Reinventing the multiplication principle

Elise Lockwood — Oregon State University
Branwen Schaub — University of Portland

Counting problems offer opportunities for rich mathematical thinking, yet students struggle to solve such problems correctly. In an effort to better understand students' understanding of a fundamental aspect of combinatorial enumeration, we had two undergraduate students reinvent a statement of the multiplication principle during an eight-session teaching experiment. In this presentation, we report on the students' progression from a nascent to a sophisticated statement of the multiplication principle, and we highlight two key mathematical issues that emerged for the students through this process. We additionally present potential implications and directions for further research.

Key Words: Combinatorics, Reinvention, Counting problems, Teaching experiment

Introduction and Motivation

The multiplication principle (MP), called by some "The Fundamental Principle of Counting" (e.g., Richmond & Richmond, 2009), is a fundamental aspect of combinatorial enumeration. Broadly, it is the idea that for independent stages in a counting process, the number of options at each stage can be multiplied together to yield the total number of outcomes of the entire process. It is generally considered to be foundational to many of the counting formulas students learn and is also a much-needed source of justification for why these counting formulas work as they do. In spite of its importance, little has been studied about the MP in and of itself. In order to better understand student thinking about the MP, we had two undergraduate students reinvent a statement of the MP over the course of eight interviews. In this paper, we describe their overall reinvention process, discussing and presenting some of their statements. We also introduce and discuss two mathematical issues that are entailed in the MP and that arose for the students (the independence of stages in a counting process and the need to count distinct composite outcomes). We seek to address the following research goals: *1) Describe a pair of students' trajectory as they reinvent a statement of the MP, and, in so doing, 2) Present the mathematical issues in the MP to which the students attended as they reinvented the statement.*

Literature Review and Theoretical Perspective

Research about the MP in Combinatorics Education Literature

Previous work has demonstrated the importance of the MP in counting, and the lack of a well-developed understanding of the MP appears to be a significant problem and hurdle for the students, particularly in terms of their ability to justify or explain formulas. We have found that students can easily assume that they completely understand the MP in counting because multiplication is a familiar operation for them. As a result, they use the operation frequently but without careful analysis, and they tend not to realize when simple applications of the operation are problematic. While some researchers have discussed multiplication within combinatorial contexts (Tillema, 2013), there have not yet been studies that explicitly target student understanding of the MP.

Lockwood, Swinyard, and Caughman (2015) had students reinvent basic counting formulas, and the students in that study did not appear to have a solid understanding of the MP. They worked with outcomes empirically but lacked the understanding of how those outcomes related to the underlying counting process involved with the MP. This work suggested the need for more research that targets students' understanding of the MP as a fundamental counting process.

In addition, Lockwood, Reed, and Caughman, (2015) recently conducted a textbook analysis that examined statements of the MP in university combinatorics and discrete mathematics textbooks. This revealed a wide variety of statements of the MP (Figures 1, 2, and 3 reveal three very different formulations).

> ***Product Rule:*** *If something can happen in n_1 ways,* ***and*** *no matter how the first thing happens, a second thing can happen in n_2 ways,* ***and*** *no matter how the first two things happen, a third thing can happen in n_1 ways,* ***and*** *..., then all the things together can happen in $n_1 \times n_2 \times n_3 \times ...$ ways.*

Figure 1 – Roberts & Tesman's (2003) statement of the MP

> ***The Multiplication Principle****: Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, ..., and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times ... \times r_m$ different composite outcomes.*

Figure 2 – Tucker's (2002) statement of the MP

> ***Generalized Product Principle:*** *Let $X_1, X_2, ..., X_k$ be finite sets. Then the number of k-tuples (x_1, x_2, ..., x_k) satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times ... \times |X_k|$.*

Figure 3 – Bona's (2007) statement of the MP

Part of the motivation for the current study, then, is to build upon the textbook analysis by actually studying how students think about mathematical issues that arose in the textbook statements of the MP. The findings from Lockwood, Reed, et al. (2015) framed and informed the mathematical issues we pursued with the students, and in the following section we briefly discuss these key mathematical issues in the MP.

