Graphing habits: “I just don’t like that”

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Students’ ways of thinking for graphs remain an important focus in mathematics education due to both the prevalence of graphical representations in the study of mathematics and the persistent difficulties students encounter with graphs. In this report, we draw from clinical interviews to report ways of thinking (or habits) undergraduate students maintain for assimilating graphs. In particular, we characterize actions constituting students’ ways of thinking for graphs that inhibited their ability to represent covariational relationships they conceived to constitute some phenomenon or situation. As an example, we illustrate that students’ ways of thinking for graphs were not productive for their representing a relationship such that neither quantity’s value increased or decreased monotonically.

Key words: Graphing, Covariational reasoning, Quantitative reasoning, Cognition, Function

“[U]nderstanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted” (Saldanha & Thompson, 1998, p. 303). Saldanha and Thompson’s call that educators not take for granted students’ ways of thinking for graphs remains relevant given the difficulties students have with topics (e.g., function, rate of change and derivatives, and variables) that involve the use of graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2007; Johnson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994, 2013; Trigueros & Jacobs, 2008; Zandieh, 2000). In this paper, we respond to the need for a better understanding of students’ way of thinking for graphs. Namely, we describe students’ graphing actions on tasks we designed to afford tracking covarying quantities. We first provide relevant background information and describe our clinical interviews and methods. We then identify students’ ways of thinking for graphs that were not productive for representing covariational relationships they had conceived some phenomenon to entail. We close by arguing that the identified ways of thinking for graphs are related to aspects that we perceive to be pervasive in U.S. school mathematics.

Background

Our core interest is characterizing students’ covariational reasoning–“the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (Carlson et al., 2002, p. 354)–particularly in the contexts of phenomena (e.g., students taking a road trip) and graphs representing covarying quantities understood to constitute the phenomena. By quantity and covariation, we do not mean numbers (or measures) and numerical patterns of two sets (cf. Confrey and Smith (1994, 1995)). Rather, we use *quantity* to refer to an attribute (e.g., length) an individual conceives to constitute some situation or phenomenon such that the individual understands the attribute as having a measurable magnitude (Thompson, 2011), possibly with respect to numerous unit magnitudes (e.g., meters or feet). We draw attention to a distinction between a quantity’s magnitude and its measure because it enables us to approach covariation in terms of the simultaneous coordination of magnitudes in flux with the *anticipation* that these magnitudes have specific measures (in an associated unit) at any moment (Saldanha & Thompson, 1998); one does not need specified measures at hand to reason covariationally.

We do not interpret this perspective to diminish the importance of understanding covariation in terms of quantities’ measures and patterns in these measures. Such an understanding is important for understanding various function classes and rate of change. Our focus on covarying magnitudes provides a complementary lens that researchers, including ourselves, have found productive in characterizing students' images of covarying quantities, particularly their conceiving quantities in flux (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 1994). Most relevant to the current study, Moore and Thompson (2015) defined *emergent shape thinking* as a way of thinking about a graph as a locus or trace that is produced by the simultaneous coordination of two quantities' magnitudes. They explained, "emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities" (Moore & Thompson, in press). As Moore and Thompson highlighted, we cannot illustrate such a way of thinking due to the static medium of print, but we do provide instantiations in *Figure 1*. We emphasize that although a displayed graph is composed of points, a student thinking emergently constructs a displayed graph in terms of a projection of two coordinated magnitudes along the axes with the anticipation that these magnitudes correspond to specified measures.

