Student resources pertaining to function and rate of change in differential equations

George Kuster
Virginia Tech

While the importance of student understanding of function and rate of change are themes across the research literature in differential equations, few studies have explicitly focused on how student understanding of these two topics grow and interface with each other while students learn differential equations. Extending the perspective of Knowledge in Pieces (diSessa, 1993) to student learning in differential equations, this research explores the resources relating to function and rate of change that students use to solve differential equations tasks. The findings reported herein are part of a larger study in which multiple students enrolled in differential equations were interviewed periodically throughout the semester. The results culminate with two sets of resources a student used relating to function and rate of change and implications for how these concepts may come together to afford an understanding of differential equations.

Key words: Differential Equations, Function, Rate of change, Resources

Differential equations form the foundation of many topics in mathematics, science, and engineering, such as biological modeling, thermodynamics, electromagnetism and fluid dynamics. Due to its central role, students in these majors are often required to successfully complete a differential equations course before enrolling in more advanced topics. The subjects within differential equations, however, present challenges to students by invoking their understanding of familiar concepts and then building on them in unique and conceptually demanding ways. For instance, Rasmussen (2001) noted that understanding certain aspects of differential equations requires a "fundamental leap" (p. 67) in one's thinking. Considering the importance of differential equations and the recent calls for increasing the number of students majoring in STEM fields (Engage to Excel Report, PCAST, 2012), research focusing on student understanding of differential equations is necessary and valuable.

The concepts of function and rate of change are necessary and important for understanding differential equations (Donovan, 2007; Habre, 2000; Keene, 2007; Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000). These concepts also transcend the subject, present in topics such as existence and uniqueness theorems (Raychaudhuri, 2007), phase planes (Keene, 2007), slope fields, and fundamental sets of solutions (Stephan & Rasmussen, 2002). In this way, function and rate of change are important for understanding what a differential equation is, and have a large impact on student understanding of many mathematical ideas embedded within a differential equations course. Though the importance of these concepts is a theme throughout the research literature, few studies explicitly focus on the role of function and rate of change with regard to student understanding in differential equations. In their review of mathematics education literature, Rasmussen and Wawro (2014) call for research that examines how the ideas of function and rate of change grow and change across a differential equations course. The goal of the research presented in this paper is to characterize how students utilize their notions of function and rate of change in differential equations, how these notions interact with each other, and how these ideas might support students in developing an understanding of differential equations.

Literature Review

The importance of the concept of function is evident in a large portion of the research literature on student learning in differential equations. For instance, Rasmussen (2001) noted students have difficulty conceptualizing solutions as functions in ways that have been documented concerning functions in general. Namely he found that students have difficulties interpreting solutions as functions with graphical representations, interpreting equilibrium solutions as functions, and interpreting the quantities represented by solutions. In her discussion of students' use of time as a parameter, Keene (2007) found that students often reason about solutions in ways that are commensurate with how they reason about functions. Concerning the development a framework for student understanding of the existence and uniqueness theorems, Raychaudhuri (2007) discussed that students' notions of continuity, function, and integration play a significant role in how they interpret and apply the theorems. For example, often times the students believed the coefficients needed to be continuous functions in order for a differential equation to have continuous solutions. A significant challenge for students is developing an understanding of the terms within the differential equation as both variables and functions. Difficult as it may be, however, it has been shown to be immensely important for students' understanding of differential equations (Donovan, 2007; Stephan & Rasmussen, 2002; Whitehead & Rasmussen, 2003), and research has suggested that constructing such an understanding requires putting together images from both concepts to create new ways of reasoning (Whitehead & Rasmussen, 2003).

Student understanding of rate of change has been shown to be connected to the ways in which students reason about various representations of differential equations. For instance, Whitehead and Rasmussen (2003) discuss student use of rate to build images of population, prediction, and function. They noted that students often used rate as a quantity that determined the behavior of a function. With regard to student reasoning with slopes, Stephan and Rasmussen (2002) documented student reasoning with regard to how slopes change over time, slopes of autonomous differential equations being horizontally invariant, and the existence of infinitely many slopes in a slope field. Additionally, it has been suggested that students can reason with rate in ways that promote the construction of new mathematical objects such as straight line solutions (Rasmussen & Blumenfeld, 2007).

