Prototype images of the definite integral

Steven R. Jones
Brigham Young University

Research on student understanding of definite integrals has revealed an apparent preference among students for graphical representations of the definite integral. Since graphical representations can potentially be both beneficial and problematic, it is important to understand the kinds of graphical images students use in thinking about definite integrals. This report uses the construct of "prototype" to investigate how a large sample of students depicted definite integrals through the graphical representation. A clear "prototype" group of images appeared in the data, as well as related "almost prototype" image groups.

Key words: calculus, definite integral, graphical representation, prototype

Mathematical representations are an essential part of doing mathematics and have been studied extensively by mathematics education researchers (e.g., Cuoco & Curcio, 2001). In particular, much attention in the mathematics education literature has been given to the *graphical* representation (e.g., Romberg, Fennema, & Carpenter, 1993). While some have advocated for increased visualization in the teaching and learning of mathematics (Cunningham, 1991; Eisenberg & Dreyfus, 1991), others have also warned about an overreliance on the graphical representation, which may lead to "uncontrollable imagery" (Aspinwall, Shaw, & Presmeg, 1997). Since graphical images can potentially be either beneficial or problematic, I believe it is crucial to understand how students make use of such representations for mathematical concepts.

This paper is intended to examine the graphical representation in the context of the calculus concept of the definite integral. Recently, several researchers have begun analyzing how students understand and conceptualize the definite integral (e.g., Jones, 2013; Kouropatov & Dreyfus, 2013; Sealey, 2014). From this research has emerged the conclusion that students tend to rely heavily on the graphical "area under a curve" interpretation of the definite integral over other potential interpretations (Jones, 2015). While there is certainly nothing wrong with graphical interpretation of the definite integral, it is important that calculus educators have an understanding of the types of graphical images that prevail in student thinking, and that we examine possibly inadvertent predilections instructors may have in presenting graphical images of definite integrals. In particular, this paper is meant to address the questions: (a) What graphical images do students and instructors tend to use to depict definite integrals? (b) Are there certain features that are common to these graphical images? Answers to these questions may help us begin to understand how certain prevalent images might help or hinder student thinking regarding definite integrals.

Prototype images and social construction

This paper uses the notion of "prototypes" (Rosch, 1973), which is built on the idea that certain categories seem to have a hierarchical nature to their membership. For example, in the category "bird," people often think of robins as better examples of "bird" than chickens, which are themselves better examples than penguins, even though the people understand that all three meet the standard scientific definition for "bird" (Lakoff, 1973). Rosch later clarified that there is not necessarily a cognitive object that *is* the prototype (Rosch, 1978), but that "prototype" represents a sort of judgment of "best fit" for possible members of a category.

Prototypes are a useful lens for this study, since its purpose is to identify commonly-used graphical depictions of the definite integral among many possible depictions. That is, one could think of a range of images that could portray the concept of "definite integral," and I am interested in documenting certain types of graphical images that students seem to use as "best depicters" or "default depicters" for the concept of the definite integral. Inherent in this is the notion that students would not necessarily believe that other images are *not* included in the idea of "definite integral," but that certain images may represent it more naturally.

While prototypes have been studied for individuals, it is clear that there is an across-individual theme to the research as well. That is, prototypes seem to extend to larger groups of people beyond individuals. In certain cases, such as prototypes for focal colors (Kay & McDaniel, 1978), there is a biological justification (the photo-receptors in the human eye), but for others, such as prototypes of birds, there does not seem to be a biological basis. In these cases, I claim that prototypes are social constructions (Ernest, 1994) in that within a community of people a certain sense of an "ideal" may emerge for a given category of objects, which becomes the dominant *shared* ideal for that particular category. Again, this is not to say that the ideal actually *exists*, but rather that judgments on prototypicality become uniform and homogenized among the community.

