Covariational and parametric reasoning

Teo Paoletti
Montclair State University

Kevin C. Moore
University of Georgia

Researchers have argued that students can develop foundational meanings for a variety of mathematics topics via quantitative and covariational reasoning. We extend this research by examining two students' reasoning that we conjectured created an intellectual need for parametric functions. We first describe our theoretical background including different conceptions of covariation researchers have found useful when analyzing students' activities constructing and representing relationships between covarying quantities. We then present two students' activities during a teaching experiment in which they constructed and reasoned about covarying quantities and highlight aspects of the students' reasoning that we conjecture created an intellectual need for parametric functions. We conclude with implications the students' activities and reasoning have for future research and curriculum design.

Key words: Covariational reasoning; Quantitative reasoning; Parametric Functions; Cognition

An increasing number of researchers have made contributions to the literature base on students' quantitative and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Jacobs, 2001; Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Johnson, 2012a; Thompson, 1994a, 1994b) with respect both to students' understandings of various content areas (e.g., function classes, rate of change, and the fundamental theorem of calculus) and to their enactment of important mental processes (e.g., generalizing, modeling, and problem solving). Although maintaining the common intention of understanding students' covariational reasoning, researchers' treatments of covariation are varied. For instance, Confrey and Smith (1994, 1995) approached covariation in terms of reasoning about discrete numerical values, finding patterns in these values, and interpolating patterns between them. In contrast, Thompson and Saldanha (Saldanha & Thompson, 1998; Thompson, 2011) approached covariation in terms of coordinating changes in two continuous magnitudes thus not constraining covariation to the availability of numerical data or specified values.

In this report, we detail results from a teaching experiment in which students conceived of simultaneously covarying quantities in ways compatible with Thompson's and Saldanha's descriptions of covariation. We focus on the students' actions during the closing sessions of the teaching experiment to discuss how the students represented relationships that constituted some situation or phenomena using projected magnitudes with an associated coordinate system, which Moore and Thompson described as *emergent shape thinking* (2015, in preparation). In characterizing the students' reasoning, we highlight the parametric nature of their reasoning including the extent that students were explicitly aware of the parametric nature of their reasoning. We close by highlighting aspects of the students' reasoning that may have supported the students in developing an intellectual need (Harel, 2007) for parametric relationships and functions.

Theoretical Background

Researchers who draw from interpretations of Piagetian and radical constructivist theories of knowing and learning have developed definitions and frameworks they have found useful when describing the mental processes and conceptual structures entailed in reasoning about relationships between quantities (Carlson et al., 2002; Johnson, 2012a, 2012b; Moore & Thompson, 2015, in preparation; Steffe, 1991; Thompson, 1994a, 2011; Weber, 2012). Of importance to this report, Carlson et al. (2002) presented a developmental framework that allows for a fine-grained analysis of students' covariational reasoning. The authors identified mental actions students engage in when coordinating covarying quantities including coordinating direction of change (quantity A *increases* as quantity B *increases*; MA2), amounts of change (the *change* in quantity A *decreases* as quantity B *increases in equal successive amounts*; MA3), and rates of change (quantity A *increases at a decreasing rate* with respect to quantity B; MA4-5).

Also of importance to this report, Saldanha and Thompson (1998) described the developmental nature of images of covariation, "In early development one coordinates two quantities' values to think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist" (p. 298). Extending this description, Thompson (2011) provided a first-order model of such an understanding in which an individual conceives of a quantity's value, x, varying over (conceptual) time, t. The individual could then conceive of covering the domain of t-values using intervals of size ε, and consider the variation of x in these intervals (i.e. considering x_ε as the set of x-values $(x(t), x(t + \varepsilon)) = x(t_\varepsilon)$). Thompson (2011) concluded his description, "I can now represent a conception of two quantities' values covarying as $(x_\varepsilon, y_\varepsilon)= (x(t_\varepsilon), y(t_\varepsilon))$. I intend the pair $(x_\varepsilon, y_\varepsilon)$ to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47). Apparent in both descriptions is the parametric nature of covariational reasoning; a student imagines two quantities varying with respect to (conceptual or experienced) time, eventually coordinating these two quantities with respect to each other to form a multiplicative unit.

