Student problem solving in the context of volumes of revolution

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The literature on problem solving indicates that focusing on strategies for specific types of problems may be more beneficial than seeking to determine grand, general problem solving strategies that work across large domains. Given this guideline, we seek to understand and map out different strategies students' used in the specific context of volumes of revolution problems from calculus. Our study demonstrates the complex nature of solving volumes of revolution problems based on the multitude of diverse paths the students in our study took to achieve the desired "epistemic form" of an integral expression for a given volume problem. While the large-grained, overarching strategy for these students did not differ much, the complexity came in how the student carried out each step in their overall strategy.

Key words: problem solving, calculus, volumes of revolution, epistemic games

Introduction

Helping students become proficient in problem solving, or the ability to complete a task where a solution is not immediately known, is an important goal in mathematics education (Lester, 2013; National Research Council, 2001; Schoenfeld, 1992). In the last several decades, researchers have studied problem solving and students' problem solving abilities (Lesh & Zawojewski, 2007; Schoenfeld, 1992), and one critical finding has been that it is difficult to determine general problem solving strategies that are useful for any situation. Researchers argue that coming up with a list of strategies for problem solving is difficult because lists are either too small and cannot account for all situations, or too broad such that students are left without a sufficient guide for which strategies to use (Lesh & Zawojewski, 2007). While it is difficult to find strategies to solve problems generally, some researchers have suggested that it may be more beneficial instead to develop strategies for particular types of problems (Lesh & Zawojewski, 2007; Schoenfeld, 1992).

Calculus problems involving volumes of revolution may provide just such a context, since they are not so broad that it becomes impractical to develop general problem solving strategies, yet are complex in that there is no single set procedure to find a solution and students cannot simply memorize one template to apply to all problems. Students need to choose between using the disk, washer, and shell methods and must choose whether to integrate with respect to *dx* or *dy*. Students must also be able to recall and apply knowledge from a range of different mathematical topics (e.g. integration techniques, geometry, and solving equations). Also, we have found no literature on student understanding of volumes of revolution, meaning that it is an area in need of exploration. Thus, in this paper we examine the strategies students use to solve volumes of revolution problems. In particular, this paper is meant to address the following questions: (1) What strategies did students use when solving volumes of revolution problems? (2) What particular features of the problems guided or focused the students in their problem solving strategies?

Epistemic Forms and Games

In investigating problem solving strategies for volumes of revolution, we employ the lens of epistemic games (Collins & Ferguson, 1993). Epistemic games provide a useful language for describing how students go from a *starting condition*, which in our case is the initial volume of revolution problem, to a desired outcome called an *epistemic form*. An epistemic form consists of "an external structure or representations and the cognitive tools to… interpret that structure" (Redish, 2004, p. 30). Collins and Ferguson provided examples of possible epistemic forms like lists, charts, and diagrams, and Redish added to the list things like an abacus or a graph. In our study, the specific "external structure" that makes up the epistemic form is an integral expression that is set up to match the volume of the particular solid given in a problem.

In order to advance from the starting condition to the epistemic form, one must make *moves*, which in our case consists of actions (mental or physical) that a student performs to achieve the desired form. The overall activity of taking the starting condition, making moves, and arriving at an epistemic form is called an *epistemic game* (Collins & Ferguson, 1993). Recently researchers in physics education have used epistemic games as a means to analyze how students problem solve in physics tasks (Black & Wittmann, 2007; Tuminaro & Redish, 2007), and we extend this to investigate students' problem solving strategies in mathematics contexts as well. Depending on the grain size one uses in analyzing the moves that constitute an epistemic game, the description one provides of a student's epistemic game can vary (Black & Wittmann, 2007). In this study, we examine both larger-grained and smaller-grained games, and we define a *global game* as one that describes the general moves a student makes and a *local game* as one that is played out within each of the moves of a global game.