Theoretical Perspective

Considering the goals of the research, the interconnected nature of rate of change and function in differential equations, and the way they are utilized to build new mathematical understandings as discussed in the literature, a theoretical perspective that is sensitive to the nature of these concepts is required. The analysis presented here makes use of the epistemological perspective, Knowledge in Pieces (diSessa, 1993; Smith, diSessa & Roschlle, 1993). Within this perspective, knowledge is characterized as a dynamic system of elements and their connections, which is shaped by the learner's interactions with their environment. These elements of knowledge are context specific, in that certain knowledge is associated (to varying degrees) with being useful in certain situations. As such, from the Knowledge in Pieces (KiP) perspective, knowledge elements are either productive or unproductive for accomplishing a certain task within a certain situation. This means the knowledge elements themselves are not evaluated as correct or incorrect (Smith, diSessa & Roschlle, 1993); the evaluation of correctness is only relevant to the application of the elements from an observer's point of view. From a KiP standpoint, learning is characterized as the reorganization, contextualization, and systematization of knowledge elements (diSessa & Sherin, 1998; Wagner, 2006). Studies in which KiP is utilized are often designed for the identification of these elements and the mechanisms that afford their systematization (Adiredja, 2014; Kapon, Ron, Hershkowitz, & Dreyfus, 2015). I contribute to this body of literature by identifying knowledge resources students utilize while completing tasks in differential equations.

There are a multitude of “pieces” used to model a learner’s system of knowledge within the KiP perspective. To complete the analysis, I make use of only a few of the various elements existing in the KiP perspective, namely, *knowledge elements*, *knowledge resources*, and *concept projections*. Knowledge elements refer to any one of the various structures within the larger system of knowledge and, as such, vary in size. While generally consistent in nature, the characterizations of knowledge resources found within the literature have been somewhat varied. Broadly speaking, knowledge resources are sets of one or more small-scale pieces and can take on many forms. In general, however, and consistent with the definition posed by Adiredja (2014) and Hammer (2000), here I take knowledge resources to be ideas consisting of small sets of small-scale knowledge elements with a specific use in a particular context. Given the large number of resources students may utilize in service to a single concept, and the context specificity of those resources, the term concept projection (diSessa, 2004; diSessa & Wagner, 2005; Wagner, 2010) is useful when discussing students’ understanding of a certain concept. Wagner (2010) defines a concept projection as “a set of particular knowledge resources that enables the knower to attend to and interpret the available information necessary to ‘perceive’ or ‘implement’ a concept within a given situation” (p. 450). In general, concept projections provide a way of illuminating the specific ideas about certain concepts that students use when encountering certain tasks. For the purposes of the research presented in this paper, concept projections serve as a way to build on previous findings by identifying how students utilize and build on their ideas about function and rate of change while completing differential equations tasks.

Methods

The research presented here is part of a larger study with the following research questions: What resources concerning function and rate of change do students utilize to complete various differential equations tasks; how do these resources change as the students progress through a differential equations course; and how do students’ resources concerning rate of change and function influence one another during the development of their understanding of differential equations? A total of 8 students participated in five one-on-one, task-based, semi-structured interviews (Clement, 2000), each spaced two to three weeks apart. The primary goal of each interview was to engage the participants in tasks centered on topics relevant to differential equations so as to elicit the students’ knowledge resources concerning function, rate of change, and differential equations. This paper focuses on a single student’s (Dominick) response to one of the tasks (see Figure 1) posed during the second interview and as such briefly addresses the first research question. Dominick was chosen because his responses explicate the interconnected nature of function and rate of change when reasoning about differential equations tasks. The task discussed here was adapted from the inquiry-oriented differential equations curriculum (Rasmussen & Kwon, 2007). The interviews were audio and video recorded and then transcribed. Student generated work was collected as a secondary data source.