The socially constructed aspect of prototype is central to this paper. As such, this investigation is not focused on individual students per se, but rather on socially shared prototypical graphical representations of the definite integral. What individual students think about definite integrals is, of course, important to this study, since individuals of students, instructors, and others make up the school mathematics community. Yet my analysis is centered more on similarities that range *across* students regarding graphical representations of definite integrals that are perpetuated through the community.

Origins of the data

This paper is an outgrowth of a series of studies regarding definite integrals (Jones, 2013, 2015, *under review;* Jones & Dorko, 2015). It is important to note that none of the studies was originally intended to produce this particular report as an outcome, and each was rather centered on trying to explore how students understand and make sense of definite integrals, or how instructors teach integration. Through the process of conducting these other studies, however, a clear and unmistakable trend began to take shape in the data. In this way, this paper admittedly represents an *a posteriori* investigation into how students from this series of studies graphically represented the definite integral.

The set of data initially used for this paper consists of interview sessions with 23 students and surveys administered to 205 students at two higher education institutions with a wide range of backgrounds and classrooms experiences. However, 67 of the surveyed students did not provide a graphical image in their responses (despite many stating "area under the curve" in words), and these 67 students were consequently removed from the data set since the study was only focused on the types of graphical images produced by the students. This left 23 interviewed and 138 surveyed students. The data set also included videotaped classroom observations from seven different instructors at these same two institutions. Since the interviews and surveys were not all done with the same purpose in mind, and therefore do not consist of exactly the same set of prompts and questions, I focused only on the parts of the overall data set that were generated from open-ended prompts in which students were asked to explain what definite integrals meant, how they understood definite integrals, or how they would describe definite integrals to others. Placing such constraints on the data is an attempt to capture in this paper how students naturally depicted definite integrals through the graphical representation. The following list provides examples of the types of prompts from the interviews and surveys that were used in the analysis for the present paper.

- Consider the expression $\int_a^b f(x)\,dx$. What does it mean? What does it represent?
- Let's say you had a friend in your calculus class who had been sick for the last week or so and missed *everything* your class learned about integrals. How would you explain integrals to them? What would you say an integral means?
- Explain in detail what $\int_a^b f(x)\,dx$ means. If you think of more than one way to describe it, please describe it in multiple ways. Please use words, or draw pictures, or write formulas, or anything else you want to explain what it means.

Analysis of the student and instructor data

The first step in analyzing the data for this study was to simply recognize many similarities between graphical representations of the definite integral used among the 161 students, as I analyzed the data for other purposes. Once these similarities were recognized (details are provided in the results section), the images were organized according to similarity. That is, images that were very similar to each other were grouped together. This organization resulted in a "web" of image clusters, since, for example, one group might differ from another in one characteristic, but then also differ from a third group through a separate characteristic. Once this organizational web had been created, the frequency of the different groups was tabulated, which led to the uncovering of one particular image group as clearly the most common. This image group was also positioned in something approximating the "center" of the web. Accordingly, I labelled this group the "prototype" group and identified what I considered to be seven key characteristics shared by the images in the group. These characteristics were then compared to the images in surrounding "similar" groups. Since there were so many other groups of images that were so close in nature to the prototype group, I decided to create a secondary label, "almost prototype." I defined an "almost prototype" as an image group that contained all but one of the seven characteristics.

Once the interview and survey data had been analyzed, I turned my attention to the videotaped classroom observations I had from seven instructors at two higher education institutions. All of these instructors had had their first two hour-long introductory lesson on integration observed, with some having had additional observations as well. With the "prototype" characteristics and "almost prototype" characteristics defined through the student data, I watched the lesson videos to identify any images that the instructors created in the classroom that matched either definition.

Results: Student data

In this section, I first display images from the student data to show examples that were included in the "prototype" group (see Figure 1). I then use these example images to highlight the seven characteristics I identified regarding the prototype image. Note that, as discussed previously, I wish to avoid the false conclusion that there exists one, single "prototype image," in the same way that Rosch (1978) clarified that "prototype" does not mean that an actual cognitive object exists that *is* the prototype. This can be seen in Figure 1, since the images are certainly not identical to each other. However, the shared features of the graphical images produced by a significant portion of the students indicates that there is clearly some sense of “best fit” features for graphically depicting the definite integral, which represents “the prototype.”

