Assessing Students' Understanding of Eigenvectors and Eigenvalues in Linear Algebra

Kevin Watson	Megan Wawro	Michelle Zandieh
Virginia Tech	Virginia Tech	Arizona State University

Many concepts within Linear Algebra are extremely useful in STEM fields; in particular are the concepts of eigenvector and eigenvalue. Through examining the body of research on student reasoning in linear algebra and our own understanding of eigenvectors and eigenvalues, we are developing preliminary ideas about a framework for eigentheory. Based on these preliminary ideas, we are also creating an assessment tool that will test students' understanding of eigentheory. This poster will present our preliminary framework, and examples of the multiple-choice-extended questions we have created to assess student understanding.

Key Words: Linear Algebra, Eigenvector, Eigenvalue, Assessment

The study of linear algebra is highly useful to students in science, technology, engineering and mathematics (STEM) fields and is often introduced in the first or second year of university. The use of linear algebra extends into upper-division university studies as well, in courses such as quantum physics. One crucial, and particularly valuable, concept encountered by students in linear algebra is that of eigentheory. As part of a larger study investigating how students reason about and symbolize concepts related to eigentheory in quantum physics (*Project LinAl-P*), we are (a) creating a preliminary framework for student understanding of eigentheory, and (b) developing an assessment to examine students' understanding of eigentheory. These research activities go hand-in-hand because, as we strive to develop a way to measure students' rich and nuanced understanding of eigentheory, our measurement tool (a collection of multiple-choice-extended questions) must be grounded in and aligned with a research-based framework that characterizes what it means to understand eigentheory (Izsák, Lobato, Orrill, & Jacobson, 2011).

Literature and Preliminary Framework

Although other researchers have examined students' understanding of eigenvectors and eigenvalues (Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Sinclair & Gol Tabaghi, 2010; Stewart & Thomas, 2006; Thomas & Stewart, 2011), a comprehensive framework encompassing and connecting the elements necessary to conceptually understand eigenvectors and eigenvalues and their uses (such as in diagonalization) does not currently exist. To begin our preliminary framework, we consulted this literature base. In particular, we drew from delineations of conceptual understanding of eigenvectors and eigenvalues through genetic decompositions (Salgado & Trigueros, 2015; Thomas & Stewart, 2011); these papers mainly focused on the mental constructs necessary to understand the standard algorithm for calculating eigenvalues and eigenvectors, rather than geometric or structural modes of reasoning. We also examined a Quantum Mechanics textbook (McIntyre, 2012) to investigate what skills related to eigentheory, such as diagonalization, were crucial to applications within that discipline.

As we progressed, we noted the compatibility of our work with that of Sierpinska (2000), who distinguished three modes of reasoning – synthetic-geometric, analytic-arithmetic, and analytic-structural – available to students in linear algebra corresponding to three interacting languages. These languages are: "the 'visual geometric' language, the 'arithmetic' language of vectors and matrices as lists and tables of numbers, and the 'structural' language of vector spaces and linear transformations" (p. 209). While our framework is still a work in progress, it will include delineations across: comprehending calculations involved in finding the eigenvalues and eigenvectors of a given matrix and why they work; understanding eigenvectors, eigenvalues, and eigenspaces geometrically when working within $\mathbb{R}^2$ and $\mathbb{R}^3$; using eigenvectors and eigenvalues in applications (e.g., diagonalization, long-term behavior of dynamical systems, Markov Chains); and drawing structural inferences from known information such as algebraic and geometric multiplicities. By the time of the conference, we aim to have this framework more fully fleshed out and organized into a structure useful for examining student understanding of eigentheory.

Assessment Instrument

Our current work in student understanding of linear algebra in physics draws its foundation from a larger research project in the teaching and learning of linear algebra; one research product from that project is an assessment instrument for measuring student understanding of key linear algebra concepts (Zandieh et al., 2015). This instrument contains closed-ended questions in an adapted multiple-choice format, which we call *multiple-choice-extended* (MCE). This format, which is just appearing in physics education research, is based on work by Wilcox and Pollock (2013), who adopt questions from the valid and reliable electricity and magnetism diagnostic to explore "the viability of a novel test format where students select multiple responses and can receive partial credit based on the accuracy and consistency of their selections," to allow for "preserving insights afforded by the open-ended format" (p. 1). Questions written in a MCE format begin with a multiple-choice element and then prompt students to justify their answer by selecting all statements that could support their choice. In *Project LinAl-P*, we have been working to create a MCE-style assessment instrument for measuring and characterizing students' understanding of eigentheory in linear algebra. Figure 1 contains an example of a MCE question from the pilot version of *Project LinAl-P*'s assessment instrument.

An eigenvalue of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is:

(a) $\lambda = 2$
(b) $\lambda = 3$

Because … (select ALL that could justify your choice)

(i) This eigenvalue is a solution to the characteristic equation of A.
(ii) This eigenvalue makes $\det(A - \lambda I) = 0$ a true statement.
(iii) This eigenvalue makes $\det(A - I)\boldsymbol{x} = \boldsymbol{0}$ a true statement.
(iv) When acted on by matrix A, all vectors in $\mathbb{R}^2$ are stretched by the amount of this eigenvalue.
(v) When acted on by matrix A, there is a line of vectors in $\mathbb{R}^2$ that are stretched by the amount of this eigenvalue.
(vi) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to have a nontrivial solution.
(vii) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to have only the trivial solution.
(viii) $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is equivalent to $(A - \lambda)\boldsymbol{x} = \boldsymbol{0}$, and this eigenvalue makes it possible for $(A - \lambda)\boldsymbol{x} = \boldsymbol{0}$ to have a nontrivial solution.

Figure 1. Example of an MCE question from the eigentheory assessment instrument.

In Fall 2015, we interviewed two students using pilot versions of eight MCE questions, which led to minor question revisions. In January 2016, we will administer the questions in written format to approximately 20 students entering a quantum physics course. Data and analysis from both of these sources will be included on the poster.

References

Gol Tabaghi, S., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: The emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning, 18*(3), 149–164.

Izsák, A., Lobato, J., Orrill, C. H., & Jacobson, E. (2011). *Diagnosing teachers' multiplicative reasoning attributes.* Unpublished manuscript, Department of Mathematics and Science Education, University of Georgia, Athens, GA.

McIntyre, D. (2012). *Quantum Mechanics: A Paradigms Approach* (1st ed.). Chicago, IL: Addison-Wesley Longman.

Salgado, H., & Trigueros, M. (2015). Teaching eigenvalues and eigenvectors using models and APOS Theory. *The Journal of Mathematical Behavior, 39*, 100–120.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 209–246). New York, NY: Kluwer Academic Publishers.

Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*, 223-240.

Thomas, M. O. J., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal, 23*(3), 275–296.

Wilcox, B. R., & Pollock, S. J. (2013). Multiple-choice Assessment for Upper-division Electricity and Magnetism. *arXiv preprint arXiv:1308.4619*.

Zandieh, M., Plaxco, D., Wawro, M., Rasmussen, C., Milbourne, H., & Czeranko, K. (2015). Extending multiple choice format to document student thinking. In T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1079-1085), Pittsburgh, PA.