Re-claiming during proof production

David Plaxco, University of Oklahoma

Abstract: In this research, I set out to elucidate the construct of Re-Claiming - a way in which students' conceptual understanding relates to their proof activity. This construct emerged during a broader research project in which I analyzed data from individual interviews with three students from a junior-level Modern Algebra course in order to model the students' understanding of inverse and identity, model their proof activity, and explore connections between the two models. Each stage of analysis consisted of iterative coding, drawing on grounded theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). In order to model conceptual understanding, I draw on the form/function framework (Saxe, et al., 1998). I analyze proof activity using Aberdein's (2006a, 2006b) extension of Toulmin's (1969) model of argumentation. Reflection across these two analyses contributed to the development of the construct of Re-Claiming, which I describe and explore in this article.

Key words: *Mathematical Proof, Conceptual Understanding, Abstract Algebra*

Mathematical proof is an important area of mathematics education research that has gained emphasis over recent decades. The majority of empirical research in proof focuses on individuals' proof production (e.g., Alcock & Inglis, 2008), individuals' understanding of or beliefs about proof (e.g., Harel & Sowder, 1998), and how students develop notions of proof as they progress through higher-level mathematics courses (e.g., Tall & Mejia-Ramos, 2012). Researchers have also generated philosophical discussions that explore the purposes of proof (e.g., Bell 1976; de Villiers, 1990). Much of this latter discussion centers on the explanatory power of proof (e.g., Weber, 2010), with the primary focus being on the techniques and methods involved in a given proof (e.g., Thurston, 1996), rather than the development of concepts or definitions (Lakatos, 1976). Few studies, however, use grounded empirical data to explicitly discuss the relationships between an individual's conceptual understanding and his or her engagement in proof (e.g., Weber, 2005). In this research I set out to explicitly explore the relationships between students' conceptual understanding and proof activity.

Methods and Analytical Frameworks

Data were collected with nine students in a Junior-level introductory Abstract Algebra course, entitled *Modern Algebra.* The course met twice a week, for one hour and fifteen minutes per meeting, over fifteen weeks. The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU) (Larsen, 2013), an inquiry-oriented, RME-based curriculum, relies on Local Instructional Theories that anticipate students' development of conceptual understanding of ideas in group theory. Three individual interviews (forty-five to ninety minutes each) took place at the beginning, middle, and end of the semester, respectively. These interviews were semi-structured (Bernard, 1988) and used a common interview protocol so that each participant was asked the same questions as the others. Unplanned follow-up questions were asked during the interview to probe students' descriptions and assertions. The goal for each interview was to evoke the participants' discussion of inverse and identity and engage them in proof activity that involved inverse and identity. I developed initial protocols for these interviews, which were then discussed and refined with fellow mathematics education researchers.

Each interview began by prompting the student to both generally describe what "inverse" and "identity" meant to them and also to formally define the two mathematical concepts. Additional follow-up questions elicited specific details about what the participant meant by his/her given statements, figures, etc. The interview protocol then engaged each participant in specific mathematical activity aimed to elicit engagement in proof or proof related activity. Participants were asked to prove given statements, conjecture about mathematical relationships, and describe how he or she might prove a given statement. As with the questions about defining, each of these tasks had planned and unplanned follow-up questions so that all participants were asked at least the same base questions, but their reasoning was thoroughly explored. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews, and all participant work and field notes were retained and scanned into a PDF format. I then transcribed all spoken communication during each interview with three of the participants (Violet, Tucker, and John), including thick descriptions of participants' gestures.

The retrospective analysis of the three participants' interview responses consisted of three stages, which I ordered so that each stage built upon the previous stages toward a resolution of the research question. This consisted of an iterative coding process to generate thorough models of the participants' conceptual understanding and engagement in proof and proof-related activity. I carried out this analysis separately for each participant, coordinating each data source chronologically so that the model of each participant's conceptual understanding corresponds with his or her conceptual development over the semester. I then investigated relationships between the participant's conceptual understanding and proof activity, exploring instances in which meaningful interactions between understanding and activity occurred.

Models of individual students' understanding

In this research I operationalize participants' conceptual understanding using Saxe et al.'s constructs of form and function (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde 2005; Saxe et al, 2009). Throughout the literature, forms are defined as cultural representations, gestures, and symbols that are adopted by an individual in order to serve a specific function in goal-directed activity (Saxe & Esmonde, 2005). Three facets constitute a form: a representational vehicle, a representational object, and a correspondence between the representational vehicle and representational object (Saxe & Esmonde, 2005). Saxe focuses on the use of forms to serve specific functions in goal-directed activity as well as shifts in form/function relations and their dynamic connections to goal formation. Through this framework, learning is associated with individuals' adoption of new forms to serve functions in goal-directed activity as well as the development of new goals in social interaction.