Using the transcription, recordings, and student work, resources related to function and rate of change were identified. Recall that resources are small pieces of knowledge that served a productive role in the student’s problem solving activity. In general, evidence of the productivity of a piece of knowledge is that the student indeed used that knowledge to attain a (not necessarily correct) solution. The first step in identifying a knowledge resource is determining the meaning behind various parts of the student’s arguments (Adiredja, 2014). To accomplish this I inquired into each of the actions and statements the student made while completing the task. More specifically, I tried to determine why these actions and statements were important to the student, and what it was from these actions and statements that he utilized to complete the task. For instance, Dominick noted that the $-2xy$ term “describes the interaction between the two [species] and has a negative effect on the initial state of one of them [species x].” At first glance this statement may seem rather straightforward; however, the various phrases, such as “negative effect,” have complex meanings for Dominick. While inquiring into the meaning of the various parts of the argument, I was also analyzing the attributes Dominick was attending to and ideas Dominick coordinated with them. The above statement includes ideas about the sign of the rate of change, how changes in y affect $\frac{dx}{dt}$, two species with a finite food source, and functions. By trying to uncover what Dominick did, the meaning behind his statements and actions, the ideas he employed to draw his conclusions, and why these ideas were important, the resources he utilized could be identified. The last step is to determine which resources Dominick associated with rate of change and function. This entire process is briefly elaborated on in the analysis section that follows.

In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning.

A	**B**
$\frac{dx}{dt} = -5x + 2xy$	$\frac{dx}{dt} = 3x - 2xy$
$\frac{dy}{dt} = -4y + 3xy$	$\frac{dy}{dt} = y - 4xy$

Figure 1: Competing/Cooperative Species Task

Analysis

This was the second time Dominick encountered the task (the first being during his first interview), and he quickly determined System B described competing species and System A described cooperative species. The analysis starts with inquiring into the meaning behind Dominick’s statements and actions. Dominick began to explain how he made the determinations by stating the $-2xy$ term “describes the interaction between the two [species] and has a negative effect on the initial state of one of them [species x].” Specifically he noted that without the interaction, the rate of change of species x by itself would be positive, “but with the competing reaction it takes away from species x.” When asked what it was that indicated a change in the initial state, he replied “so this here [points to $-2xy$], the interaction, is basically the constraint that there is a finite food source and if one species gets so much, then the other species can’t.” He then said that $\frac{dx}{dt}$ is “the rate of change of the population of species x” and that it describes *how* the population changes, while the right hand side of the equation tells you *why* the population is changing. When prompted to elaborate on what he meant, he stated “well, so there is a finite food source and there are only two species pulling from that food source…Species y gets more food than species x, species x will, population will decrease. So, it [the population of species x] will have a negative rate of change.” For Dominick this meant the population of species y is increasing (because it is getting more food) which in turn causes species x to decrease, as indicated by the negative rate of change. This was determined after considering one of his final responses to a question about the relationship between $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in the systems of differential equations. Dominick replied “…if there is a relationship between x and y, then the change in one is definitely going to affect the change in the other, so if the population of x gets exponentially bigger…

that means that x is getting more food… which would cause y to get even smaller and x to get even bigger." Here, for Dominick, "getting more food" means an increase in population.

Identifying the resources Dominick used requires determining what attributes he attended to and how he made use of them. Much of Dominick's argument revolved around the $-2xy$ term. From this we can see that he attended to the $-2xy$ term, interpreting it as "the interaction" between species x and species y. Additionally he attended to $\frac{dx}{dt}$, referring to it as "the rate of change of the population of species x." He concluded that $-2xy$ has a negative effect on the population of species x by coordinating "the change in the initial state" (a decrease in x) with ideas relating negative rate of change to decreases in population. In other words, Dominick considered how "the interaction" affected the value of the rate of change of x and how this in turn affected the value of x. Concluding that the interaction caused the population of species x to decrease, these ideas came together to signify two competing species with a finite food source. It seems that for Dominick the population of species x is decreasing because of the existence of a finite food source from which species y is getting a higher proportion of food. This is the "why it's changing" he made reference to concerning the right hand side of the equation. With these attributes identified, the focus of the discussion shifts toward identifying the resources utilized in the construction of his argument.

To identify the small-scale knowledge elements he coordinated with these attributes to construct his argument, consider his responses to a follow up question that explicitly inquired into how he thought about x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Dominick noted, "Each one of these variables [x and y] could be functions of time, so how much, like the current population at that time. So how these [x and y] change affects the total rate of change." Dominick was then asked to elaborate on what he meant by x and y being functions and variables. He replied "they are functions so they have a graph, but if you plug in a certain, you plug in their independent variable that is going to give you a number, which you would then plug in for the variable." Describing this procedure, he went on to say " if you plug in t_0 into each y and x, you get x_0 and y_0 and you plug that into this differential equation, you are gonna get the rate of change of x at time t_0." Dominick was then asked what he meant by rate of change, which he explained as the "slope of the line tangent to the curve at the point t_0, x_0" and that it describes how the function is behaving: a positive slope means the function is increasing, a negative slope means the function is decreasing and a zero slope means the function "is transitioning."

