was conducted in which two post-calculus undergraduate students developed techniques to approximate the root of a polynomial. They then adapted those techniques into a (rough) proof of the IVT.

Key words: *RME, instructional design, design experiment, real analysis, limit*

Introduction

The concept of limit has served as the theoretical foundation for the calculus and its applications ever since the work of Cauchy, Bolzano, and others in the early and mid-19th century (Grabiner, 1981). Specifically, in a standard analysis class, the ideas of limit and convergence lie at the heart of such topics as sequences, continuity, derivative, integral, and the completeness of the real numbers. It follows that a formal understanding of the limit concept is essential to any investigation of the theoretical underpinnings of the calculus.

Presented here is a description of how the context of the Intermediate Value Theorem (IVT) can serve as a natural launching point for many topics in a real analysis course, starting with formalizing the concepts of limit and convergence. The IVT provides such a context in two ways: 1) using said theorem to approximate the root of a polynomial and 2) adapting that approximation technique into a formal proof. I will report on an RME-based design experiment, the goal of which was to investigate the following questions:

- *What student strategies anticipate the formal limit concept?*
- *What problems or tasks can be used to elicit these strategies?*
- *How can these strategies be leveraged to develop more formal understandings of the limit concept?*
- *What student strategies suggest avenues for developing other real analysis topics?*

Literature Review

Student understanding of the limit has received a great deal of attention from the mathematics education research community. A great deal of research has focused on investigating the struggles students face in working with limits and the tools they use to deal with those struggles (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Moru, 2009; Oehrtman, 2009; Sierpińksa, 1987; Szydlik, 2000; Tall & Schwarzenberger, 1978). Briefly, students employ intuitive metaphors (e.g. “limit as motion”, “collapsing dimension”, “limit as unreachable boundary”) that can be problematic in more formal endeavors. The other main area of focus has been investigating the process of students formalizing their understanding of limit (Cottrill, et al., 1996; Oehrtman, Swinyard, & Martin, 2014; Swinyard & Larsen, 2012; Williams, 1991); that is, coming to understand and work with limits in a way that is consistent with the standard formal definition(s)[1]. One important development in our understanding has

1 There are many logically equivalent formulations of these definitions, so saying “the standard formal definitions” is perhaps misleading. By “standard” I refer to the ε-δ (or ε-N) characterizations found in most

been the recognition that formal definitions of convergence, and therefore formal work with limits, serve a markedly different purpose than informal work with limits (Swinyard & Larsen, 2012). Specifically, tasks in the calculus sequence generally involve *finding* or *evaluating* limits, while more formal tasks focus on *verifying* limit candidates, or constructing proofs given the existence of certain limits. Motivating this shift in character, while still building on intuitive knowledge gained in the calculus sequence, heavily influenced the development of the task sequence and local instructional theory for this design experiment.

Theoretical Framework

This paper reports on an RME-based design experiment, which represents the early stages of curriculum development for a real analysis course. Design experiments should inform both instructional design and theory development (Cobb, et al., 2003; Gravemeijer, 1998). The design heuristics of Realistic Mathematics Education (RME), namely *guided reinvention, emergent models*, and *didactic phenomenology,* guided the development and implementation of the experiment as well as the underlying theory. The contribution of each of these heuristics will be discussed briefly below.

Guided Reinvention

On a macro level, the heuristic of guided reinvention motivated my overall instructional goal of having the students develop their own formal definitions of convergence, rather than working to make sense of the standard formal definitions. In RME, the goal is not that everything be strictly reinvented by the students, but rather that, "formal mathematics would be experienced as an extension of [students'] own authentic experience" (Gravemeijer & Doorman, 1999). That is, instructional activities should be designed and sequenced so that the formal mathematics emerges from students' informal mathematical activities, so that students feel a sense of ownership over the mathematics developed. While guided reinvention provides a macro-level structure for instructional design, other RME heuristics are more useful at filling in this structure.

For actual task generation, sequencing, and refinement, I relied largely on the design heuristics of *didactic phenomenology* and *emergent models*.