The form/function analysis for participants' understanding consisted of iterative analysis similar to Grounded Theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). This analysis is differentiated from Grounded Theory most basically by the fact that the purpose of this specific analysis was not to develop a causal mechanism for changes in the students' conceptual understanding, but rather that it was used to develop a detailed model of students' conceptual understanding at given moments in time. For each interview transcript, I carried out an iteration of open coding targeted towards incidents in which the concepts of inverse and identity were mentioned or used. In this iteration, I focused on the representational vehicles used for the representational objects of identity and inverse and pulled excerpts that afforded insight into the correspondence that the participant was drawing between the representational vehicle and object in the moment. Along with the open codes, I developed rich descriptions of the participants' responses that served as running analytical memos. After the open coding, I carried out a second iteration of axial coding using the constant comparative method, in which open codes were compared with each other and generalized into broader descriptive categories. These categories emerged from the constant comparison of the open codes and were used to organize subsequent focused codes until saturation was reached. Throughout this process, I wrote analytical memos documenting the decisions that I made in forming the focused codes and, in turn, providing an audit trail for the decisions made in the development of the emerging categories. This supports the methodology's reliability (Charmaz, 2006).

Documenting engagement in proof

In order to model the participant's proof activity, I use Aberdein's (2006a) adaptation of Toulmin's (1969) model of argumentation. Several researchers have adopted Toulmin's model of argumentation to document proof (e.g., Fukawa-Connelly, 2013). This analytical tool organizes arguments based on the general structure of claim, warrant, and backing. In this structure, the claim is the general statement about which the individual argues. Data is a general rule or principle that supports the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use backing, which supports the warrant; rebuttal, which accounts for exceptions to the claim; and qualifier, which states the resulting force of the argument (Aberdein, 2006a). This structure is typically organized into a diagram, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

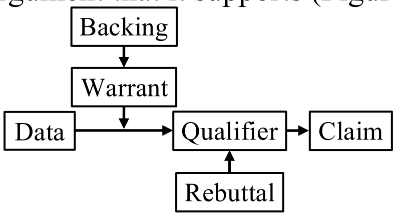

Figure 1. Visual representation of Toulmin models

Aberdein (2006a) provides a thorough discussion of using Toulmin models to organize proofs, including several examples relating the logical structure of an argument to a Toulmin model organizing it. Using "layout" to refer to the graphic organization of a Toulmin model, Aberdein includes a set of rules he to coordinate more complicated mathematical arguments in a process he calls combining layouts: "(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout" (p. 213). The first two rules are relatively straightforward – the first focuses on the treatment of the graphical layout, as for the second, one can imagine including multiple data sources in the same data or claim node. The third and fourth rules provide a structure for combining different layouts and rely on organizational principles that Aberdein uses. He provides examples of combined layouts (Figure 2).

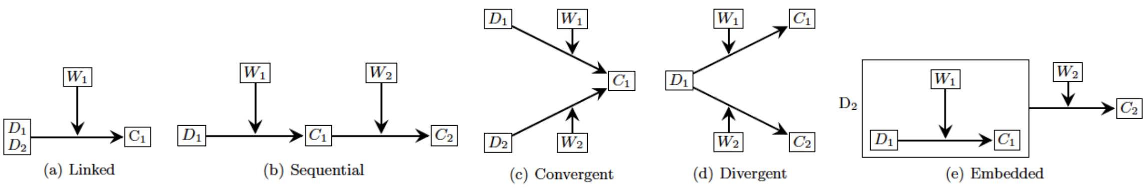

Figure 2. Five Ways of Combining Layouts (Aberdein, 2006a, p.214)

In this second stage of analysis, I first separated statements that conveyed a complete thought, initially focusing on complete sentences and clauses. I then reflected on the intention of each statement, focusing on prepositions and conjunctions that might serve to distinguish the intentions of utterances that comprise the sentence or clause. Following this, I compared these utterances to the model's constructs, focusing on which node an utterance might comprise. I constantly and iteratively compared each utterance relative to the overarching argument in order to parse out how the utterance served the argument in relation to other statements within the proof. For each proof, I then generated a working graphic organizer (i.e., a figure with the various nodes and how they are connected), including corresponding transcription highlighting the structure of the participant's argument. I then iteratively refined the graphical scheme to more closely reflect the structure of the argument as the participant communicated it. After this process, I completed a final iteration in which I compared the scheme to the participant's communication of the proof in its entirety to ensure that the model most accurately reflected the participant's communication of the proof. An expert in the field then compared and checked the developed Toulmin schemes against transcript of the interview in order to challenge my reasoning for the construction of the scheme, supporting the reliability of the constructions of the Toulmin schemes.