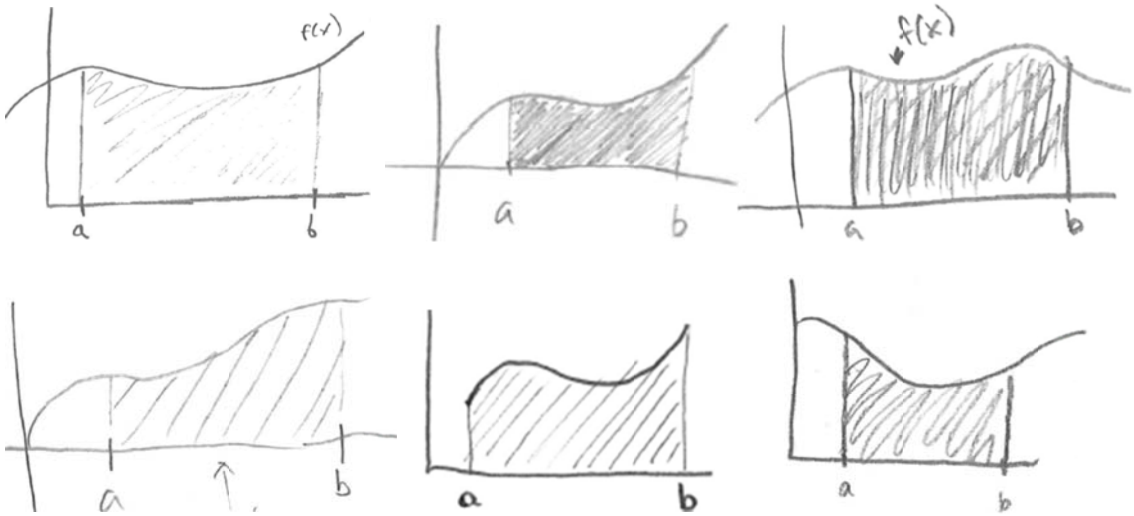

Figure 1. Typical examples of images in the “prototype” group

While artistically distinct, the images shown in Figure 1 have strikingly similar features, especially given the range of students involved in the various studies. In particular, I identified seven major characteristics the images in this group shared, as follows:

1. Neither *x* nor *f*(*x*) are negative over the interval [*a,b*]
2. The graph does not touch the *x*-axis nor *y*-axis over the interval [*a,b*]
3. Vertical lines (either solid or dashed) extend vertically from *a* and *b* to the graph to form the left and right sides of a region
4. The graph is “wavy” in that it contains at least one change in concavity over [*a.b*]
5. There are no apparent “steep” slopes for the graph over [*a,b*]
6. The graph seems smooth in that it visually appears differential at all points in [*a,b*]
7. The graph over [*a,b*] does not deviate significantly from the average function value over [*a,b*]

A couple of these characteristics merit clarification. First, I initially had lumped the first and second features together as “*x* and *f*(*x*) are non-negative.” However, in light of certain related image groups (compare Figure 2b and Figure 3, for example), it seemed that having a graph simply *touch* the *x*- or *y*-axis had a notably different qualitative feel than having an interval that extended to the left of the *y*-axis or a graph that went below the *x*-axis. As a result, it seemed necessary to split these into separate characteristics. Second, the feature of having no “steep” slopes is subject to visual interpretation, since it is not possible to accurately calculate the slope of a hand-drawn graph at every point. To make this judgment, I approximated what the greatest slope might be if an evenly-scaled coordinate system was placed directly on top of the drawing. For all the images in the prototype image group, there appeared to be no slopes (positive or negative) having magnitude greater than two. That said, the vast majority of these images even appeared to have slopes of magnitude no greater than one. Third, it is also difficult to precisely identify what counts as “deviating significantly” from the average function value. In this paper, however, I only attempt to highlight this particular characteristic and do not attempt to define a precise quantitative measure to provide a cut-off for “significant deviation.” For an example of an image that violates this feature, see Figure 2d. Finally, I note that for convenience I have defined these characteristics in terms of the symbols x, $f(x)$, a, and b to match the expression $\int_a^b f(x)dx$, but that these symbols could easily be switched for different symbol, such as t, $g(t)$, t_1, and t_2, for example.