Methods

For this study, eight students from a second semester calculus course were invited to participate in a one-hour interview regarding volumes of revolution problems. The students came from a course in which the instructor had attended to a conceptual development of the disk and shell methods and had also given examples of some cases where each method would not work, given the techniques available to the class. In order to get a range of participants, students were chosen based on their responses to four quiz problems given in class. Three students (Gabi, Doug, Trevor) got all four quiz problems correct, two students (Sarah, Frank) got three correct, two students (Bryan, Claire) got two correct, and one student (John) got only one correct. In the interview, the students were asked to set up integrals for four standard, textbook-style volumes of revolution problems, where a region bounded by certain curves is rotated around either the *x*- or *y*-*axis* (such as in Stewart, 2012, p. 430-445). The problems indicated in words to take the region bounded by the following curves and rotate it around the specified axis, and to determine the volume of the generated solid: (1) $y=\sqrt{1-x}$, $y=2x$, $y=0$ around *x*-axis; (2) $y=\sqrt{1-x}$, $y=2x$, $y=0$ around *y*-axis; (3) $f(x)=4x-\frac{1}{2}x^4$, $y=0$ around *y*-axis; (4) $f(x)=4x-\frac{1}{2}x^4$, $y=0$ around *x*-axis. Graphs of the functions in each problem were also provided to the students. The students were asked to discuss their thinking aloud and the interviewer asked clarifying questions while the student worked.

The videotaped interviews were first watched to record the steps the individual students took in setting up the integrals. These steps were used to define a set of "moves" for the global epistemic games the students played. Once these moves had been described, the videos were watched again in their entirety to code each student's work according to these moves. These gave rise to patterns in the types of global games the students played. Next, each global move was analyzed to determine the specific local game each student played within each move. The local games were then compared and contrasted across students.

Results

In the following subsections we discuss the global games we observed and then present our findings on the local games played within each global move. We acknowledge here that slight differences occurred among some students, but the following is an attempt to synthesize the main patterns found in the data.

Global games

Six of the eight students played quite similar global games, with two variations (compare Figures 1a and 1b). The main difference seemed to be whether the students began setting up the integral before finding the bounds for the integral, or vice versa. Each move within these global games are explained in the following subsections as we examine each move more closely. Two students played different global games, that we do not portray in Figure 1. When solving shells problems, Gabi played the game in Figure 1a, but when solving disk/washer problems, she skipped the "visualizing slice" and "finding volume of slice" moves, since she seemed to have the related formulas memorized and could use them effectively. The other student, Bryan, rarely got past the visualzing volume move and had difficulty in making further progress.

(a)


(b)

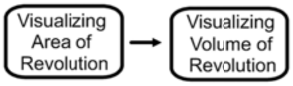

Figure 1. Global games played by (a) Claire, Frank, Gabi, John, Sarah, and (b) Doug, Trevor

Local games

While the majority of students had very similar global games, variations of students' problem solving strategies were more evident in the local games played within each move, especially as seen in the *choosing* game. In the following subsections, we explore each move in more detail.

Visualizing volume

The visualizing move involved an attempt to visualize the three-dimensional object whose volume was being calculated. There seemed to be two types of visualizing local games played by the students. All of the students started this local game by visualizing the area that would be revolved around one of the axes. For most students, this involved determining the curves that bounded the region and shading the region between the curves. All but two students then proceeded to visualize the revolution of this region in some way. The majority of the students drew a reflection of the area on the other side of the axis of rotation and gave descriptions of what the solid would look like. It is interesting to note, however, that the two students (Frank and Trevor) made no attempt to visualize the region's actual revolution. These two students also had more difficulty in setting up the integral expressions than some of the other students. This suggests that visualizing the rotated solid may be a factor that influences students' ability to solve these types of problems.

(a) 

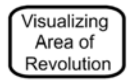
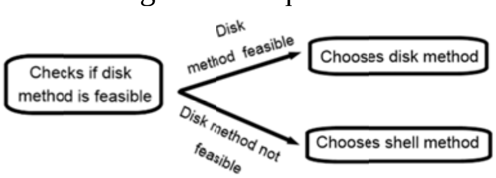
(b) 

Figure 2. Visualizing game for (a) Bryan, Claire, Doug, Gabi, John, Sarah, and (b) Frank, Trevor