His descriptions of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are connected to the attributes he attended to, as well as various parts of his argument and their meanings. For instance, the combination of ideas indicating that x and y are variables whose values can be substituted into the differential equation to find the rate of change at certain population values, provide insight into how he was able to determine how increases in y affect x. Namely, based on his description of x and y as variables, he was coordinating the effect of larger and larger y values on $\frac{dx}{dt}$. Additionally, Dominick utilized these ideas as he reasoned about the "competing interaction," the "changes in the initial state," and when making conclusions about the finite food source. For example, when he noted that the competing interaction takes away from species x, he considered how different values of y impact the value of $\frac{dx}{dt}$. Here he was treating y as a variable. In other words, he was treating $\frac{dx}{dt}$ as if it was dependent on y. Both of these are small-scale ideas Dominick utilized to complete the task, in other words, resources.

Results

Dominick utilized many resources associated with function and rate of change as he completed the task. These resources were used in coordination with certain attributes from the task, which served as affordances for him. In other words, the attributes of the task that Dominick perceived and attended to both influenced and were influenced by the knowledge resources Dominick had available to him at that particular time. Some of these attributes were terms within the task such as “interaction,” terms within the differential equation such as $-2xy$, and the problem situation (two competing or cooperating species). Additionally he was attending to and interpreting these attributes in ways that promoted the coordination of multiple resources related to function and rate of change with these various attributes. By analyzing his argument I was able to construct his concept projections for these constructs (see Figure 2 and Figure 3, respectively). It should be noted that there might be additional resources Dominick utilized while performing the task which are not captured within the concept projections. Therefore they should not be thought of as a representation of the totality of Dominick’s thinking during the task.

- The terms x and y are variables that could be functions of time.
- x and y represent the current populations of species x and y respectively, at some time, t.
- As time changes, so does the population/function value.
- Functions have graphs.
- Plug in an independent variable (in this case t) and that gives a number (in this case population).

Figure 2: Dominick’s concept projection of function

- $\frac{dx}{dt}$ is a change, the rate of change of the population of species x.
- The value of $\frac{dx}{dt}$ is dependent on the value of y.
- Population decreases when the rate of change is negative.
- Rate of change is the slope of the line tangent to the curve at a point on the curve.
- Rate of change tells you how the function behaves. (+) implies increasing (-) implies decreasing.
- The function/variable y affects the rate of change of x.

Figure 3: Dominick’s concept projection of rate of change

All of these resources where utilized to enable Dominick to determine which system of equations represented a competing relationship. Namely, these are ideas Dominick saw as useful and productive for completing the task and provided different affordances for him at different times. For instance, the idea that x and y are variables allowed Dominick to consider multiple values for y and what happens when y “gets more food.” However, attending to x and y as functions allowed him to reason about $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In short, depending on how he was attending to x and y at the different times he used resources that allowed him to treat x and y as static quantities in some cases and dynamic quantities in others. This is evident for example in his statements about y increasing or “getting more food,” and y representing the population value at a certain time. These resources, among others, allowed Dominick to treat $\frac{dx}{dt}$ as a quantity that depended on the values of x and y, to treat x and y as quantities that depended on t (and hence make sense of $\frac{dx}{dt}$), to coordinate changes in y with changes in $\frac{dx}{dt}$, and to utilize the sign of $\frac{dx}{dt}$ to determine the behavior of x.

The concept projections found in Figure 2 and Figure 3, directly address the first research goal. To address the interaction between the resources related to function and rate of change, consider the resources in Figure 4. Each of these resources could be categorized as both resources relating to function or resources relating to rate of change. Additionally, each of these resources seem reasonably related to Dominick’s understanding of differential equations. Take for example the resource, “after evaluating $x(t)$ and $y(t)$ at some value t_0, you get x_0 and y_0, this then gets plugged into the DE in place of x and y respectively." This is indicative of an understanding that x and y in the differential equation are both functions and variables, an idea researchers have noted as being key to understanding differential equations (Donovan, 2007; Stephan & Rasmussen, 2002). In other words, Dominick was able to simultaneously coordinate resources relating to function and resources relating to rate of change in ways that afforded him the ability to interpret and implement ideas associated with differential equations to complete the task.