Didactic Phenomenology

In order to find an intuitive context that could evoke potentially useful student strategies, the heuristic of didactic phenomenology suggested that I look to the origins of the formal definition, paying particular attention to the didactic implications (i.e. consequences for instruction) of those origins. From where did our modern formal definition of convergence come? What problems did it solve for mathematicians at the time? Approximations of various kinds played a pivotal role in the historical development of the limit concept (Grabiner, 1981). Mathematicians (especially Lagrange) of the late 18th and early 19th centuries had made great strides in techniques of approximation and error-bounding in applied contexts. Cauchy is credited with developing the first ε-δ style definitions of convergence, and there is strong evidence to suggest that he took inspiration from these approximation techniques (Grabiner, 1981). Further, both he and Bolzano developed formal proofs with these definitions by adapting those same approximation techniques[2]. Prior to these developments, the mathematical community, including Newton and Leibniz, had only been able to justify limits with vague (by

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analysis textbooks, e.g.: *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $0 < | x - a | < \delta$ *then* $| f(x) - L | < \varepsilon$.

2 For a clear, thorough demonstration of this, see Grabiner, 1981, especially pp. 69-76.

today's standards) statements about “vanishing quantities” and “infinitesimals”. The work of Cauchy and Bolzano put calculus on a firm, well-defined foundation for the first time.

Additionally, Gravemeijer and Terwel interpreted didactic phenomenology to suggest that, “situations should be selected in such a way that they can be organized by the mathematical objects which the students are supposed to construct” (2000, p. 787). That is to say, in order to support students in reinventing a formal definition of convergence, a curriculum designer should seek contexts and tasks in which the students would able to reason intuitively, and in which a formal definition would have power to organize and solve problems. Inspired by the works of Cauchy and Bolzano, I conjectured that approximating the roots of a polynomial using the Intermediate Value Theorem (IVT), and then constructing a formal proof of the theorem[3], would be just such a context.

The Intermediate Value Theorem:
If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then for every y between $f(a)$ and $f(b)$ there exists a $c \in (a, b)$ so that $f(c) = y$.

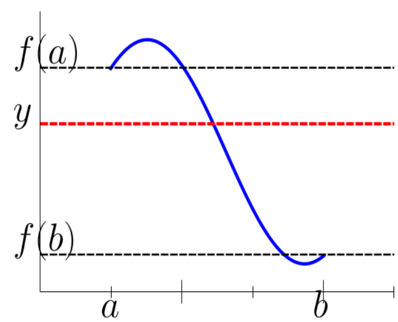

Figure 1: The IVT for continuous functions.

There are a few features that make the IVT such a context. First, the IVT is a fairly intuitive result which students will likely assume even if they have never been exposed to the formal theorem (Figure 1). This context requires students to draw on their concept images of functions and limits, and so builds on their intuitive knowledge gained from the calculus sequence. Second, the IVT provides an incredibly rich context for investigating the properties of real numbers and convergence. If we follow Cauchy's example and adapt our proof from the approximation techniques of Lagrange, then a rigorous proof of the IVT requires formal definitions of sequence convergence, continuity, and the limit of a function at a point. Some form of the Completeness axiom of the real numbers is also necessary, and so this context could motivate investigations in that direction as well. (Exploring these possibilities will be the focus of ongoing analysis.) In this way the context of approximating roots using the IVT, and then constructing a proof of the IVT for continuous functions, is a mathematically rich context that provides the students with a need to develop the desired formal definitions of convergence, as

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3 Technically, if we restrict ourselves to establishing the existence of *roots* of continuous functions, then we are only proving a special case of the IVT (sometimes referred to as *Bolzano's Theorem*). But the proof is easily adapted to the general case by a simple vertical shift.

well more formal understandings of continuity and the completeness of the real numbers. The specific development of students Brad and Matt in this design experiment will be outlined below.