Choosing

The greatest variety among the students occurred in the *choosing* move, which involved determining whether to use disks, washers, or shells methods. While many of the students shared similar reasonings for choosing to use one method or another, each student seemed to play their own unique local game during this move. The most common move in the choosing local game was checking the feasibility of the disk/washer or shell methods, or in other words checking whether the resulting integral could *not* be evaluated given the techniques available to the students. Six of the eight students made the move "check feasibility" during the choosing local game, with four of the students' making it their first move (see Figure 3). In fact, Trevor based his entire decision solely on whether the disk/washer method was feasible or not (see Figure 3a). Claire also started off checking feasibility, but if both methods were feasible she would also check the number of integrals each method used (i.e. whether the region had to be split up due to intersection points or other features). In Doug's game, if both methods were feasible, he would look check to see if there was a square root term, and if so, he said he would always choose disks/washers, since squaring the function would eliminate the square root. If there was not he would choose the method that requires the fewest integrals (see figure 3c). Sarah had the most complicated choosing game (see figure 3d). If both methods were feasible she would randomly choose a method and keep going with that method until she ran into a problem. Like Doug, the number of integrals and square root terms seemed to be important in her choosing local game.



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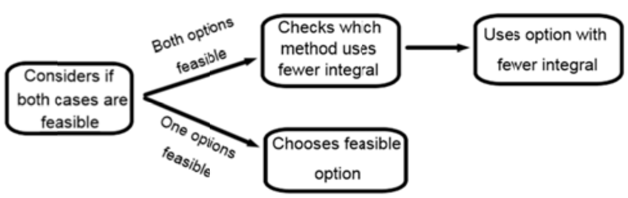

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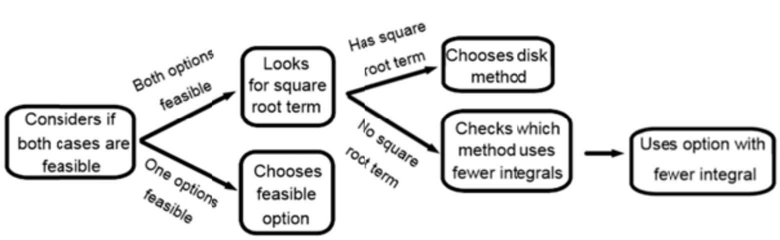

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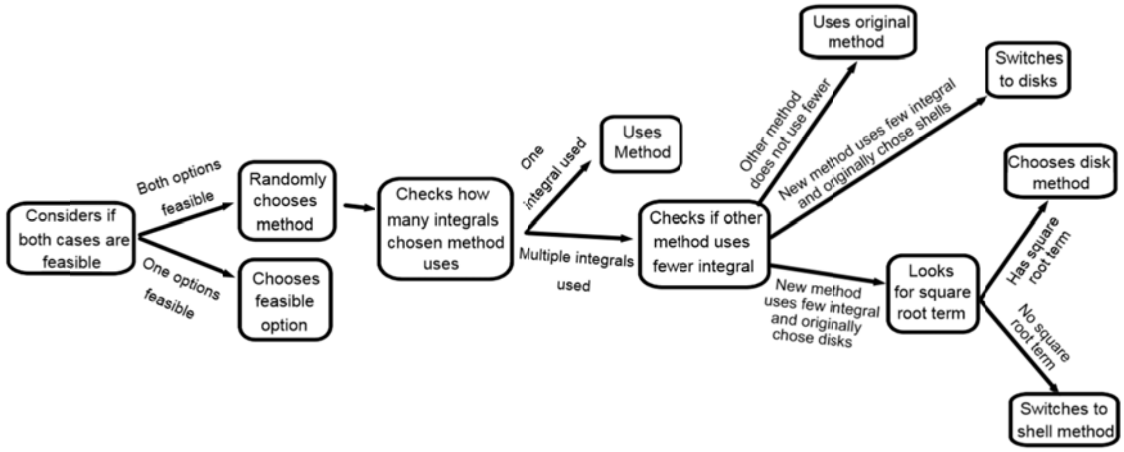
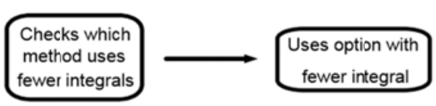


(d)
Figure 3. Choosing local game for (a) Trevor, (b) Claire, (c) Doug, and (d) Sarah

As mentioned in the previous paragraph, the number of integrals required (i.e. whether the region had to be split up) was a defining factor for many students in deciding whether to use the disk/washer or shell method. Five of the eight students made the move “check number of integrals,” with two of them making it their first move within this local game (see Figure 4). In fact, determining “number of integrals” was the *only* move made by Gabi in this local game. If disks/washers or shells resulted in the same number of integrals, Frank would always default to the disk/washer method (see figure 4b), using the shell method only if the disk/washer method was not feasible. Another student who preferred disks/washers to shells was Bryan. In fact, when asked what method he would likely use in solving the problems, he indicated he would *always* choose disks or washers (see Figure 4c) because he was more comfortable working with them.



