Effects of dynamic visualization software use on struggling students' understanding of calculus: The case of David

Julie M. Skinner Sutton
The University of Texas at Dallas

James A. Mendoza Epperson
The University of Texas at Arlington

Using dynamic visualization software (DVS) may engage undergraduate students in calculus while providing instructors insight into student learning and understanding. Results presented derive from a qualitative study of nine students, each completing a series of four individual interviews. We discuss themes arising from interviews with David, a student exploring mathematical relationships with DVS who earns a C in calculus. David prefers to visualize when solving mathematical tasks and previous research suggests that such students, while not the 'stars' of their mathematics classroom, may have a deeper understanding of mathematical concepts that their non-visualizing peers. Using modified grounded theory techniques, we examine evidence of uncontrollable mental imagery, the need to refocus David on salient aspects of the animations, instances when David's apparent conceptual knowledge is neither fully connected to nor supported by procedural knowledge, and David's failure to transfer knowledge when DVS was not offered during assessment.

Key words: calculus, visualization, dynamic visualization software

The urgent need for the United States to produce an additional one million graduates studying science, technology, engineering and mathematics (STEM), the fact that students to choose to leave the sciences often cite uninspiring introductory courses as the reason (President's Council of Advisors on Science and Technology (PCAST), 2012) and our knowledge that students who fail to obtain a deep understanding of calculus abandon their quest for a STEM degree (Carlson, Oehrtman, & Thompson, 2007) lead us to consider innovative methods for student engagement in college calculus that may also provide insight into student learning of the subject. Dynamic Visualization Software (DVS) facilitates visual investigation of important mathematical relationships and may assist students in exploring topics in a way that promotes conceptual understanding. Presmeg (2006) found that students in the high school mathematics classroom who prefer to visualize when solving mathematical tasks are not the 'superstars' of the classroom but that their understanding of the concepts and ideas of mathematics may be stronger than those of their non-visualizing peers. Presmeg defines visualization as including, "the processes of constructing and transforming both visual and mental imagery and all the inscriptions of a spatial nature that may be implicated when doing mathematics," (Presmeg, 2006, p. 3). This case study of David, is part of a larger study that explores how student interactions with DVS influence the learning and understanding of calculus for undergraduate STEM majors (Sutton, 2015).

Study Design

In this study we investigate how the use of DVS influences student understanding of derivative as a rate of change of one quantity with respect to another and how the experiences with DVS affect student understanding of derivative at a point as well as a student's graphical, analytical and conceptual understanding of derivative. We also explored student understanding of the relationship between continuity and differentiability.

We completed this study during the Fall 2013 semester at a large university in the Southwestern United States. Participants came from a single section of calculus with 110 students enrolled. Fifty students enrolled in this section also participated in an intervention program that met twice weekly and focused on problem solving. Funding guidelines required that students in the intervention program be first-time, first-semester freshmen US Citizen or permanent resident, majoring chemistry, engineering, physics or mathematics.

During the first week of classes, all students enrolled in the selected section of calculus completed a background demographic survey and Presmeg's (1985) Mathematical Processing Instrument (MPI). Scoring of the MPI only provides information about the student's preference to visualize when solving mathematical tasks. However, when reviewing the MPI we saw that not all students had adequately solved the problems in the instrument and we decided to assign each student two scores: one indicating the preference to visualize (0-24) and an additional score showing the number of questions that student correctly answered (0-12). We did this in an effort to select participants most likely to complete the course successfully. Students with a correctness score less than eight were not invited for participation. We invited eight visualizers (MPI score above 15) and seven non-visualizers (MPI score less than 8) to participate in a series on four individual interviews. Nine students (five visualizers) completed three one-hour interviews and a thirty-minute Exit Interview.

We placed each student in one interview group: DVS or Static. Students experiencing the DVS interviews explored the mathematical relationships highlighted during each interview using pre-designed visualizations called sketches. Students in the static interview group worked on problems adapted from a calculus textbook and answered questions analogous to those from the DVS interviews. We provided students in both groups a basic scientific calculator, paper and a writing instrument. Interviews were video recorded and smartpen technology captured real-time voice and written data. We transcribed the interviews and analyzed common themes using open coding techniques (Corbin & Strauss, 2007).