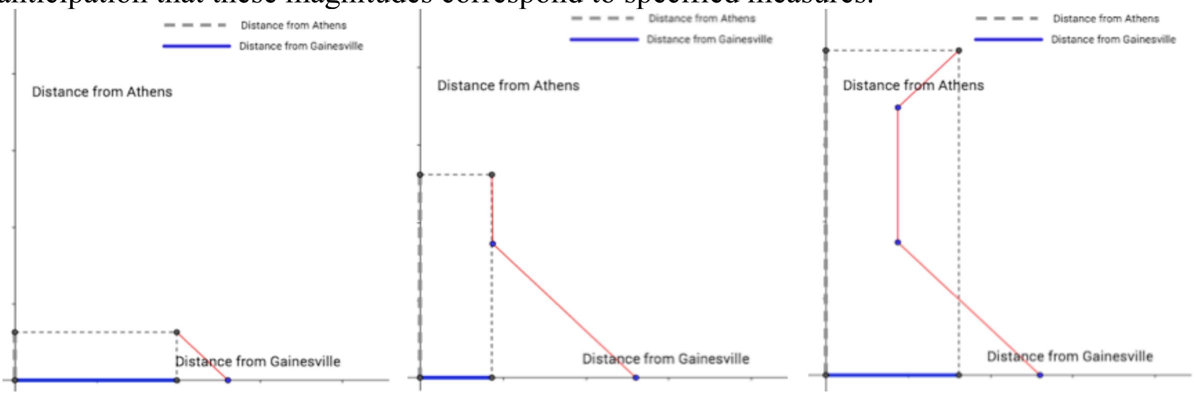

Figure 1. A graph as an emergent (snapshot) coordination of two magnitudes.

Subjects, Setting, and Methods

Our subjects were ten prospective secondary mathematics teachers (hereafter referred to as students) enrolled in an undergraduate secondary mathematics education program in the southeastern U.S. The students ranged from juniors to seniors in credits taken, and they had completed at least one mathematics course beyond an undergraduate calculus sequence. We chose the students on a volunteer basis. We conducted three semi-structured clinical interviews (Ginsburg, 1997; Goldin, 2000) with each student. Each interview lasted approximately 75 minutes. The lead author facilitated each interview, often with the aid of a member of the author team. We asked the students to talk-aloud as much as possible, and we asked open-ended questions during their progress to gain insights into the students' thinking with minimal guiding. The interviews occurred throughout one semester with approximately 1.5 months passing between each interview. The time between the interviews enabled us to design subsequent interviews based on retrospective analyses of prior interviews.

We video- and audio-recorded all interviews and digitized student work after each interview. The lead author and fellow interviewer also recorded observation notes after each interview. We analyzed the data using selective open and axial methods (Corbin & Strauss, 2008) in combination with *conceptual analysis*–an attempt to build models of students' mental actions that explain their observable activity and interactions (Thompson, 2008; von Glasersfeld, 1995). First, members from the research team identified instances that provided insights into the students' thinking. The research team then viewed these selected instances in order to characterize the students' thinking. As we developed these characterizations, we continually returned to previously viewed instances (across all students) to revise or provide alternative characterizations if necessary. We generated themes among our characterizations through this iterative process, including the ones that we report in this paper.

Task Design

We designed a series of six tasks (two per interview, with each student receiving the same sequence of tasks). Each task entailed a different context, but were similar in that we: (1) provided a dynamic, albeit often simplified, phenomenon through video; (2) did not include numerical values for attributes of the phenomenon; (3) prompted the student to graph a relationship between two quantities; and (4) often prompted the student to create a second graph, either between different quantities or the same quantities under different axes orientations. To illustrate, the second interview with each student included *Going around Gainesville* (*Figure 2*; see *Figure 1* for a solution to Part II), which entails a video depicting a car traveling back-and-forth between Athens and Tampa Bay. Reflecting (1) and (2), the task consists of a dynamic phenomenon depicted by a video without numerical information. In Part I, we prompted the students to graph a particular relationship between two quantities (i.e., (3)). In Part II, which we presented after the students completed Part I, we prompted the students to graph a different relationship with an imposed axes orientation (i.e., (4)).