Having discussed the features present in the images in the prototype group, I return to the idea of “almost prototype,” which was defined in the previous section as an image having all but one of the seven characteristics listed for the prototype. Figure 2 shows examples of images from four different “almost prototype” image groups. I note that by far the two most common almost prototype groups were those in which the graph contained no inflection point (Figure 2a) and those in which the graph touched one of the axes, but still contained no negative values for x or $f(x)$ over $[a,b]$ (Figure 2b).

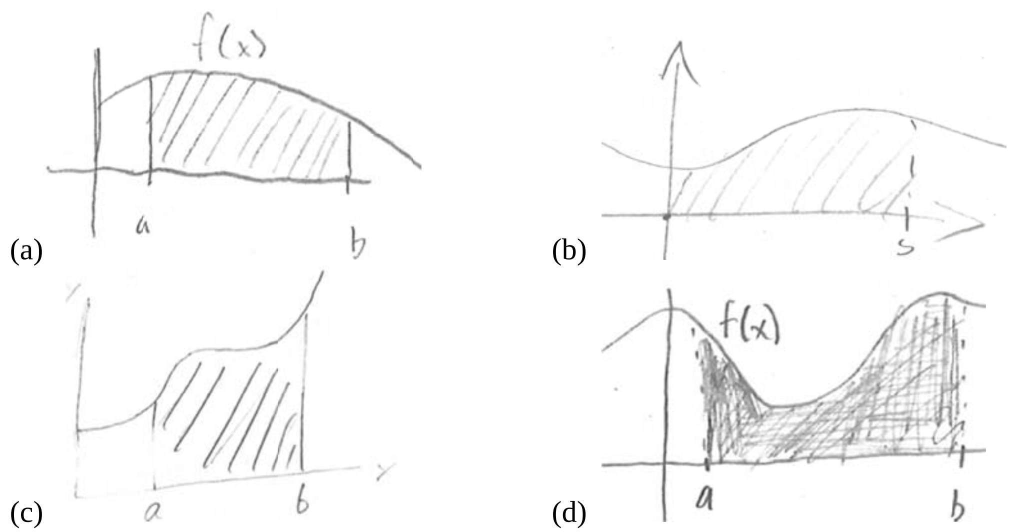

Figure 2. Example “almost prototype” images, including (a) no inflection point, (b) touches axis, (c) steep slopes, and (d) significantly deviates from average

As defined, any image that violated two or more of seven characteristics were excluded from being an “almost prototype.” For example, a continuous graph over an interval $a < 0 < b$ (in violation of #1 and #2) was excluded from being an “almost prototype.” However, many of these types of images were still quite *related* to the prototype group, but were not counted as being in either the “prototype” or “almost” groups. Figure 3 shows examples of these kinds of images.

Figure 3. Example images NOT counted as “prototype” nor “almost prototype” for violating at least two characteristics, but that are still related to the prototype group

Having discussed the prototype group and the almost prototype image group, I now provide overall summary percentages from the student interview and survey data. Note that in

Table 1, each student is counted *only once*, regardless of how many images they produced in the interview or survey. A student is counted in “produced prototype” if they created an image satisfying all seven characteristics, regardless of whether they also drew “almost” images or other images. Conversely, a student is *only* counted in “almost prototype” if they did not create an image that fit in the prototype group. The “neither” group consists of students who produced at least one image, but none of which fit into the “prototype” or “almost prototype” groups.