David

David, an eighteen year-old black male majoring in mechanical engineering, graduated from a large (more than 2100 students) urban high school in 2013. The only Advanced Placement course David completed was Calculus AB. He correctly answered 10/12 items on the MPI and has an MPI visualization score of 18/24, classifying him as a visualizer. David participated in the intervention program offered for calculus and he earned a C in the course. He participated in the DVS interview group for this study.

Interview I

During Interview I David, explored relationships between tangent and secant lines and how they corresponded to the relationship between average rate of change over an interval and instantaneous rate of change at a point contained within the interval.

The first sketch in Interview I presented David with the graph of a quadratic function. The sketch includes a fixed point, A, corresponding to $\left(x_a, f(a)\right)$, a dynamic point, B, corresponding to $(x_b, f(b))$, and the secant line containing both A and B. During the interview, David manipulated B as he collected data in a dynamic table. The data in the table included the values for $x_a, x_b, f(x_a), f(x_b)$, and $\frac{f(x_b)-f(x_a)}{x_b-x_a}$. Eventually, David moves B sufficiently close to A and the secant line disappears from the screen. The interviewer asks David why he thinks this happened, "The change in x is equal. When x - - I mean, when $A = B$ the change is 0 'cuz… the same thing so… there is no average speed." When asked if there is a relationship between the function's average rate of change over $[x_a, x_b]$ when the points are close together and the function's instantaneous rate of change at point A, David says that there is a relationship, but the instantaneous rate of change at point A is undefined.

The disappearance of the secant line when points A and B are sufficiently close together leads David to believe that the instantaneous rate of change of the function at point A is undefined.

The final sketch of Interview I focused on investigations related to $f(x) = e^x$. Before he began exploring, the researcher asked David about his knowledge regarding such functions. David responds that, "… like, it's never touching, whatever, like at the 4 and 0." While making this statement, David makes hand gestures indicating that the function has a vertical asymptote at $x = 4$ and a horizontal asymptote at $y = 0$. The researcher asks for more information regarding his statement that it never touches at 4. David continues to gesture and states, "it's like it's going up, so there's an asymptote and, and asymptote right there." He clarifies that the function has a horizontal asymptote at $y = 0$ and, "…a vertical asymptote at… I'd say 4 'cuz it passed right through it."

David's ability to use the software did not hinder his exploration of the graph $f(x) = e^x$. In fact, on a previous sketch he asked if he could explore and used a dynamic point on the function graph that he found interesting to do so. His assertion that the graph of $f(x) = e^x$ has a vertical asymptote at $x = 4$ is a powerful referent that he continues to hold through this and subsequent interviews. Prior to beginning Interview II, the researcher asks David what he remembers from Interview I; he states that the exponential function has a vertical asymptote.

While exploring the exponential function in a similar manner as he did with the quadratic function, David appears to gain some insight into the relationship between the function's average rate of change over an interval and its instantaneous rate of change at a point within the interval. He states that when two points A, corresponding to $(x_a, f(x_a)$ and C, corresponding to $(x_c, f(x_c))$, on a graph are "really close" together the secant line connecting them could represent, "the tangent line. Oh! The speed! The speed." The researcher continues to ask probing questions and David says that the slope of the tangent line corresponds to the instantaneous speed of a particle whose position with respect to time is determined by the exponential function. Using another dynamic table to collect data, and through some probing questions from the researcher, David eventually relates, "the y value of A," to the particle's speed point A. "So, yeah. If you, if you have your y value then it would be equal to the instantaneous speed of the particle [at that time]."

During Interview I, David experiences several transitions in thinking about the relationship between average rate of change over an interval and instantaneous rate of change at a point within the interval. Initially, David believes that the instantaneous rate of change at point A does not exist, or at least he believes that he cannot find it, when points A and B coincide. However, after further exploration (with the initial quadratic function, a quartic function and, finally, the exponential function) David appears to make a connection between the decreasing size of $[x_a, x_c]$ and the slope of the secant line as an appropriate estimate for the instantaneous rate of change at point A. After stating this connection, David discusses the slope of the tangent line at point A as the value of the function's instantaneous rate of change at the point. Though he may be building upon previous conceptual knowledge, it was not evident that he possessed this knowledge prior to the interview. It may be that the DVS evoked this knowledge that he did not demonstrate initially. We did not observe evidence of procedural knowledge or skills during this interview.