Drawing on the parametric conceptions of covariation described by Thompson (2011) and Saldanha and Thompson (1998), researchers (Moore & Thompson, 2015, in preparation; Weber, 2012) have described *emergent shape thinking* as a student conceiving graphs in terms of an emergent trace constituted by covarying (projected) magnitudes. We use Figure 1 to represent instantiations of an emergent image of a graph representing the height and volume of liquid in a bottle covarying as liquid is poured into the bottle (i.e. $h_\varepsilon = h(t_\varepsilon)$ and $v_\varepsilon = v(t_\varepsilon)$ both increase as time, t_ε, increases). A student with such an image of a graph understands that the magnitude of the blue segment represents the height of liquid in the bottle and the magnitude of the red segment represents the volume of liquid in the bottle at a certain moment of (experiential or conceptual) time, and that the resulting trace is a product of tracking how these quantities covary with respect to (experiential or conceptual) time (i.e. understands the graph as representing $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$).

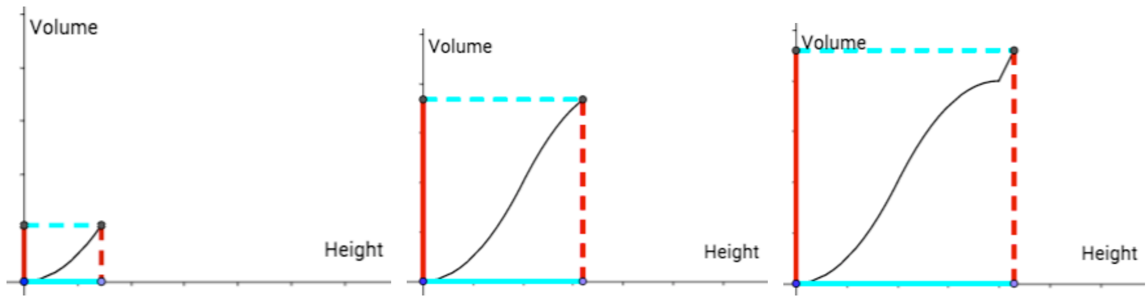


Figure 1: A representation of an emergent conception of covarying quantities.

Methods

We conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with two undergraduate students, Arya and Katlyn (pseudonyms), to explore the ways of reasoning students engage in during activities intended to emphasize reasoning about relationships quantitatively and covariationally (e.g., if the students engaged in emergent shape thinking, what ways of reasoning supported this?). The students were enrolled in a secondary education mathematics program at a large state institution in the southern U.S. Both were juniors (in credit hours taken) who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The teaching experiment consisted of three individual semi-structured task-based clinical interviews (per student) (Clement, 2000) and 15 paired teaching episodes (Steffe & Thompson, 2000). Each clinical interview and teaching episode lasted approximately 1.25 hours. We video and audio recorded the sessions and we captured and digitized records of the students' written work at the end of each episode.

When analyzing the data we conducted a conceptual analysis–"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 60)–to develop and refine models of the students' mathematics. With this goal in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Clement, 2000; Strauss & Corbin, 1998). Initially, we identified instances of Arya's and Katlyn's behaviors and actions that provided insights into each student's understandings. We used these instances to generate tentative models of the students' mathematics that we tested by searching for confirming or contradicting instances in their other activities. When evidence contradicted our constructed models, we made new hypotheses to explain the students' ways of operating and returned to prior data with these new hypotheses in mind for the purpose of modifying previous hypotheses or characterizing shifts in students' ways of operating.

Task Design

Throughout the teaching experiment, we provided Arya and Katlyn tasks that included prompts asking them to represent relationships between covarying quantities. We followed certain principles when designing these tasks. First, we designed tasks to include situations that would be familiar and accessible to the students, with most tasks including videos, applets, or images of phenomena (e.g., circular motion). Second, we avoided tasks that provided specific values for quantities to support the students in developing images of covariation that were magnitude based. Finally, we often asked students to construct multiple graphs related to a situation to explore if, and if so how, the students would leverage their images of the quantities and covariation between quantities when creating multiple graphs that may or may not differ in appearance.