Relating conceptual understanding and proof

During the third and final stage of analysis, I focused on the participants' use of forms and functions within nodes of the Toulmin scheme, comparing the roles that specific forms and functions served in various nodes within the argument. I also focused on the shifts in which the participants' generated new, related arguments, specifically attending to concurrent shifts in forms and functions. I compared across arguments, looking for similarities and differences between the forms upon which the participant drew and the functions that the forms serve within the respective arguments. As in the previous stages, the analysis across conceptual understanding and proof centered on an iterative comparison of the patterns emerging across the analyses of the three participants' argumentation. In this comparison, I noted differences and similarities in the overall structures of Toulmin models for arguments. Further, I attended to the aspects of form/function relations that served consistent roles across similar types of extended Toulmin models. I continuously built and refined hypothesized emerging relationships through constant comparative analysis and memos. Through this process, I characterized constructs that unify the patterns found between the roles forms and functions of identity and inverse served across Toulmin schemes for the three participants.

Results

In this section, I discuss data from Tucker's second (midsemester) interview in order to demonstrate a broader construct of Re-Claiming that emerged during the third stage of analysis. I first discuss specific aspects of the form/function model of Tucker's understanding of inverse and identity relevant for discussing a selected part of his response to Question 7 of the protocol, which asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G (Figure 3). Specifically, Tucker's discussion throughout the interviews supported the development of three functions of inverse served by various forms of inverse (in this instance, the "letter" form of inverse): an "end-operating" *function* of inverse in which Tucker operates on the same end of both sides of an equation with a *form* of inverse, a "vanishing" *function* of inverse in which an element and its inverse are described as being operated together and are removed from an algebraic statement, and an "inverse-inverse " *function* of inverse characterized by an element serving a *function* of inverse in relation to its inverse. Throughout his proof activity in this excerpt, Tucker draws on the "letter" form of inverse to serve these functions.

"Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 3. Asking participants to prove about the normalizer of h

During part of his response to this part of the protocol, Tucker reads over his work and says, "I- you know what I might do actually?" (line 1078). He then begins an explanation, but pauses and restarts in order to explain his thinking more clearly, saying,

> So, right now, we have *g* star *h* star *g* inverse is equal to *h*. We want to get to somewhere that looks like- … Want to show. *g* inverse star *h* star *g* is equal to *h*. In

order for the inverse of *g* to satisfy this (points to definition of *H*) right here. Cause that's what you do when you put in the *g* inverse. (lines 1084-1086).

With this excerpt, Tucker begins a subargument (Figure 4) for his broader, overarching proof in which he attempts to show that the set *H* contains inverses of its elements. He begins with the equation used to define *H*, saying, “right now, we have *g* star *h* star *g* inverse is equal to *h*” (line 1085), which serves as initial data (Data1.1) for the argument. He then describes wanting to show that $g^{-1}*h*g = h$, which serves as the claim in the subargument (Claim1). He supports this claim by explaining that this goal means that g^{-1} satisfies the given equation, saying, “Cause that's what you do when you put in the *g* inverse” (line 1087). This warrants the claim by reflecting Tucker’s previous activity in which he replaced *g* in the equation used to define *H* with its inverse and drew on the “inverse-inverse” function of inverse to rewrite the equation ($g^{-1}*h*g = h$). This constitutes a shift in Tucker’s description of what it would mean for the set *H* to contain inverse elements, anticipating a manipulation of the definition of *H* to result in the same equation.