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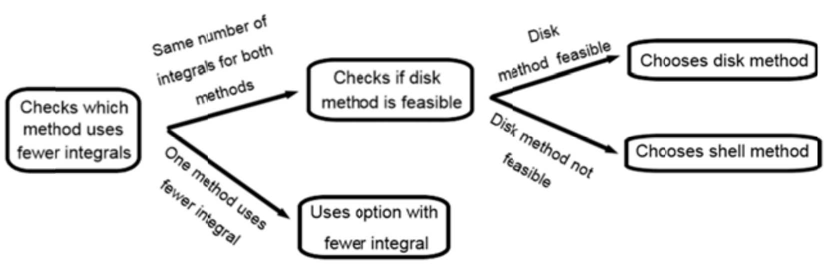

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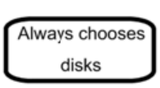


(c)
Figure 4. Choosing local game for (a) Gabi, (b) Frank, and (c) Bryan

John had a rather unique choosing local game (see Figure 5). His initial strategy depended mainly on whether the solid of revolution had a “hole” through the middle of it or not. If the solid did have a “hole” in the middle he would always try to use the shell method first. If not, then he would always try using the disk method first.

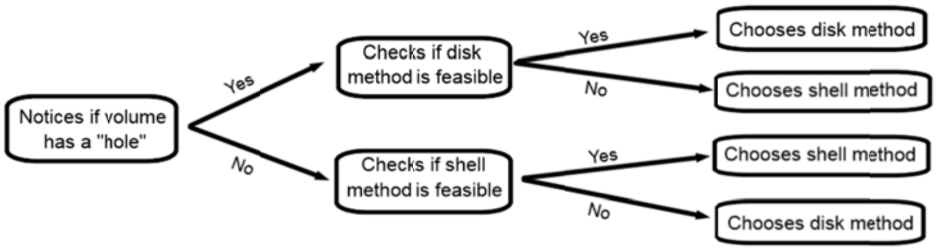
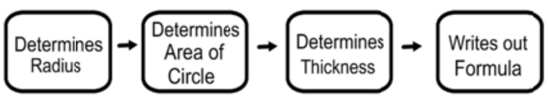
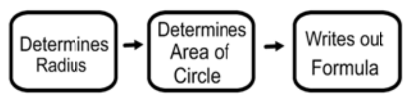


Figure 5. Choosing local game for John

Visualizing and finding the volume of one slice (disk/washer)

In the "finding the volume of one slice" game for disks/washers, which consisted of determining the integrand and differential to use for the integral, several students seemed to take the same general approach, though some students skipped some of the steps that others explicitly took. Sarah and Doug explicitly found all of the parameters needed for obtaining the volume of the disk, including the radius, the area of the disk's face, and the thickness/differential. They then put all this information together in a volume formula for one disk. Trevor also focused on finding different parameters involved in find the volume of the disk, but never explicitly attended to finding the thickness/differential of the disk, only adding the differential in later when he began to write the formula. Four of the students did not explicitly attempt to find any of the parameters, but would begin directly applying a memorized formula for the volume of disks or washer. These are summarized in Figure 6.

(a) 

(b) 

(c) 

Figure 6. Finding volume of one slice local game for (a) Doug, Sarah, (b) Trevor, and (c) Claire, Frank, Gabi, John

Visualizing and finding the volume of one slice (shells)

The students all followed essentially the same basic local game when determining the integrand and differential for the shell method (see Figure 7). They would draw a generic shell and then draw a rectangular prism that represented a shell being cut and flattened, as had been shown in class during the conceptual development of the shell method. Students would then find the radius, circumference, height, and thickness/differential (although the ordering of these differed from student to student) and put these together to write down the integrand and differential. The fact that students always seemed to explicitly find the different parameters associated with shells, while being able to skip steps or use the formulas for disks, leads us to conclude that students were generally more familiar or comfortable with the disk method.