Interview II

Prior to beginning Interview II, David recalls the relationships he explored during Interview I. David uses his hands to illustrate that he understands relationships between secant lines, tangent lines, average rate of change over an interval and instantaneous rate of change at a point contained in the interval and he draws a sketch (prompted by the researcher) illustrating that he understands the role of interval size in this relationship. However, when asked what he knows about derivatives David's response is rambling and nonsensical. He says that, "derivative is the velocity… and the derivative of velocity is… well…"

The first sketch in Interview II presents the graph of a cubic function, a dynamic point P, corresponding to $(x_p, f(x_p))$, and a line tangent to the graph at point P. As David manipulates P, he mentions that the value of $f'(x_p)$ corresponds to, "the slope of the point on the graph at that instant." He also collected data in a dynamic table listing the values $x_p, f(x_p)$, and $f'(x_p)$. When asked what information from the table would be needed to plot a point lying on the graph $y = f'(x)$, he struggled to answer. After answering some probing questions, he eventually realizes that the point $(x_p, f'(x_p))$ lies on the graph of the derivative of f, though he struggles to equate $f'(x_p)$ with the instantaneous rate of change at point P. Using the software, David checks his hypothesis and, by manipulating point P along the function graph, he traces out the derivative graph of f.

The final sketch presented during Interview II revisits the graph of $f(x) = e^x$. He explores the function using a dynamic point P corresponding to $(x_p, f(x_p))$. He notes that, "it [the instantaneous rate of change at point P] is always positive," and that, "as it moves farther [in the positive direction of the x-axis] it changes faster." David collects data in a dynamic table listing the values of $x_p, f(x_p)$ and $f'(x_p)$. He immediately notices, "wherever it is on the y-axis it's the same as the slope—the slope of the tangent line," and states that the point $(x_p, f(x_p))$ lies on the graph of f'. David says that he learned in class that, "the derivative of e^x is just e^x. When asked if other exponential functions, say $f(x) = 5^x$, have this same property, David is unsure. He reasons that, "if you have e^x and you add an ln it would just be x," but he remains unsure what this means mathematically or how to even write it. In the end, David says that he, "just knows" that if $f(x) = e^x$ then $f(x) = f'(x)$.

During Interview II, David continues to make conceptual connections about the relationship between a function's instantaneous rate of change at a point and the derivative value at the point. He struggled, but succeeded, in giving the coordinates of points lying on the graph of f' when provided with the graph of f. Evidence of David's weak procedural knowledge emerges in his inability to show or explain why $f(x) = f'(x)$ for $f(x) = e^x$.

Interview III

Unlike Interviews I and II, Interview III consisted of a single sketch showing the graph of a piecewise-defined function on a closed interval (see Figure 1). This interview focused on the Extreme Value Theorem and the relationship between continuity and differentiability.

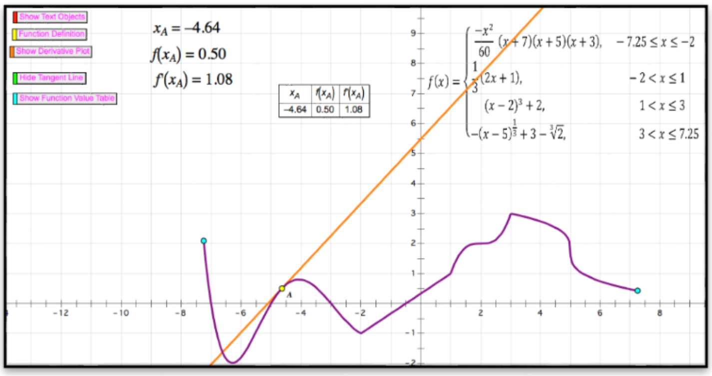

Figure 1 Screenshot of DVS Interview III.

Prior to working with the DVS for Interview III, the researcher asks David what it means for a function to be continuous on its domain. He responds, "… that it will go through all the

x values. The positive ones and negative ones… there's no holes and asymptotes or no, like stops in the graph." He relates differentiability to a lack of "corners or cusps" on the graph of the function. When the researcher probes about what David means by this he responds that, "it's like 0 at the corner, I'm guessing. You can't find the derivative of 0. I just know like - - I just remember that if there's a corner or a cusp you can't… it's not differentiable."