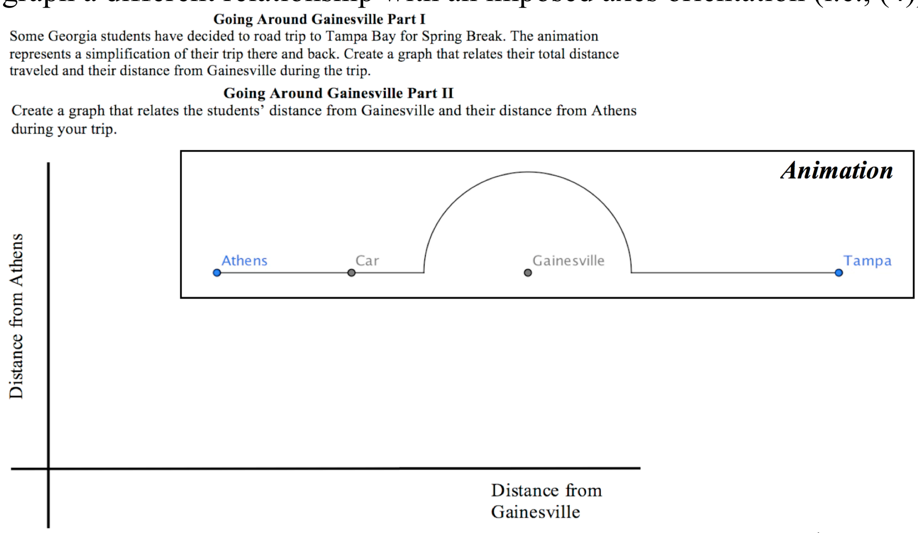

Figure 2. The *Going around Gainesville* task and video.[1]

In general, (1)-(3) each reflects our interests in students' covariational reasoning with particular attention to their coordinating magnitudes. Design goal (4) stems from two major findings from our previous work. First, we identified that students' ways of thinking for function and their graphs led to perturbations when graphing relationships in different axes orientations (Moore, 2014; Moore, Silverman, Paoletti, & LaForest, 2014). Although the quantities are slightly modified in Part II above (with us intending that one distance be accumulative and the other be displacement), we intended to gain additional insights into the perturbations that arose (or did not arise) when graphing the relationships in various orientations. Second, we designed the tasks to be what we perceived as non-canonical; our previous work led us to conclude that students encounter difficulties with such graphs (Moore, 2014). With respect to Part II, and because U.S. students nearly exclusively work with graphs such that the quantity represented on the vertical axis is a function of the quantity

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[1] This task is a modification of the task provided by Saldanha and Thompson (1998). We strongly suggest that the reader work these tasks before continuing to read.

represented on the horizontal axis, we hypothesized that the students might experience perturbations graphing a relationship that did not have this property, thus providing us additional insights into their ways of thinking for graphs.

Results

We structure the results section around three interrelated ways of thinking for graphs: *graphs 'start' on the vertical axis*, *graphs are drawn or read left-to-right*, and *graphs pass the 'vertical line test'*. We present these ways of thinking separately but do not intend to imply they exist independently. For instance, we describe students' anticipating graphs drawn or read left-to-right, as well as their anticipating that graphs 'start' on the vertical axis. These schemes were related in that some students' ways of thinking for graphs involved the sequence of marking an initial point on the vertical axis and then drawing a graph from that point to the right. However, if one of these schemes constituted a student's ways of thinking for graphs, then it was not necessarily the case that the other scheme also constituted a student's ways of thinking for graphs.

Graphs 'start' on the vertical axis

We inferred from some students' activities that a necessary action constituting their ways of thinking when drawing a graph was beginning their drawing on the vertical axis. At times, 'starting' their graph on the vertical axis led to what *we conceived* to be contradictions between the relationship that the students claimed or intended the graph to represent and the relationship that we perceived the graph to represent. At other times, this way of thinking led to the students experiencing perturbations as *they conceived* 'starting' their graph on the vertical axis as incompatible with the relationship they intended the graph to represent.

As an example, we return to *Going around Gainesville, Part II*. A normative graph includes a point on the horizontal axis corresponding to paired magnitudes when starting the trip in Athens; the graph includes no points representing a magnitude of zero for the distance from Gainesville (see *Figure 1*). Upon orienting to a task, some students immediately marked a point on the vertical axis and anticipated drawing a graph from that point (Excerpts 1).

Excerpts 1. Two students 'starting' graphs on the vertical axis.