Table 1
Frequencies from the student interview and survey data

	Interviewed students (n=23)	Surveyed students (n=138)
Produced prototype image	18 (78.3%)	69 (50.0%)
No prototype, but produced “almost prototype” image	4 (17.4%)	42 (30.4%)
Produced images, but neither “prototype” nor “almost”	1 (4.3%)	27 (19.6%)

Results: Instructor data

We can see in the previous section that a large percentage of students produced prototype or almost prototype images. All but one of the interviewed students produced either a prototype or an almost prototype image and about 80% of the surveyed students produced one of these kinds of images. As such, the characteristics that define the prototypicality of definite integral images seem to be shared in the community of calculus students. The next obvious question is: from where do students adopt this shared sense of prototypicality?

To provide a partial answer, I relate the results from the seven observed calculus instructors at two institutions of higher education. Of the seven instructors, five of them produced exact matches for the prototype group that emerged from the student data. Examples of these images are shown in Figure 4.

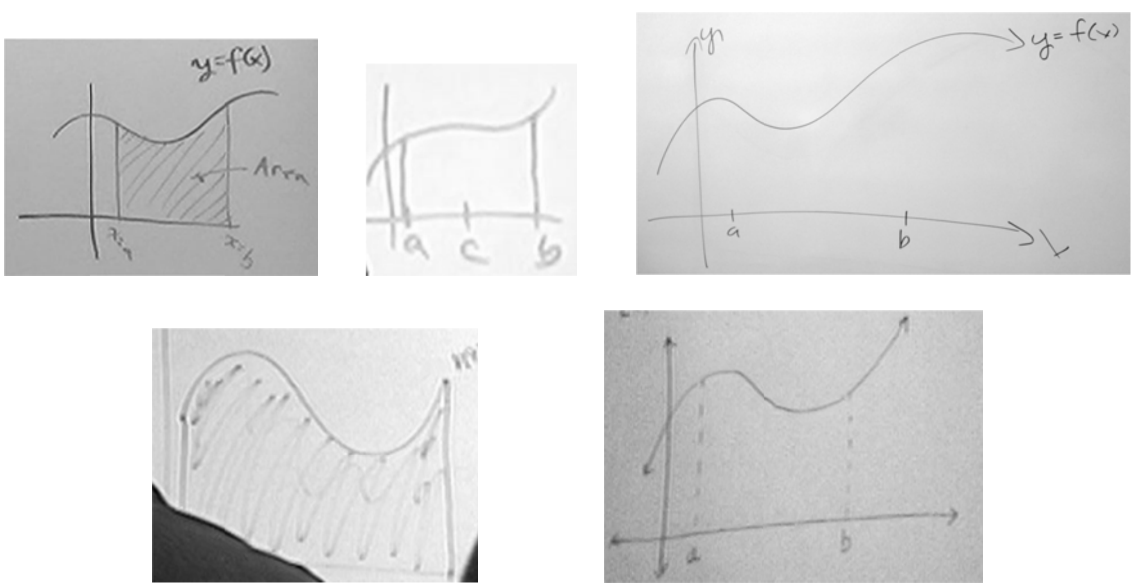

Figure 4. Examples of images produced by five of the seven instructors

These images are directly related to the student-produced images seen in Figure 1. Note that one of the images does not contain vertical lines at *a* and *b*, but the instructor did use his hands to indicate vertical lines at these two points. These results suggest that students may be inducted into the usage of these kinds of "best representational fit" images from their calculus instructors. While the underlying idea of teachers inducting their students into a shared practice is obviously neither new nor revelatory, this portion of the data does reveal, though, that it is more than students who share a sense of "prototypicality" of graphical representations of definite integrals. It seems to be shared by instructors as well. As such, it appears deeply embedded in the calculus education culture.