Key Mathematical Issues

Here we describe two mathematical issues in the MP, both of which are seen in Tucker's (2002) statement (Figure 2). In the Results section we will describe the students' reasoning about these key ideas, and so we briefly introduce them here to facilitate subsequent discussion. First, there is the notion of *independence* of stages in the counting process, which captures the idea that a choice of options at a given stage does not affect the number of outcomes in any subsequent stage. This is a necessary condition in order to apply the MP, or else overcounting may occur. Second, the MP must yield *distinct composite outcomes*, which means that when applying the MP we want to ensure that there are no duplicate outcomes. This qualification, too, prevents instances of overcounting. In the Results section we highlight two counting problems that demonstrate the need for each of these mathematical issues in statements of the MP.

Reinvention

Gravemeijer, Cobb, Bowers, and Whitenack (2000) describe the heuristic of *guided reinvention* as "a process by which students formalize their informal understandings and intuitions" (p. 237). From this perspective, students can formalize ideas through generalization of their previous mathematical activity. We had students reinvent statements of the MP because we felt this would allow students to meaningfully understand and articulate a statement, giving us insight into how students come to understand the MP. This is in line with other researchers who have used principles of reinvention to gain insight into students' reasoning about a particular concept or definition (e.g., Oehrtman, Swinyard, & Martin, 2014; Swinyard, 2011).

Methods

Data Collection

We conducted a teaching experiment (as described by Steffe & Thompson, 2000) in which a pair of undergraduate students solved counting problems over eight hour-long sessions. The students were enrolled in vector calculus in a large university in the western United States, and they were chosen because they had not been explicitly taught about the MP in their university coursework (and thus would not simply try to recall it). The interviews took place outside of class time over a period of four weeks. Broadly, the students solved a series of counting problems, and they were asked periodically to write down and characterize when they were using multiplication as they solved these problems. They wrote down several iterations of statements of the MP. Throughout the study the interviewer selected tasks to highlight various aspects of the MP and regularly asked clarifying questions.

For the sake of space we provide only a sampling of tasks from the teaching experiment. Broadly, the students engaged in three kinds of activities: solving counting problems that involve multiplication, articulating a statement of the MP, refining their statements of the MP, and evaluating given textbook statements of the MP. Although there was some overlap of activities in each session, Table 1 gives the overall structure of the teaching experiment by outlining the session number (and total number of tasks in each session), a sample task given in that session, and the predominant activity that occurred in each session.

Session	Sample Tasks for Each Session	Emphasis of Session
1 (6 tasks)	You have 4 different Russian books, 5 different French books, and 6 different Spanish books on your desk. In how many ways can you take two of those books with you, if the two books are not in the same language?	Solving counting problems that involve multiplication
2 (5 tasks)	How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f: (a) with repetition of letters allowed? (b) without repetition of any letter? (c) without repetition and containing the letter e? (d) with repetition and containing e?	
3 (5 tasks)	In a standard 52-card deck there are 4 suits (hearts, diamonds, spades, and clubs), with 13 cards per suit. There are 3 face cards in each suit (Jack, Queen, and King). How many ways are there to pick two different cards from a standard 52-card deck such that the first card is a face card and the second card is a heart?	Articulating a statement of the MP
4 (2 tasks)	How many ways are there to flip a coin, roll a die, and select a card from a standard deck?	
5 (2 tasks)	There are 7 professors and 5 grad students. In how many different ways could an advisor and a grad student be paired up?	
6 (3 total tasks)	How many 6-character license plates consisting of letters or numbers have no repeated character?	Refining their statement of the MP

7 (6 tasks)	How many rearrangements of the letters in the word DYNAMIC start with a vowel?	
8 (7 statements)	Please read the following statement [such as Tucker's (2002) in Figure 1]. How is it similar to or different from your own statement?	Evaluating given textbook statements

Table 1 – Overall structure of the teaching experiment

Data Analysis

The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We used prior understanding of the MP that had emerged from the textbook analysis to guide our focus of particular mathematical issues. Key episodes involving mathematical issues were flagged and reviewed, and we scrutinized the students' statements of the MP and their explanations for insights about their reasoning.

Results

We organize the results into two sections. First we provide an overview of their progress and offer several of the statements that they developed as they reinvented the MP. This should demonstrate their overall progress and provide a broad narrative of what transpired during the teaching experiment. Then, we present their handling of two key mathematical issues (independence and distinct composite outcomes), highlighting student thinking about important aspects of the MP.

Overall Progression of Statements

The students went through between 20-25 statements (depending on how one defines a statement, as some were verbally articulated, and some involved minor adjustments from previous statements). Here, due to space, we provide seven statements (exactly as the students had written them on the board), which emphasize development from a nascent to sophisticated statement of the MP. Statements 2a and 2b, 3a and 3b, and 4a and 4b each represent minor changes that reflect the students' realization about a key mathematical issue.

