Analyzing Classroom Developments of Language and Notation for Interpreting Matrices as Linear Transformations

Ruby Quea — Florida State University

Christine Andrews-Larson — Florida State University

As part of a larger study of students reasoning in linear algebra, this research analyzes how students make sense of language and notation introduced by instructors when learning matrices as linear transformations. This paper examines the implementation of an inquiry-oriented instruction that consists of students generating, composing, and inverting matrices in the context of increasing the height and leaning a letter "N" placed on a 2-dimensional Cartesian coordinate system (Wawro et. al., 2012). I analyzed two classroom implementations and noted how instructors introduced and formalized mathematical language and notation in the context of this particular instructional sequence, and then related that to the ways that language and notation were subsequently taken up by students. This work was conducted in order to enable me to build theory about the relationship between student learning and the ways in which language and notation are introduced.

Keywords: inquiry-oriented instruction, linear algebra, linear transformations, language and notation

The topic of matrix transformations is commonly taught in introductory linear algebra classes offered at many community colleges and four-year universities (Carlson, Johnson, Lay & Porter, 1993). The interpretations students need to develop and coordinate in the context of matrix transformations have been detailed in the literature (Larson and Zandieh, 2013). Studies have shown that students experience difficulties understanding functions (e.g. Oehrtman, Carlson, & Thompson, 2008); this work has the potential to inform our understanding of the difficulties students experience in coming to understand matrix transformations. This study has the potential to build theory about the relationship between how instructors introduce language and notation and how students make sense of that in the context of learning about matrix transformations.

Background

In the mathematics education community, researchers continue to find ways to support students' learning of concepts in ways that can be formalized into general definitions and theorems with instructional guidance. A hypothetical learning trajectory (HLT) is a theoretical model that is comprised of three components: learning objectives, a series of learning tasks, and a theorized learning process (Simon, 1995). This paper examines implementation of a particular HLT that consists of getting students to generate, compose, and invert matrices. This work is set in a 2-dimensional, geometric setting where students work to transform a letter "N" into a tall and leaner "N". Intuitively, students are investigating function mappings through a matrix transformation. Students' work in this context draws on ideas of matrix multiplication, noncommutativity of matrices, and invertible mappings.

Research Questions

When and how did instructors introduce and formalize language and notation in the context of this instructional sequence? How is the use of language and notation subsequently taken up by students?

Data Sources and Context

In order to explore these research questions, I analyzed two HLT classroom implementations that took place where students explored how to italicize the letter "N". Data sources include video recordings of two different instructors implementing the instructional sequence at different institutions. The sequence took about three to four class periods, each of which were fifty minutes in length. My focus was on portions of the class in which there was whole class discussion and lecture.

Before students began working on the task sequence, the instructors both provided a review of three ways one can interpret the product of a matrix with a vector: as a linear combination, a system of equations, or a transformation. The third of these interpretations was highlighted and analogized to students' previous work with functions as they begin to work on the three tasks. The first task consists of having the student figure out what matrix transforms a regular "N" to a tall and lean "N." This task is aimed at supporting students to consider how the matrix representation of a transformation can be found by coordinating input vectors with output vectors. The second task requires students to consider the transformations from the previous task in two parts: one that stretches the "N" to make it taller and one that then skews/leans the taller "N" to make it look "italicized". Students must coordinate this two-part transformation in a way that helps them conceptualize the composition of matrix transformations. The third task requires students to undo the italicization of the "N" by two ways: using a single matrix transformation and by using two separate matrix transformations. This is intended to give rise to the concept of invertible matrices, as students were instructed to find a matrix that 'undoes' the original transformation. In other words, students make sense of the definition of the inverse of an invertible square matrix A as that matrix B that "undoes" A so that $AB = I$ and $BA = I$ where I is the identity matrix.

Methods of Analysis

The first phase of my analysis involved developing content logs as I watched videos of both classroom implementations that were recorded. These logs contain detailed descriptions of the interactions between the students and the instructor. This information was organized in a table with time stamps for each key event. The type of interaction was be categorized as whole class discussion, student, group work, and lecture. I made note when language and notation was first introduced during the sessions. From there, I started a timeline of instances that summarizes when terminology and notation were introduced by the instructor and how those were subsequently used by students.

In the next phase of my analysis, I will first identify key terms and notation introduced by the two instructors, and develop categories for ways in which terms and notation were introduced, as well as categories for ways in which terms and notation were taken up by students. I will then trace the development of student thinking across the four days of instruction in each of the units. Finally, I will consider similarities and differences in the themes relating the categories for development of language and notation in the two classes. I will then discuss implications for when and how instructors might introduce definitions in order to bridge the gap between informal and formal mathematical language. I will also provide examples on my poster of the instances mentioned and speculate on patterns within and across these instances in order to address my research questions.

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