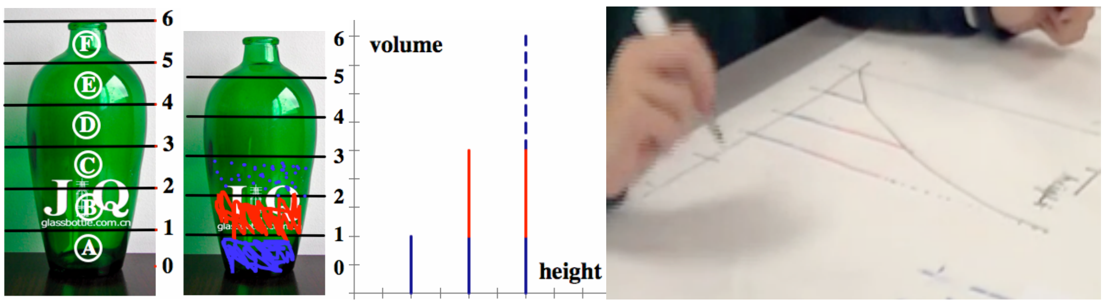
To illustrate, we used a variation of the *Bottle Problem*, which was designed by the Shell Centre (Swan & Shell Centre Team, 1985) and used by researchers investigating students' covariational reasoning (e.g., Carlson et al. (2002), Carlson et al. (2001), Johnson (2012, 2015)). We provided the students with a pictured bottle and asked them to imagine the experience of filling the bottle with liquid. We then asked them to graph the relationship between volume and height of liquid in the bottle as it filled with liquid. After they constructed a graph for a given bottle and a bottle for a given graph, we altered the prompt to ask the students to imagine liquid evaporating from the bottle. We then asked the students to represent the relationship between height and volume of liquid in the bottle for this new scenario.

Results

We first summarize the students' actions when creating graphs to represent how the height and volume of liquid covaried as a bottle filled with liquid. We then present their activities addressing liquid evaporating from the bottle in order to illustrate the students representing an additional aspect of the situation in their graph: the direction in which they imagined the graph being traced out. We conclude by highlighting the students' activities on a task that we implemented during the last clinical interview in which we explicitly asked the students to discuss a parametrically defined function for a graphed relationship.

Overview of students' activities addressing the Bottle Problem

As the teaching experiment progressed, the students exhibited activities indicative of reasoning about graphs as emergent traces representing two covarying quantities they conceived as constituting some situation (i.e. emergent shape thinking). For instance, during the first part of the Bottle Problem, each student coordinated how the volume and height of liquid in a bottle covaried in terms of direction of change (MA2) and amounts of change (MA3); each student conceived that the two quantities increase in tandem and then determined how the volume of liquid changes for equal successive increases in liquid height. Each student then created a graph while maintaining an explicit focus on how all drawn points and traces represented the relationship she conceived between the height and volume of liquid. As an example, consider Katlyn's activity as she created her graph (see Figure 2(c)). Describing why she was drawing the red segment longer than the blue segment, Katlyn stated, "'Cause this [*pointing to* (B) *in the picture of the bottle recreated in* Figure 2(a)] is so big compared to this [*pointing to* (A) *in the picture of the bottle*]." Katlyn then shaded in parts of her bottle (Figure 2(b)) corresponding to the segments in her graph, adding a dashed blue segment to represent the volume contained between tick 2 to tick 3 in her bottle (Figure 2(c)). Katlyn reasoned about the magnitudes of color-coordinated segments she constructed as representing amounts of volume within specific height intervals, understanding that each added segment corresponded to an amount of volume added to the total volume. Underlying this was Katlyn's understanding of the trace of her curve representing projected magnitudes as represented in Figure 1 (i.e. using Thompson's (2011) notation she conceived her graph as composed of the coordinate points $(h_\varepsilon, v_\varepsilon) = (h(t_\varepsilon), v(t_\varepsilon))$ with $h(t_\varepsilon)$ and $v(t_\varepsilon)$ representing height and volume as experiential or conceptual time, t_ε. elapses).