Session	Statement
2	#1 – Use multiplication in counting problems when… there is a certain statement shown to exist and what follows has to be true as well.
4	#2a – For each possible pathway to an outcome there is an equal number of options leading to that path.
	#2b – For each possible pathway to an outcome there is an equal number of options leading to that path but without repeating the same pathway more than once.
6	#3a – For every selection towards a specific outcome, if one selection does not affect any subsequent selection, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.
	#3b – For every selection towards a specific outcome, if one selection, no matter the previous selections, is no difference in the number of options, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.

6	#4a – If for every selection towards a specific outcome there is no difference in the number of outcomes, regardless of previous selections, then you multiply the number of all the options in each selection together to get the total number of possible outcomes.
7	#4b – If for every selection towards a specific outcome, if there is no difference in the number of options, regardless of the previous selections, then you multiply the number of all the options in each selection together to get the total number of possible unique outcomes.

Table 2 – The students' progression toward a statement of the MP

We began the interviews by simply having students solve counting problems that involved multiplication, the motivation being to give them some experience using multiplication so they might extrapolate some key principles of when to use multiplication in counting. By the end of Session 2 we gave them the following prompt: *Can you take a stab at characterizing when you use multiplication when you're solving these problems?*, and they produced Statement #1. In Sessions 3 through 7 we gave them more counting problems and asked them to refine their statements. As they developed more sophisticated language in talking about the statements, we targeted key mathematical ideas by giving counting problems that would elicit certain ideas.

We note a couple of important observations about their progression in Table 2. First, we highlight the lack of sophistication in Statement #1 compared to Statement #4b (their final statement). Statement #1 is not well formed, and this suggests that the task of characterizing when to use multiplication is not a trivial one. By the end of the teaching experiment, however, they had developed a rigorous statement. We also point out the shift in language and emphasis from pathways and paths in Statements #2a and #2b to language of outcomes, selections, and options in Statements #3a, #3b, #4a, and #4b. This shift reflects an intentional pedagogical move that occurred after Session 5. In Session 4 the students had come up with Statements #2a and #2b. Session 5 saw no progress or refinement of their statement, and so we decided to redirect the students' attention away from the pathway language and toward more general language. This led them to introduce language of options, selections, and language.

Key Mathematical Issues

We now offer data examples that exhibit how the students' statements developed over time, and in particular how they adjusted their statements to address some key mathematical issues. These are meant to demonstrate the nature of the students' interactions with each other and with the interviewer and also to show how students' reasoning developed as they interacted with particular tasks.

Independence. In Session 6, the students had come up with Statement #3a from Table 2. We draw attention to the phrase "if one selection does not affect any subsequent selection then you multiply…" This is a valuable insight in and of itself, because it ensures that one only multiplies when the stages of a counting process are independent of one another. However, as stated, Statement #3a is not quite accurate, because multiplication is still appropriate if the actual options change from stage to stage. Instead, it is the *number* of options that cannot change (as the following episode demonstrates).

To draw the students' attention toward this issue, we presented them with the following problem: *How many 6-character license plates consisting of letters or numbers have no repeated character?* They immediately recognized that they could use multiplication on this problem –

they knew the answer would be 36*35*34*33*32*31, and they had the following exchange as they tried to justify the answer.

Pat: *It's still multiplication, but it's not the same as multiplication that we were thinking of. So as we, it has to change things now doesn't it?*

Caleb: That one definitely put a damper on our – [cut off]

They then thought about the problem a bit more, and Pat had the following realization.

Pat: *I'm just concerned about the idea that we're saying, that the selection is affecting the next selection, because technically in this case, the selections…affect subsequent selections, but it still is multiplication…*

Pat saw that their wording would not allow for the multiplication in a problem like the License Plates problem, because their statement #3a was too restrictive about how subsequent selections might be affected. The two students then had the following exchange (the A and 9 refer to choices for a license plate character):

Caleb: Maybe you pick like A and then 9, and you can't do either one of those again.

Pat: *Yeah but it's like A or 9 won't affect the number of next selections. 'Cause no matter what it's gonna be 36, 35, 34, 33…See what I'm saying? So like how do we incorporate that? Because like, so as long as it – as long as the next, the subsequent selections still have the same number of selections, it's okay.*

The students then proceeded to write down language that might help address this issue, now emphasizing the *number* of options to which they had not previously attended, ultimately writing statement #3b. This exchange provides evidence of how the students reasoned about a subtle aspect of independence, and the License Plate task was carefully chosen to refine an already-existing idea – that the *number* of options (not the options themselves) must be independent.