- Changes in the values of the variables x and y affect the rate of change of x.
- After evaluating $x(t)$ and $y(t)$ at some value t_0, you get x_0 and y_0, this then gets plugged into the DE in place of x and y respectively.
- The DE gives you the rate of change at a certain time on $x(t)$.
- The function/variable y affects the rate of change of x.

Figure 4: Resources relating function and rate of change

Conclusions and Implications

The results generated from the analysis of Dominick's reasoning during the task indicate that he utilized many ideas about function and rate of change. Particularly, it is important to note that he was not solely using knowledge strictly pertaining to what a function or rate of change is, he also utilized much knowledge about these concepts as a tool for recognizing and implementing them. For example, the ability to recognize x and y as functions and variables and attend this as a useful piece of information afforded him a productive line of reasoning about "the interaction term." This highlights the importance of knowledge for implementing and utilizing certain concepts with regard to student thinking, and points to the value of including this type of knowledge in the analyses of student learning.

More importantly, the results enlighten why understanding x and y as both variables and functions, and recognizing the differential equation as a function are so important. Dominick's ability to attend to the dependence of $\frac{dx}{dt}$ on y and to coordinate this with the resources related to y being both a variable and a function provided him with powerful ways of reasoning about the systems of equations. Specifically he was able to coordinate changes in the value of x and y with respective changes in the value of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. This formed the basis on which he was able to draw his conclusions.

In light of the importance the research literature places on function and rate of change for understanding differential equations, looking at resources that overlap both concept projections may provide insight into how students construct an understanding of differential equations. The results suggest that supporting the development of students' understanding of differential equations also requires supporting their abilities to attend to relevant attributes and implement mathematical ideas from which differential equations are built.

References

Adiredja, A. (2014). *Leveraging Students' Intuitive Knowledge About the Formal Definition of a Limit* (Unpublished doctoral dissertation). University of California, Berkeley CA.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and instruction*, *10*(2-3), 105-225.

disessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change? *International Journal of Science Education*, *20*(10), 1155-1191.

diSessa, A. A. & Wagner, J. F. (2005). What coordination has to say about transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 121-154). Greenwich, CT: Information Age Publishing.
Donovan, J. (2007). The importance of the concept of function for developing understanding of first-order differential equations in multiple representations. In Hauk, Shandy (ed), *Electronic Proceedings for the Tenth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.* San Diego: San Diego State University. Retrieved from: http://sigmaa.maa.org/rume/crume2007/papers/donovan.pdf
Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior, 18*(4), 455–472.
Hammer, D. (2000). Student resources for learning introductory physics. *American Journal of Physics*, *68*(S1), S52-S59.
Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*(3), 230-246.
Keene, K.A. (2008), Ways of reasoning: Two case studies in an inquiry oriented differential equations classroom. Psychology of Mathematics Education Annual Conference, Mexico: Morelia.
Kapon, S., Ron, G., Hershkowitz, R., & Dreyfus, T. (2014). Perceiving permutations as distinct outcomes: the accommodation of a complex knowledge system. *Educational Studies in Mathematics*, 1-22.
President's Council of Advisors on Science and Technology. (2012). *Engage to excel*. Washington, DC: White House.
Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.
Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, *26*(3), 195-210.
Rasmussen, C. L., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(2), 161-172.
Rasmussen, C., & Wawro, M. (2014). *Post calculus research in undergraduate mathematics education.* Manuscript submitted for publication.
Raychaudhuri, D. (2007). A layer framework to investigate student understanding and application of the existence and uniqueness theorems of differential equations. *International Journal of Mathematical Education in Science and Technology*, *38*(3), 367-381.
Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459-490.
Smith III, J. P., diSessa, A. A., & Roschelle, J. (1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of the learning sciences*, *3*(2), 115-163.
Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, *24*(1), 1-71.
Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *The Journal of the Learning Sciences*, *19*(4), 443-479.
Whitehead, K. & Rasmussen, C. (2003) Undergraduate students' mental operations in systems of differential equations. In Dawson, S., (Ed.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education.* Honolulu, HI