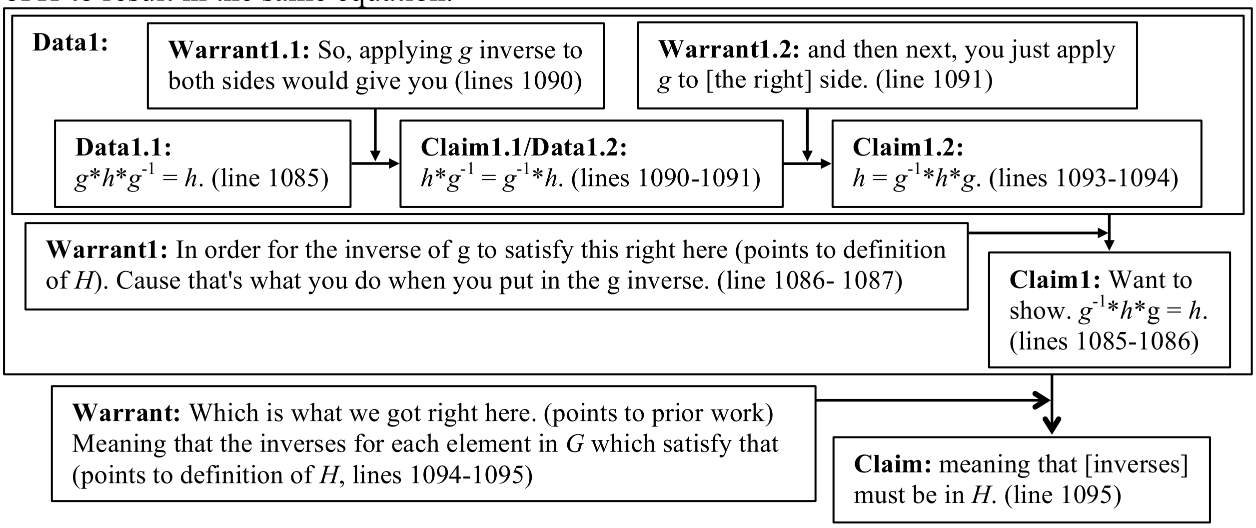

Figure 4. Tucker’s inverse subproof in response to Interview 2, Q7

Tucker then continues, explaining how he might manipulate the first equation so that it looks like the second equation. Tucker begins by left-operating with g^{-1}, saying, “let's apply the g inverse to that. So, applying g inverse to both sides would give you h star g inverse is equal to g inverse star h” (Warrant1.1, lines 1089-1091). This process comprises a warrant that draws on the “end-operating” and the “vanishing” functions of inverse to support the claim that a new equation (Claim1.1/Data1.2) can be produced. This equation then serves as data as Tucker describes right-operating with g to produce the equation $h = g^{-1}*h*g$ (Claim1.2). Similar to the left-operation with g^{-1}, this draws on the “end-operating” and “vanishing” functions of inverse to warrant the new claim. However, this action also subtly draws on the “inverse-inverse” function of inverse in that Tucker is using the element g as the inverse of its own inverse in order to cancel the g^{-1} on the right end of the left-hand side of the equation. Tucker then interprets the result of this activity, saying, “Which is what we got right here. Meaning that the inverses for each element in G which satisfy that (points to definition of H), mean that must be in H” (lines 1093-1095), which comprises a warrant and claim for the overarching argument that H contains the inverses of its elements.

Tucker’s work in this instance exemplifies a broader construct of re-claiming (Figure 5), which I define as the process of reframing an existing claim in a way that affords an individual the ability to draw on a specific form of identity or inverse and the functions that this form might be able to serve. In this study, it was often the case that re-claiming occurred when a participant was asked to prove or disprove a general statement and, in response, interpreted the general statement using a specific form to produce a new claim in terms of this form. An important part of successfully re-claiming is the consistency between the original claim and new claim. The individual must also be able to interpret any possible hypotheses or assumptions of the original claim with respect to the new form upon which they draw. Once the individual generates appropriate initial data from the given hypotheses and assumptions, he or she is then able to draw on the new form to serve specific functions, which affords the development of meaningful argumentation toward the new claim. Finally, after supporting the new claim, the individual should be able to provide a warrant for how or why this claim supports the original claim. More concisely, participants reinterpret a general claim by generating initial data in a specific form based on the original claim (in this case, being a proof by contradiction, they each draw on the "letter" form of identity to necessarily produce data contradictory to the original claim). They then draw on available functions of identity and inverse that this form serves in order to generate new. Finally, each participant interprets this claim to argument that it supports the original conjecture.