Distinct composite outcomes. The progression from statements #4a to #4b reveal an evolution in the students' thinking about distinct composite outcomes. This came about through the Three *e*'s problem: *How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f with repetition and containing e?*. In this problem, a common, tempting incorrect answer is to argue that there are 3 places in which to place an *e*, and then once that *e* is placed, since repetition is allowed there are 6 options for the next spot and then 6 for the remaining spot, yielding an answer of 3*6*6. Indeed, this is what the students first answered. However, in this answer an outcomes like *eee* gets counted more than once, because the process described above could generate *eee* both when *e* is placed in the first spot in the first stage (and then *ee* are in the last two spots in the second and third stages), and also when *e* is placed in the third spot in the first stage (and then *ee* are in the first two spots in the second and third stages). The students had written statement #4a, and then they revisited this problem. As they worked through the problem, they realized that someone could use their statement #4a to solve the Three E's problem but would end up overcounting. Caleb talks about wanting to disallow this kind of overcounting in their statement of the MP, but he acknowledges the difficulty of how to articulate that.

Caleb: Well you have to make it seem like you can't over count something without saying don't over count it. Because the reason is, we're not wanting to over count it.

Int.: *Okay, okay. So what do you mean? Yeah, say more about that.*

Caleb: So our problem here is over counting, and you can't just like put in a clause of like don't over count. […] 'Cause right now we sort of have the difference in the number of options but that doesn't, it's not necessarily specific to over counting. We were sort of thinking of less.

We encouraged them to think more about the problem, and they had the following exchange:

Int.: *Okay, cool. I might just let you guys think about this for a couple minutes.*

Caleb: *We could say without any repeated outcomes.*

Int.: *Okay. Say more about that.*

Caleb: *So our problem here is where we're getting like a repeated outcome. If we say, um.*

Pat: *Oh hey, we already have specific outcome in there [in statement #4a].*

Caleb: *Yeah.*

Pat: *So how about we say specific unique outcome?*

They then added the word “unique” to their statement, resulting in their final statement, #4b. This addition of the word “unique” is their way of addressing the possibility of overcounting, and ensuring that the outcomes must be unique, is equivalent to Tucker’s (2002) clause of “distinct composite outcomes.” By specifying that they only want to count unique outcomes, their statement technically does not allow for this kind of overcounting to occur. Thus, again we see an instance in which the students refined their statement after considering a particular mathematical task. This sheds light on how their thinking developed about the MP and the mathematical aspects of the statement on which they focused.

Conclusion and Implications

By having students reinvent a statement of the MP, and by closely analyzing aspects of multiplication to which they attend, we gain insight both into how students reason about the MP, and also how productive reasoning about the MP might be developed. In particular, by engaging with particular tasks, the students we worked with were able to come to reason about key mathematical aspects of the MP (such as independence and unique outcomes) that they wanted to include in their statement of the MP. In addition to insights about how they come to understand particular mathematical ideas, we can draw a couple of key conclusions from their overall progression from to a final statement. First of all, we have an existence proof that it is possible for students to develop, on their own, a mathematically rigorous statement of the MP. It is not trivial to characterize many of the subtle mathematical details of the MP, and it is impressive that the students were able to do so. Second, we see that although they were able to accomplish this task, it was not a trivial activity to characterize when to use multiplication in solving counting problems. This is demonstrated most clearly in their first statement, which shows that even after they had successfully used multiplication in counting problems for two sessions, they still struggled with articulating a statement about it.

Our findings suggest a couple of implications. First, as a fundamental aspect of counting, the MP is invaluable, yet potentially challenging, for students to understand well. Although it deals with a familiar operation, there are subtle mathematical features that it involves, which might take time and effort for students to learn. More work is needed to more carefully evaluate how best to teach the MP to students in a classroom setting, but our work suggests that it may be worthwhile to unpack some key mathematical issues with students. Instructors should appreciate the mathematical details in the MP and should help students think carefully about when multiplication properly applies in counting situations. In terms of research, we plan to continue to explore what might be entailed in a robust understanding of the MP, which includes interviews with more students and also with mathematicians. Based on our findings from this study, especially insights about understanding independence and distinct composite outcomes, we can look to design instructional interventions that might draw students’ attention toward such ideas.

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