PHYSICS STUDENTS' CONSTRUCTION AND USE OF DIFFERENTIAL ELEMENTS IN MULTIVARIABLE COORDINATE SYSTEMS

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As part of an effort to examine students' understanding of non-Cartesian coordinate systems when using vector calculus in the physics topics of electricity and magnetism, we interviewed four pairs of students. In one task, developed to force them to be explicit about the components of specific coordinate systems, students construct differential length and volume elements for an unconventional spherical coordinate system. While all pairs eventually arrived at the correct elements, some unsuccessfully attempted to reason through spherical or Cartesian coordinates, but recognized the error when checking their work. This suggests students' difficulty with differential elements comes from an incomplete understanding of the systems.

Key words: Coordinate Systems, Differential Elements, Physics, Vector Calculus

Introduction

Various physics education researchers have explored student difficulties with the mathematics applied in Electricity and Magnetism (E&M). These studies have assessed student understanding of integration and differentials (Doughty *et al.,* 2014; Hu & Rebello, 2013; Nguyen & Rebello, 2011); have identified difficulties in applying Gauss's and Ampère's Laws, two integral components of E&M courses that involve a surface integral and line integral, respectively (Guisasola, 2008; Manogue, 2006; Pepper, 2012); and have addressed calculation, understanding, and application of gradient, divergence, and curl in both mathematics and physics settings (Astolfi & Baily, 2014; Bollen, 2015).

A key factor in the application of these mathematical concepts and operations in E&M is a working understanding of the spherical and cylindrical coordinate systems appropriate for the symmetry of most physical situations. In order to solve problems, students are expected to use differential line, area, and volume elements, as well as position vectors that describe the locations of charges distributed over volumes, surfaces, and lines, in order to set up appropriate integrals. A further complication is that the differential line and area elements are vector quantities and thus have a specific direction, while the volume elements are scalar. Given the importance of these differential elements – in different coordinate systems – to the calculations, the main research questions of this study are:

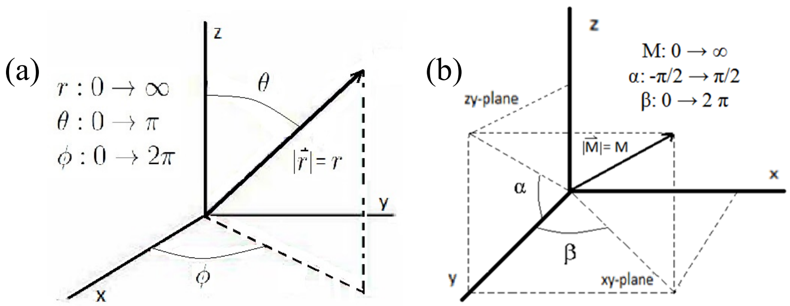
- How do students make sense of and work with coordinate systems, specifically cylindrical and spherical coordinates?
- How do students construct differential vector elements within a given coordinate system?

While disciplinary conventions (e.g., ϕ and θ angle labels switch from math to physics) can be an obstruction to student understanding early in the course (Dray and Manogue, 2003; 2004), even when these are addressed, students have difficulty constructing these differential elements.

Methods

Clinical think-aloud interviews were conducted with pairs of students (N=8) at the end of the first semester of a year-long, junior-level E&M sequence. Pair interviews allowed for a more authentic interaction and sharing of ideas between students with minimal influence from the interviewer. This report focuses on a task in which students were given an unconventional spherical coordinate system. Students were asked to conclude whether the system was feasible, and to build and verify the differential line and volume elements. As students work through these tasks, we are able to see how they reason about the differential elements in a specific coordinate system, thus giving insight into the choice and use of these elements in their problem solving.

Our initial analysis has identified student specific difficulties (Heron, 2003) and successes. We are currently connecting these to aspects of student *concept images* (Tall & Vinner, 1981) of the differential elements and of the non-Cartesian coordinate systems. Similar analysis has been done for student difficulties with divergence and curl in electrodynamics (Bollen *et al.,* 2015).



(c)

$$\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta} + rsin\theta d\phi\hat{\phi}$$
$$d\tau = r^2 sin\theta dr d\theta d\phi$$

(d)

$$\overrightarrow{dl} = dM\hat{M} + Md\alpha\hat{\alpha} + Mcos\alpha d\beta\hat{\beta}$$
$$d\tau = M^2 cos\alpha dM d\alpha d\beta$$

Figure 1: (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

Results

Results shed a unique light on how students build differential elements within a coordinate system. None of the interview pairs determined the correct elements at first, often attempting, incorrectly, to map from a conventional coordinate system rather than constructing the necessary differential elements geometrically. However, all of the pairs correctly attempted to verify the volume element with a spherical integral, which is when they recognized any error(s) in their differential elements.

Some pairs attempted to recall how to decompose the vector **M** into its Cartesian components. Two pairs, trying to map directly to a spherical system, incorrectly included a *sin*(*α)* in the *β* component of their differential length rather than the appropriate *cos(α)*; this is reminiscent of "*x,y* syndrome" (White & Mitchelmore, 1996), wherein students remember expressions in terms of symbols used rather than in terms of the concept. Another pair had no trigonometric function in their components.

Regardless of the elements determined, all pairs attempted integration to obtain the spherical volume formula. All pairs eventually realized the need for *cos(α)* because of the projection into the *xy*-plane. In some cases the cosine term arose in an attempt to obtain the correct formula by integration, while in other cases the need for the vector to project into the *xy* plane was recognized first, and the cosine term was inserted or substituted into the differential.

Our results suggest students do not have a robust understanding of how to build differential elements, but are able to check the validity of these elements and adjust terms appropriately.

This work is preliminary; subsequent data interpretation will use perspectives that have been productive in describing student understanding of mathematics in physics contexts, including *layers* (Zandieh, 2000; Roundy *et al.*, 2015) and *symbolic forms* (Sherin, 2000; Jones 2013). Additional plans are to develop instructional resources that improve student understanding of the construction of differential elements in multivariable coordinate systems in physical contexts.

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