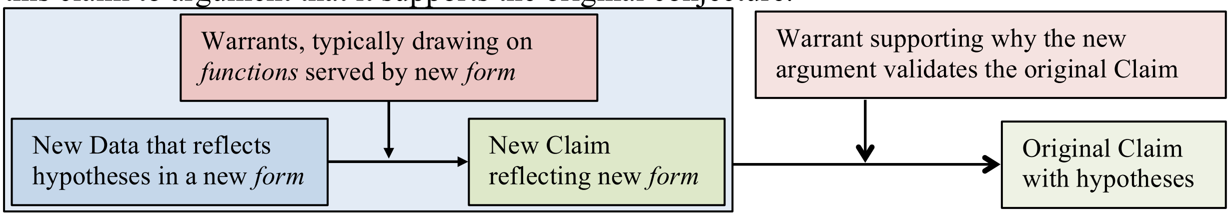

Figure 5. Toulmin scheme reflecting the general structure of re-claiming

A sense of the various facets involved in re-claiming can be drawn from the discussion of Tucker's proof activity. Specifically, in re-claiming, it is not sufficient, to only reframe a claim. Rather, one must likely also reframe its related (often hidden) hypotheses. These aspects of reclaiming reflect the frequently taught proof mantras of "what do I know?" and "what do I want to show?" In this case, Tucker describes needing to show that $g^{-1}*h*g = h$ and begins with the equation $g*h*g^{-1} = h$, which reflects the assumption that g satisfies the definition of H. In the context of the form/function framework, these restated hypotheses serve as initial data (drawing on a specific form of identity or inverse) in a new argument in which the participant is able to draw on the form of identity or inverse with which the data is reframed to serve appropriate functions of identity and inverse in support of the new claim. The individual should then be able to reason that this new argument supports the original claim. In this sense, Re-Claiming provides a type of proof activity in which an individual's conceptual understanding (forms upon which an individual draws and the functions that these forms are able to serve) informs the his or her proof approach. Specifically, the access to a form that is able to serve specific functions affords the individual an opportunity to generate a meaningful argument that he or she would likely not have been able to produce without Re-Claiming the initial statement. This activity is not necessarily an inherent necessity of a given conjecture, but rather depends on the individual's understanding in the moment. This reflects the importance of Balacheff's (1986) call to focus on students' understanding when considering their proof activity.

Conclusions

The current research was constrained by several factors. First, my focus on three students' responses to individual interview protocols limits analysis of the relationships between conceptual understanding and proof activity, warranting further analysis of different participants' conceptual understanding and proof activity. Also, although this analysis was informed by the broader contexts of the classroom environment, the focus on the individual interview setting affords insight into a specific community of proof in which argumentation develops differently than in other communities. For instance, the structure of the interview setting necessitated that participants developed their arguments solely relying on their own understanding in the moment and for the audience of a single interviewer. My early observations of and reflections on the development of argumentation in the classroom and homework groups included the mutual development of argumentation in which participants' argumentation was informed by their interactions. Accordingly, analysis of the classroom and homework group data is warranted.

This research contributes to the field by drawing on the form/function framework to characterize students' conceptual understanding of inverse and identity in Abstract Algebra. This affords insight into the forms upon which students participating in the TAAFU curriculum might draw as well as the various functions that these forms are able serve. The broader research also contributes to the field by providing several examples of how Aberdein's (2006a) extension of Toulmin's (1969) model of argumentation might be used to analyze proofs in an Abstract Algebra context. Further, this research draws attention to an aspect of the relationships between individuals' conceptual understanding and proof activity. These results situate well among the work of contemporary mathematics education researchers. For instance, Zazkis, Weber, and Mejia-Ramos (2014) have developed three constructs that also draw on Toulmin schemes to model students proofs in which the researchers focus on students development of formal arguments from informal arguments. These constructs provide interesting parallels with the three aspects of relationships between conceptual understanding and proof activity developed in the current research. Zazkis, Weber, and Mejia-Ramos (2014) describe the process of rewarranting, in which an individual relies on the warrant of an informal argument to generate a warrant in a more formal argument. However, the current research focuses more on the aspects of conceptual understanding that might inform such activity.

Moving forward from this research, I intend to analyze the data from other participants' individual interviews in order to develop more form and function codes for identity and inverse, affording deeper insight into the various form/function relations students in this class developed. Such analysis should also explore the proof activity of the other participants in the study, which would provide a larger sample of proof activity, in turn affording new and different insights into the relationships between mathematical proof and conceptual understanding. I also intend to analyze the sociomathematical norms and classroom math practices within the classroom. This will afford insight into the sociogenesis and ontogenesis of forms and functions at the classroom and small group levels in order to support and extend the individual analyses – which are focused on microgenesis – in the current research.

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