Paula: Your distance from Athens starts at zero [*plots point at origin*] because you're in Athens. Um, so as you get. Mmm, no, you're gonna start up here [*plots point on vertical axis but not at origin*]. Ignore that [*covering origin*]. 'Cause, oh wait, no, stop [*crosses out second plotted point*]. No, you're here [*points to origin*].

Annika: We're in Athens [*moves to paper, marks point at origin*], as we're moving away from Athens we're getting closer to Gainesville [*draws segment from the origin going up and to the right, Figure 3*].

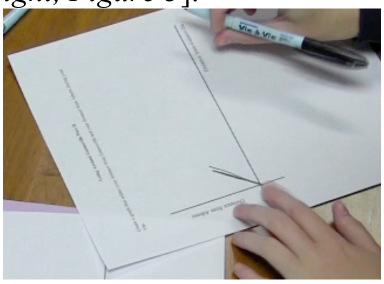


Figure 3. Annika starting the graph from the origin.

In anticipating the graph 'starting' on the vertical axis, the students' initial actions were to plot a point identifying the appropriate initial distance from Athens (and not the corresponding distance from Gainesville). Although Paula identified that the initial distance from Gainesville was non-zero, she maintained an alternative ‘starting’ point on the vertical axis and she quickly returned to plotting a point corresponding to the initial distance from Athens. As the students moved forward, they experienced perturbations due to their ‘starting’ point. Annika eventually noted that she did not take into account the initial distance from Gainesville and how to represent that distance as decreasing when first creating her graph. Paula experienced a more sustained perturbation, which we explain in the following section.

Graphs drawn or read left-to-right

Another action constituting some students’ ways of thinking entailed students understanding graphs as drawn or read left-to-right. When constructing a graph, students anticipated drawing the graph by starting at a point and exclusively moving their pen to the right while allowing for movements vertically. The vertical movements either connected previously plotted points (regardless of the order that these points were plotted) or captured some relationship that they intended the graph to represent (see Annika, Excerpts 1). To illustrate, we present two students’ activities (Excerpts 2). Karrie was working the task presented by Saldanha and Thompson (1998), which includes a prompt to graph how a traveler’s distances from two cities covary (see *Figure 4* for animation and a graph). Paula’s work is a condensed continuation of that in Excerpts 1.

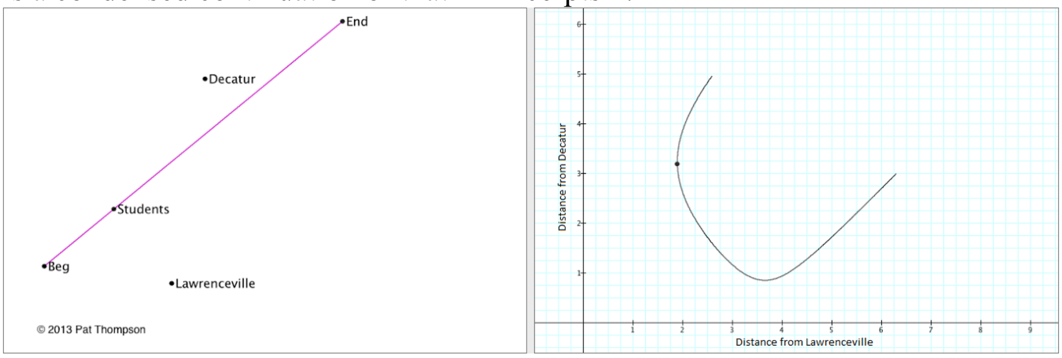


Figure 4. *City Travels* animation and graph (modified from Saldanha and Thompson, 1998).

Excerpts 2. Two students drawing graphs left-to-right.