Discussion

As discussed in the beginning of this paper, graphical representations play an important role in mathematics education, and it is consequently important to understand how graphical images are used by students. In this study I have presented a set of seven characteristics that define a measure of "prototypicality" for graphical representations of the definite integral that seems pervasive in calculus education. Given that this paper, together with past studies (Jones, 2015), suggest this type of image may be a "default" image for students (and instructors), and since graphical images can potentially override other forms of representation (Aspinwall et al., 1997), it is important to understand the ways in which this particular type of image may benefit or hinder student thinking in relation to definite integrals. The results of this study provide some initial insight into possible benefits and hindrances.

On the positive side, this type of graphical image is simple, free of visual clutter, and contains a function that both increases and decreases and whose slope continuously changes. These characteristics may provide individuals with a quick image in which to check the plausibility of certain integral properties or to imagine the quantities involved in a real-world-based integral. Yet, on the negative side, it seems problematic that neither the "inputs" nor "outputs" (i.e. x and $f(x)$) attain negative values, which may have important ramifications for both integral properties and real-world quantities. Also, the fact that the graph has no dramatic rises or drops and is always continuous and smooth may oversimplify the nature of definite integrals if this kind of image is too dominant in a student's thinking.

In stating these possible benefits and hindrances, I wish to be clear that I am not taking the position that this default image is bad. However, I am advocating that we, as calculus educators, should take a careful look at the types of graphical images we use in connection with the definite integral in order to develop a more robust catalogue of images that could serve more flexibly in a wider range of situations. Having a single graphical image that is so prominently culturally embedded may be problematic for thinking about definite integrals. By contrast, if this image were included as just one in a set of easily-accessible graphical images, students may possibly develop a more robust understanding of definite integrals. Since no alternative images came up with nearly as much frequency in this study, it may be that many students might not have such a catalogue of useful images, and may be overlying on this one particular type of image. If this is the case, we, as calculus instructors, might wish to emphasize a greater variety of graphical images in connection with integrals.

References

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*(3), 301-317.

Cunningham, S. (1991). The visualization environment for mathematics education. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. Washington, DC: Mathematical Association of America.

Cuoco, A., & Curcio, F. (Eds.). (2001). *The roles of representation in school mathematics: 2001 NCTM yearbook*. Reston, VA: NCTM.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. Washington, DC: Mathematical Association of America.

Ernest, P. (1994). Social constructivism and the psychology of mathematics education. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (pp. 68-77). London, UK: Falmer Press.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Jones, S. R. (2015). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann-sum based conceptions in students' explanations of definite integrals. *International Journal of Mathematics Education in Science and Technology, 46*(5), 721-736.

Jones, S. R. (*under review*). Teaching Integration: How certain instructional moves may undermine the potential conceptual value of the Riemann sum and the Riemann integral.

Jones, S. R., & Dorko, A. (2015). Students' understandings of multivariate integrals and how they may be generalized from single integral conceptions. *The Journal of Mathematical Behavior*, *40*(B), 154-170.

Kay, P., & McDaniel, C. K. (1978). The linguistic significance of the meanings of basic color terms. *Language, 54*(3), 610-646.

Kouropatov, A., & Dreyfus, T. (2013). Constructing the integral concept on the basis of the idea of accumulation: Suggestions for a high school curriculum. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 641-651.

Lakoff, G. (1973). Hedges: A study of meaning criteria and the logic of fuzzy concepts. *Journal of Philosophical Logic, 2*(4), 458-508.

Romberg, T. A., Fennema, E., & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representation of functions*. Mahwah, NJ: Lawrence Erlbaum Associates.

Rosch, E. H. (1973). Natural categories. *Cognitive Psychology, 4,* 328-350.

Rosch, E. H. (1978). Principles of categorization. In E. H. Rosch & B. B. Lloyd (Eds.), *Cognition and categorization* (pp. 27-48). Hillsdale, NJ: Lawrence Erlbaum Associates.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*(1), 230-245.