Emergent Models

The heuristic of emergent models provides one way to describe the process by which formal mathematics might emerge from informal student activity in these contexts. The use of "models" in RME is not restricted to physical drawings or tools. In describing a local instructional theory for the development of the quotient group concept, Larsen conjectured that, "the quotient group concept could emerge as a model-of students' informal mathematical activity as they searched for parity in the group D8 (the symmetries of a square)" (Larsen & Lockwood, 2013). Thus "model" in this sense can also refer to a concept or structure that the teacher or researcher recognizes as a model of the students' mathematical activity, but of which the students themselves may not be aware. Continuing with Larsen's example, once the students had begun to reflect on their activity with parity and other group-like partitions of groups, conjecturing and verifying common properties, the concept of quotient group became a model for their reasoning in this new mathematical reality; a "model of" informal mathematical activity had become a "model for" more formal mathematical reasoning. "This shift from model of to model for concurs with a shift in the students' thinking, from thinking about the modeled context situation to a focus on mathematical relations" (Gravemeijer, 1999, p. 162). In RME-based instruction, this progressive mathematization is the primary mechanism by which students develop more formal mathematics and create new mathematical realities for themselves.

The modern formal definition of convergence can be seen as a model of the approximating activity of the mathematics community in the 18th and 19th centuries. A formal definition of convergence emerged from these activities of approximating and error-bounding, first for Cauchy and then for the rest of the mathematical community. In this way the historical development of the concept of limit suggested that a formal definition of convergence could emerge as a *model of* student activity centered around approximations. By reflecting on and organizing this approximating activity, such a formal definition could emerge from their activity and serve as a *model for* more formal mathematical reasoning about limits and convergence.

Students' informal understandings of approximations and error-bounding have also been used as a foundation for instruction of the calculus sequence. Research suggests that this foundation has supported students in formalizing their concept of limit (Oehrtman, 2008; Oehrtman, Swinyard, & Martin, 2014). In this way formal characterizations of convergence can be seen as a useful model for describing and supporting students' progressive mathematization.

Methods

The design experiment involved two students, Brad and Matt, working together on a sequence of tasks over the course of 10 sessions, approximately 60-minutes each. Data consisted of the video/audio recordings of each session, researcher notes, and student-generated summaries from the conclusion of each session. After each session an outline of the students' progress was made, with key segments being analyzed in greater depth. This analysis focused on finding student strategies and statements on which to build toward the larger goal of formalizing their understanding of limits, which in turn supported the ongoing development of the task sequence.

Brad and Matt begin their investigation by working on the following task:

Does p(x) have a root in [0,3]?

$$p(x) = x^4 - 4x^3 - 7x^2 + 22x + 10$$

This polynomial was intentionally constructed to have only irrational roots, so that students would not be able to use algebraic tools (e.g. factoring, the quadratic formula, polynomial division, the rational roots theorem, etc.) to find the exact roots and would have to find a way to approximate. Subsequent tasks had the students approximating the root to different degrees of accuracy, and then working to generalizing their technique. The task was then to prove a version of the IVT which they had postulated, which in turn motivated the development of formal definitions of multiple types of convergence.

Preliminary Analysis and Results

On the first task, Brad and Matt developed an approximation strategy wherein they iteratively bifurcated the given interval to get more and more accurate approximations for the root. Through the course of constructing a proof of the IVT from this approximation technique, Brad and Matt were tasked with developing their own formal definitions of sequence convergence, function limit at infinity, continuity, and function limit at a point. Below I have included their first and their final definitions for what it means for a function to have a limit of zero as x tends to infinity.

Def 1b: $\forall\ 1/\varepsilon$, $\exists\ n$ s.t. $f(n) < 1/\varepsilon$. ε, n in **R.**

Def 3: $\forall\ 1/\varepsilon\ \ \exists$ an interval (x_a, ∞) s.t. $|f(x)| < 1/\varepsilon\ \ \forall\ x$ in (x_a, ∞)

Current analysis is focusing on explaining how these reinventions were supported by the students' activity in the starting task.

Subsequent analysis will focus on identifying fruitful starting points, within the proof of IVT task, for follow-up tasks investigating other real analysis topics. Brad and Matt had some very interesting conversation about continuity which were not fully capitalized on. Further, their approximation strategy suggested many possible approaches to the idea of the completeness of the real numbers, including the Monotone Convergence Theorem, the Nested Interval Property, and even the Least-Upper Bound Property. Designing and implementing these tasks will also be the focus of future design experiments in the further development of this real analysis curriculum and local instructional theory.

Questions:

- What do you consider to be central topics in an introductory real analysis course?
- What student strategies presented here suggested possible paths for further development of other topics?
- What role should counter/pathological examples play in an introductory real analysis course?

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