Once he begins exploring properties of the function graph using the dynamic point A corresponding to $(x_a, f(x_a))$, David easily identifies the maximum function value and minimum function value on the given domain and to state that the function is defined on a closed interval. He is unable, however, to write an inequality guaranteed by the EVT for all function values compared to the maximum function value. After further questioning from the researcher, David eventually concludes that $3 \geq f(x)$ for all x in $[-7.25, 7.25]$, though he in unable to explain an analogous inequality for the function's minimum value. He also states that if $f(x_d)$ is the function's minimum or maximum function value the $f'(x_d) = 0$, because, "… it changes from increasing to decreasing… or the other way."

The researcher asks David to use the DVS capability to collect data in a dynamic table and to mark points on the graph where he estimates that the instantaneous rate of change is greatest given several closed intervals. For each x_i he indicated, David is asked about the value of $f'(x_i)$. He states that the points $(x_i, f'(x_i))$ would correspond to "relative maxes," on the graph of f' and that $(x_i, f''(x_i))$ would correspond to zeros on the graph of f''.

David struggles to understand why the derivative at a point corresponding to a sharp corner on a function graph does not exist. Initially, David believes that, "you can't set the derivative equal to zero," at such a point. However, after investigating on the graph (see Figure 1) near $x = -2$ he realizes, "… so the derivatives from both sides aren't equal." David continues to investigate his notion near $x = 1$ and $x = 3$ on the same graph and concludes that his statement also applies there. The final question in Interview III required David to explain the relationship between continuity and differentiability; he responds that, "… like it can be continuous but that doesn't mean that it is differentiable everywhere."

The transcripts for Interview III contain several examples of David acquiring conceptual knowledge, or experiencing transitions to existing conceptual knowledge. He demonstrates how changes in one quantity result in changes in another quantity as he relates $f(x_i), f'(x_i)$, and $f''(x_i)$. However, it is unclear if he adjusted his conceptual knowledge relating a point, $x_a, f(x_a)$, that corresponds to a maximum function value to include that $f'(x_a)$ may equal 0 or be undefined. Through probing questions from the researcher and exploration with DVS, David makes conceptual connections about the derivative at sharp corners of the graph of f.

Exit Interview

The Exit Interview for all students was administered in a static fashion. No DVS was offered for exploration and the interview protocol was identical for all student participants.

The first question asks students to say what comes to mind when they hear the word derivative. David's responses only include lists of specific derivative rules and examples. "x^2 derivative equals $2x$. $\cos x$ derivative is $-\sin x$…"

The second task included the graphs of two polynomial functions with the point (2,3) marked on each graph. David is asked to compare the instantaneous rate of change at $x = 2$ of each function. His response, "you take the derivative and plug in 2," while correct, relied upon the function definitions, but only graphs were given. He does make some comparisons, "… where it goes from increasing to decreasing, the point where it does that, the point where it switches the 0, the instantaneous rate of change, which is the derivative of the function would be 0." He reasons that the instantaneous rate of change at $x = 2$ is the same value as the slope of the line tangent to each graph at (2,3) and attempts to find this. Yet, for one graph he chooses (2,0) and (2,3) to find the slope of the tangent line and becomes confused.

David also struggles with the idea that a continuous function may not be differentiable over the entire interval. He states that "…it's continuous and differentiable," when asked about what continuity implies about differentiability. He also struggles with the relationships between the function value at a point and the derivative at a point and, at times, is unsure which he is referencing. After drawing a graph similar to the graph of $y = |x|$, David appears to clear up his confusion and he states that the derivative is undefined at a point making a sharp corner, but he amends this statement at the end of the interview and states the derivative would be zero. When asked why at a point, $\left(x_a, f(x_a)\right)$, corresponding to a sharp corner on the graph of f, $f'(x_a)$ would not exist, David replies, "I've never thought about that before." His understanding of why the derivative does not exist at the point he indicated is limited to an incorrect, rudimentary procedural understanding that, "… Because it's - - it's a 0 or not 0, undefined so. Never thought about that. Cuz the - - what's it called? The slope at that point is like, no - - I guess it would be 0. And you can't find the derivative of 0 so - -."

Overall, David exhibited a weak ability to complete the tasks presented to him during the Exit Interview. He was, in general, able to make correct, or partially-correct, conceptual statements. Even when his statements suggested that he possessed the procedural knowledge necessary to complete a task it was a challenge for him to do so. David's inability to accurately explain why $f'(x_a)$ exists when $(x_a, f(x_a))$ corresponds to a point making a corner on the graph of f is puzzling, as he was able to explain this during Interview III.