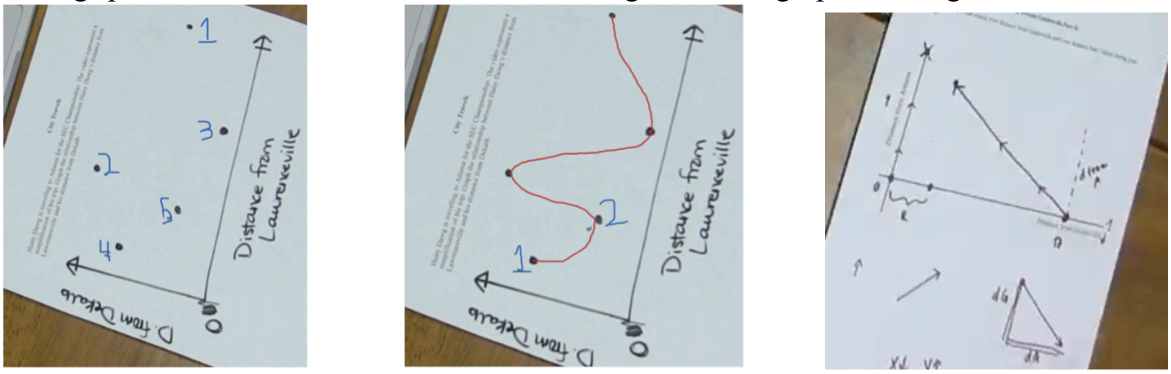
[*Karrie has plotted five points corresponding to locations during the trip in the order we have annotated in Figure 5a*]

Karrie: Okay, wait. This one [*pointing at the leftmost point she plotted*] was when he’s closest to Lawrenceville, which happens first [*labels the point ‘1’*], then this one [*labels the next leftmost point ‘2’, moves pen to the third leftmost point*] so it’s something like that [*making a sweeping motion indicating a curve connecting the points from left-to-right in the order we have annotated in Figure 5b*].

[*Paula is now focused on the initial point on the vertical axis that is not at the origin–see Excerpts 1–and anticipating drawing a segment sloping downward left-to-right from her initial point that she later crosses out–see Figure 5c*]

Paula: I wanted to show that the distance was decreasing [*motioning diagonally down and to the right from the point plotted on the vertical axis*], but that means that your distance from Athens is decreasing [*tracing along the vertical axis from the initial point to the origin*]…But your distance from Athens is growing. But your distance from Gainesville is decreasing. So, if that’s growing [*draws arrow pointing upward beside the vertical axis labe*] and that’s decreasing, so [*draws arrow pointing downward beside horizontal axis label and then an arrow pointing upwards beside the vertical axis label*]…[*the student works for six additional minutes before having an insight*]…Oh, what if I started it like here [*plots point on the right end of the horizontal axis*]…But I don’t want to start like, like I don’t like starting graphs. You know, I don't know. Work backwards. That's weird…[*draws in what we perceive to be a correct initial portion of the graph over the next minute and a half*]… my graph is from right-to-left, which is probably not right…Backwards is traveling from right-to-left. But I think my graph is just, I think I'm just not clicking. I think I'm missing something.

In Karrie's case, we highlight her immediate move to anticipate connecting the points from left-to-right after ordering two points from left-to-right (*Figure 5b*). Paula, on the other hand, did not plot points other than her initial point. Instead, she anticipated drawing a graph left-to-right from the initial point to indicate a decreasing *distance from Gainesville*. However, as she reflected on her anticipated graph with respect to the axes, she understood that such a graph would indicate a decreasing *distance from Athens*. Over the next seven minutes, Paula produced what we perceived to be a correct graph by thinking of the graph emergently (*Figure 5*c), but her resulting graph continued to perturb her due both to the 'starting' point on the horizontal axis and to having to draw a graph from right-to-left.



(a) (b) (c)

Figure 5. Karrie's annotated work (a-b) and Paula's work (c).