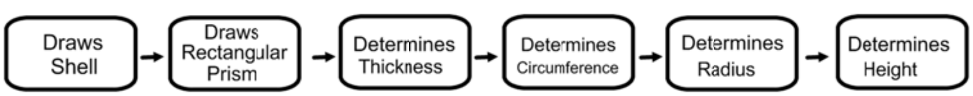


Figure 7. Finding volume of one shell local game played by most students

Setting up the integral and finding bounds

The setting up the integral and the finding bounds local games, which we discuss together due to space limitations, consisted of writing out the actual integral expression for the volume of revolution problem, with some moderate variation existing in how the students played this local game (see Figure 8). Four students first found all the information they needed to set up the integrand, began to write the integral expression, and then would return to the problem to find limits of integration. Sarah, however, began setting up an integral *before* determining any of the parameters she would needed and would then fill in the integral exprsesion as she found more information. Doug and Trevor on the other hand waited until *all* necessary information had been found before they began working on writing out the integral expression.

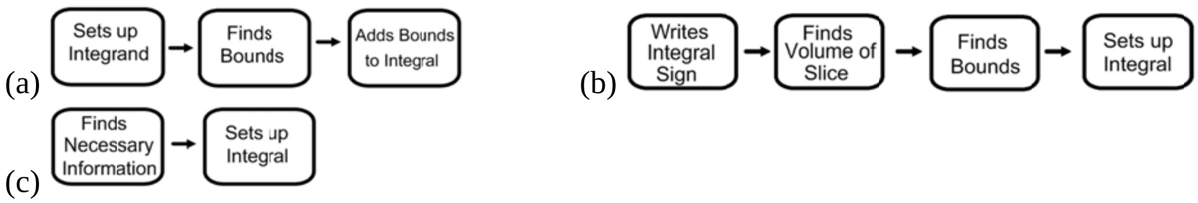


Figure 8. Setting up the integral local game played by (a) Claire, Gabi, Frank, John, (b) Sarah, and (c) Doug, Trevor

Discussion

In this study, we have analyzed and mapped out different epistemic games that students played in the volumes of revolution problems. We identified both global games for the entire problem solving process, as well as the local games that made up the individual moves of the global games. In this way, we have attended to both the larger-grained and smaller-grained strategies enacted by the students in this study as they solved volumes of revolution problems. The most striking outcome of this analysis is how it complexifies, to us, the nature of solving volumes of revolution problems. What could easily be viewed as a problem solving context with only six particular outcomes (one each for associating the disk, washer, and shell methods to a choice of *dx* versus *dy* integration), has been shown to have a much richer variety. The students in this study employed a range of moves to create the desired epistemic form of a completed integral expression. Especially in the *choosing* local game, we were surprised by the amount of diversity and complexity in how the students made moves toward the epistemic form.

Interestingly, however, this range did not seem to occur at the global level, though we acknowledge that this may be due to the common classroom experience for the eight students. Our study found that six of the the students played global games with essentially the same moves and with essentially the same move ordering. The only real difference was whether students would find all necessary information to set up the integral first, or whether they would begin to set up the integral and then go back to the problem to find missing information. We note that this type of overall global game resembles the *pictoral anaylsis* epistemic game described by Tuminaro and Reddish (2007) in physics problem solving contexts. As such, it is possible that learning to solve volumes of revolution problems could have potential effects on some aspects of students' physics problem solving.

We also conclude that solving volumes of revolution problems may be at the right level of complexity according to the problem solving literature, which suggests that contexts that are neither too broad nor too narrow may provide the best setting for developing problem solving strategies (Lesh & Zawojewski, 2007). The volumes of revolution problem context was clearly shown in this paper to be anything but trivial, but is also confined to a more narrow domain, so that strategies *can be* (and were!) developed by students. No student had a memorized template for all volume problems, meaning that they all engaged in problem solving at some level.

In regards to being able to solve volumes of revolution problems specifically, our data suggests that some moves may be especially useful, including visualizing the volumes of revolution, being equally comfortable with all available methods, checking to see which methods are feasible, determining which method uses fewer integrals, and examining which methods involve simpler algebra. Consequently it may be useful for calculus instructors to spend time developing some of these local games. For instance, an instructor may wish to have their students draw out solids of revolution so that they can become comfortable visualizing the desired object, or to have students examine the number of integrals required to work out a problem with respect to *dx* versus *dy*. While this list certainly does not contain all useful strategies in solving volumes of revolution problems, we believe that understanding which moves are useful can help students develop flexible problem solving strategies regarding volumes of revolution.

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