Course Performance

The course grades for calculus at this institution are based heavily (80%) on departmental exams. These exams included minimal visualization and very few conceptual questions. Instead, the exams are heavily procedurally based. Given the evidence of David's weak procedural ability, his grade of C in the course is unsurprising.

Literature Review and Discussion

We interweave the supporting research literature into the discussion of the themes emerging from the open coding of our interview transcripts.

Throughout David's interviews the researcher refocused his attention toward the particular mathematical relationship highlighted in the sketch when he seemed unsure where to direct his attention, when he overlooked the mathematical relationship presented or when he simply needed further guidance. During Interview III, the researcher asks David to investigate near $x = -2$, using the dynamic point and data table. Even when David explains why $f'(-2)$ is undefined, he is again refocused toward places on the function graph with similar characteristics. This need to refocus the learner's attention to the highlighted mathematical relationship is called focusing phenomena (Lobato & Burns-Ellis, 2002). Without such refocusing it is possible that many of the conceptual gains noted in David's interviews would either be fewer in number or quality or not present at all. This underscores the importance of the instructor's role in refocusing attention when needed, especially in an environment where greater numbers of students may interact with dynamic visualizations as part of online homework while working alone.

David's struggle during Interview II to give the coordinates of a point on the graph of f' illustrates how scaffolding, instructor probing, and refocusing resulted in the student successfully completing the task. Only after David answers probing questions can he state that the points on the graph of f' all had the form $(x_p, f'\left(x_p\right))$. He is then able to validate his hypothesis using the software and additional scaffolding included in the sketch. This supports Henningsen & Stein's (1997) work on the need and importance of scaffolding during problem-solving tasks and suggests that it may play an equally vital role in DVS exploration as well as work regarding student validation routines (Walter & Barros, 2011).

A balance of both conceptual and procedural knowledge is necessary for student success in calculus (Gray, Loud, & Sokolowski, 2009; White & Mitchelmore, 1996; Hardy, 2009; Lithner, 2004; Szydlik, 2000). David made many statements in each interview suggesting that he either possessed conceptual knowledge relevant to the topic being discussed, or he made statements of a conceptual nature in a procedural manner.

There are instances where David makes a statement showing evidence of conceptual knowledge that is either not replicable or that does not transfer to newly encountered situations. These episodes suggest that, for David, the interactions with DVS are possibly not resulting in the creation of connected schema between concepts. It is possible that his lack of access to DVS for exploration during the Exit Interview also contributed to this. Had DVS been allowed, his responses may have reflected the conceptual knowledge present in earlier interviews. However, it is possible that David's isolated conceptual remarks that were unsupported by procedural knowledge may be statements learned from lecture, lab or the intervention workshops but forgotten due to the lack of connections with which to form schema (Cooley, Baker, & Trigueros, 2003). Possibly for David, working with DVS enabled him to communicate his understanding of concepts, but the absence of the tool, limited his access to connections needed to complete the task in the Exit Interview (Lobato, Rhodehamel, & Hohensee, 2012). David's weak procedural knowledge failed when he was unable to access his understanding of mathematical relationships in the absence of DVS and he could not apply his previous knowledge to the new situation.

David's experience with uncontrollable mental imagery (Aspinwall & Shaw, 2002; Aspinwall, Shaw, & Presmeg, 1997) is important to note as he carried the incorrect notions regarding the graph of $f(x) = e^x$ having a vertical asymptote at $x = 4$ with him throughout the interviews. Though David's previous statements suggest his comfort with exploring using DVS, he chose not to do so when faced with probing questions regarding his observation about the graph. Situations where this occurs should be carefully discussed and addressed to address student thinking and understanding in an effort to address uncontrollable mental imagery and to limit possible misconceptions introduced by DVS.

Conclusions

The case of David, a struggling C student in calculus, raises important issues regarding the use of DVS in calculus learning. DVS accompanied by instructor guidance or embedded scaffolding questions may enhance conceptual gains and limit possible drawbacks in using DVS. Also, static assessments may not accurately reflect understanding for students who use DVS in learning the concepts. We observed that David needed consistent refocusing and additional probing questions from the researcher throughout the interviews. Without the presence of scaffolding or the focusing phenomena, it is unlikely the outcomes regarding conceptual knowledge would be the same. We also observed that, when not offered DVS as a tool, David's assessment results indicate a below average understanding of calculus (his grade of C in the course) and that he may be unable to transfer his knowledge.

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