Graphs pass the 'vertical line test'

Some students' ways of thinking for graphs involved students anticipating that their drawn graph must pass the 'vertical line test' (i.e., a graph such that each abscissa value only corresponds to one ordinate value). In some cases, their anticipation was related to drawing graphs exclusively left-to-right. However, this way of thinking for graphs also emerged when students could anticipate graphs not drawn in this way. To illustrate, consider Angela's work on *Going Around Gainesville, Part II* (Excerpt 3). She did not encounter issues while drawing part of the graph right-to-left to indicate the distance from Gainesville decreasing as the distance from Athens increases. However, drawing a vertical segment, which she understood as representing the distance from Gainesville remaining constant as the distance from Athens increased, perturbed her to the extent that she both hesitated drawing a vertical segment and continued to question this vertical segment throughout working the task.

Excerpt 3. Angela anticipating that a drawn graph pass the 'vertical line test'.

[*Angela has plotted three points corresponding to positions on the semicircular path*]

Angela: So, that's weird [*motions pen indicating a vertical segment connecting the points*]. I don't wanna connect those dots, but, [*laughs softly*] I really don't like that.

Int.: What don't you like?

Angela: I just don't like that [*draws in a correct graph with a vertical segment*] my graph looks like this…I dunno. If I was taking a test and I drew that [*quickly motions the pen over the graph in the direction she had connected the points*] I'd feel like my answer was wrong. But I feel [*quickly motions pen back over the graph in the reverse direction*] like I graphed my points correctly…

Discussion

We find the above results notable for a few reasons. First and foremost, it is significant that undergraduate students who have completed mathematics courses beyond a calculus sequence hold ways of thinking for graphs that inhibit their ability to represent covariational relationships. Moreover, the students' difficulties did not stem from underdeveloped images of phenomenon, as is sometimes the case (Carlson et al., 2002; Moore & Carlson, 2012), but instead stemmed from ways of thinking for graphs that limited their ability to represent conceived relationships. We interpret this finding to corroborate researchers' (Moore, Paoletti, & Musgrave, 2013; Moore & Thompson, 2015) conjecture that ways of thinking that do foreground graphs as emergent traces of covarying quantities are more productive for accommodating novel phenomena and relationships than those ways of thinking that do not. A student thinking of a graph emergently maintains a focus on simultaneously coordinating magnitudes along axes with a trace emerging from this coordination; the mental operations that generate a trace are essentially equivalent regardless of the resulting trace and properties of its shape. On the other hand, ways of thinking for graphs that foreground recalling a repertoire of shapes and properties of these shapes (e.g., graphs passing the 'vertical line test'; graphs being traced left-to-right) are constrained to those phenomena or situations that are compatible with these shapes and properties.

We also find the results notable given the extent that some students remained perturbed after they had constructed graphs they conceived to represent a relationship compatible with the relationship they conceived to constitute some phenomenon. In these cases, and due to their resulting graphs being incompatible with particular ways of thinking they had previously constructed for graphs, the students questioned the correctness of their graphs (see Paula and Angela). We find the students' inability to reconcile their states of perturbation significant, especially because the students are prospective secondary mathematics teachers. Both researchers and policy authors (Carlson et al., 2002; Ellis, 2011; Johnson, 2015; Moore et al., 2014; National Governors Association Center for Best Practices, 2010; Thompson, 2013) have argued that covariational reasoning should underpin middle and secondary school mathematics (including precalculus and calculus). Our results raise questions about the extent that prospective teachers' ways of thinking support their capacity to heed this argument.

Closing Remarks

We close by noting that the aforementioned findings are, in retrospect, unsurprising given the traditional focus of U.S. mathematics curricula. For instance, U.S. mathematics curricula nearly exclusively limit the study of relationships to those relationships that are functions even if not explicitly defining such relationships as functions (e.g., the study of linear relationships in middle school). Such curricula afford students repeated opportunities to construct and re-construct ways of thinking compatible with those described here, possibly to the extent that these ways of thinking become habitual responses to graphing situations. For instance, if students only experience graphs such that the quantity represented along the horizontal axis is monotonically increasing, then there is little need for the student to maintain attention to variations in this quantity's value while attending to another quantity's value; students can merely focus on the latter quantity while assuming the other quantity's value is increasing. We conjecture that our findings have implications for curricular approaches to functions, relationships, and their graphs. Namely, students might benefit from opportunities to graph a wider range of relationships between covarying quantities.

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