Root of Misconceptions – the Incorporation of Mathematical Ideas in History

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The evolution of a mathematical concept in history has been the process of merging different ideas to form a more rich, general, and rigorous concept. Ironically, students, when learning such well-developed concepts, have similar difficulties and make the same misconceptions again and again. To illustrate, despite the well-developed and defined concept of real numbers, many students still have difficulties in comparing fractions or doing basic operations on irrational numbers. In this poster, the incorporation of different ideas to form a general and rigorous mathematical concept in history is examined. Students' struggles and misconceptions in learning the concepts are investigated from the perspective of the incorporation process. Finally, a model for differentiating and validating the variations of a general mathematical concept is suggested for resolving learning difficulties and misconceptions.

Key words: Misconceptions, History of Mathematics, Formation of Concepts.

Misconceptions due to no differentiation

Things don't always turn out the way you want, and don't always work the way you expect. One common kind of mathematical misconception is no differentiation (Schechter, 2009), for example, adding variables and numbers together (e.g., 5x+3=8) or adding fractions like integers (2/3 + ½ = 3/5). Some no differentiation cases are about properties. For example, everything is additive (e.g., 1/(x+y) = 1/x + 1/y; $\sqrt{(x+y)} = \sqrt{x} + \sqrt{y}$; sin(x+y)=sinx+siny). Everything is commutative (e.g., log2x=2logx, sin2x=2sinx). It seems students are lost in the bigger misconception of "general" in mathematics and overlook the variations of operations, properties, or methods embedded in a "general" mathematical concept.

The possible reason of the misconceptions

Generality is emphasized in mathematics. For example, mathematics is applicable to different fields or mathematical methods work for all cases. However, rather than generality, different mathematical ideas were incorporated in history in terms of extending or modifying existing mathematical structures, or creating general construct to encompass different ideas, for instance, the real number line and the concept of function (Benson, 2003; Kleiner, 1989; Kline, 1972; Ponte, 1992). The incorporation of different ideas to form a general and consistent concept or system may make it difficult for students to differentiate differences. For example, there is rich mathematics education literature about students' struggles in differentiating different operation rules regarding to different kind of numbers (whole numbers, fractions, irrational numbers). Therefore, knowing how different ideas were incorporated in history may help recognize the threads of differences embedded in a general concept of mathematics (e.g., different rules, properties, or methods).

The formation of a mathematical concept in history

In this study, the formation of the concept function and real number line in history were examined. In particular, the original meanings of real-life contexts, operations, and methods that have been lost or hidden in the current meaning of the two concepts were examined. In history, the concept of function started with "tables" (e.g., the values of square roots). The concept of function then was developed as the corresponding values on a graph in analytical geometry in

16th and 17th centuries. After 17th century, with the development of algebra, the focus of function was shifted to analytical expressions (e.g., algebraic expressions), departed from graphs. Since functions as analytical expressions are only a small subset of all functions, the idea of function was gradually changed to the correspondence between sets, numerical or non-numerical, to replace the perspective of analytical expression.

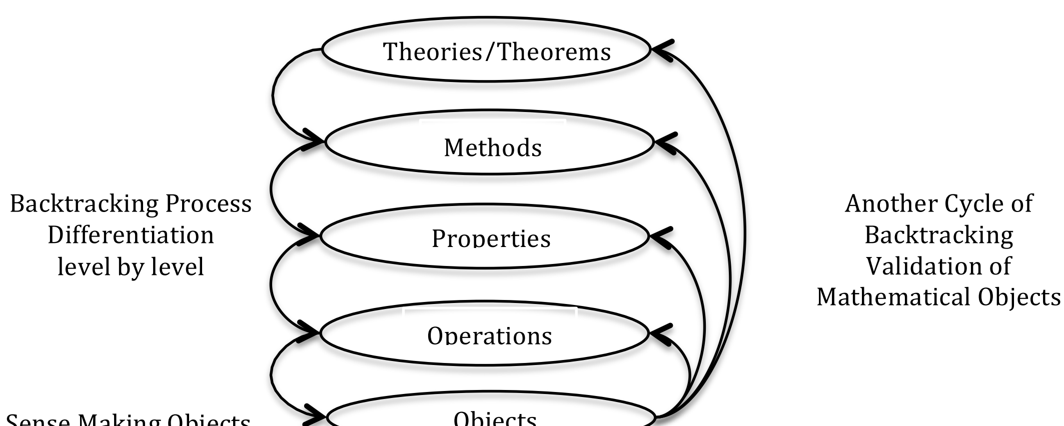
Fractions were invented as a method for dealing with real-life problems (taxes, commercial exchange) in ancient Egypt and China. A fraction was not a number, but a method. Every natural thing exists in the form of natural numbers. The operations on fractions (e.g., adding or multiplying fractions) were based on the idea of ratio, not "numbers". For example, Pythagoreans took fractions as commensurable ratios. Moreover, a square root of a number was not a "number", but a magnitude that could not be explained for a long time in history.

In summary, there are different contextual meanings, rules, and properties incorporated to the concept of function and real numbers, as we have seen in the history. Students' difficulties and misconceptions regarding to the concepts, to some extent, are related to the incorporation process.

The model of differentiation and validation

A model was constructed based on the incorporation of ideas in history. The model (See Figure below) is the backtracking process regarding to the five levels of mathematical entities: (1) mathematical object (2) operation (3) property (4) method (5) theorem/theory. There are different purposes in the backtracking process for avoiding or correcting misconceptions. The special feature of multiple cycles of backtracking process is possible if needed.

Misconceptions on the mathematical object level (e.g., negative numbers, irrational numbers) were corrected by backtracking to the physical world or the existing mathematical models to search for meanings or explanations for new mathematical objects. Misconceptions on the property level (e.g., additive or commutative property) were corrected by backtracking to the mathematical object level to validate new mathematical objects (e.g., quaternions, matrices), and to differentiate new mathematical objects and their operational rules from the existing ones. Misconceptions on the method level (e.g., an infinitesimal is a fixed number) were corrected by backtracking to the operation level (e.g., a variable approaching to a point) and property level (e.g. a continuous function) to refine and replace the idea of infinitesimal. Misconceptions on the theorem/theory level (e.g., continuous functions are not differentiable only at some points) were corrected by counterexamples, which were new mathematical objects with new properties (e.g., a continuous function with nowhere differentiable, concave polyhedrons).



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