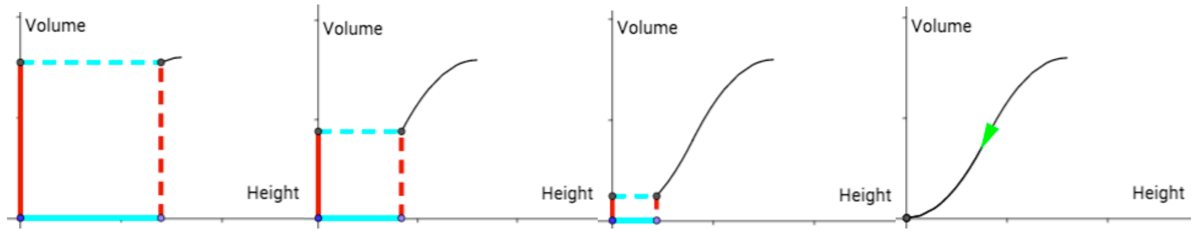


(a) (b) (c) (d)

Figure 2: (a) Katlyn's bottle (numbers and letters added for referencing), (b-c) Katlyn representing total volume with respect to height in the situation and a graph, and (d) Katlyn's resultant graph from tick 0 to tick 3.

Addressing water evaporating from the bottle

After the students had constructed a graph for the bottle in Figure 2(a), we asked them to graph the relationship between height and volume of liquid in the bottle as the liquid evaporated from the bottle. We asked them to complete the graph on the same whiteboard as a graph representing the relationship between height and volume of liquid in the bottle as the bottle filled. Indicating they did not anticipate that their previously completed graph might represent the posed relationship, the pair first drew a new set of axes. As they continued to consider the new scenario, Arya noted they should start at “full volume, full height.” Katlyn then added, “It’s going to look backwards… We can literally just travel this way instead [*motioning over the completed prior graph from the top-right most point back to the origin*]. [*To the interviewers*] We’re done, we’re just going to travel this way [*again motioning over the original curve from the top-right most point back to the origin*].” As the interaction continued, Katlyn’s actions suggested she now conceived the prior graph as $(h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $h(t_{2\varepsilon})$ and $v(t_{2\varepsilon})$ decreasing as experiential or conceptual time in this second situation, $t_{2\varepsilon}$, elapses (recreated in Figure 3(a)-(c)).



(a) (b) (c) (d)

Figure 3: (a)-(c) A recreation of the students’ graph as an emergent trace and (d) a recreation of their graph with the added arrow representing the direction of the trace.

To investigate if using the same curve for a new context created a perturbation for the students, we asked, “Is the situation the same? You’re ending up with the same graph.” Katlyn responded, “No, I just want to draw little arrows... we’re going this way now [*draws an arrow on the curve pointing towards the origin, recreated in* Figure 3(d)].” As she addressed the displayed graph representing two (experientially) different situations, Katlyn differentiated the two situations by adding an arrow to indicate the direction in which the graph is traced out with respect to the second situation; Katlyn parameterized her graph (from our perspective) with respect to (experiential or conceptual) time to differentiate how it is traced out with respect to how the previous graph is traced out. Adopting Thompson’s (2011) notation, Katlyn understood the displayed graph as composed of points (h, v) representing the appropriate magnitudes of height and volume of liquid in the bottle, regardless if liquid is entering or leaving the bottle. In the first scenario, she understood $(h, v) = (h_{1\varepsilon}, v_{1\varepsilon}) = (h(t_{1\varepsilon}), v(t_{1\varepsilon}))$ with $t_{1\varepsilon}$ representing (experiential or conceptual) time as liquid enters the bottle. In the second scenario she understood $(h, v) = (h_{2\varepsilon}, v_{2\varepsilon}) = (h(t_{2\varepsilon}), v(t_{2\varepsilon}))$ with $t_{2\varepsilon}$ representing (experiential or conceptual) time as liquid evaporates from the bottle.

The Car Problem

We conjectured that the students’ actions addressing the Bottle Problem had the potential to support them in becoming explicitly aware of the parametric nature of their reasoning as well as possibly bringing to the surface parametric functions. We intended to explore the extent that we could support the students in bringing this reasoning to the forefront as they addressed the *Car Problem* that Saldanha and Thompson (1998) designed to investigate students’ covariational reasoning. This task involves the students representing the relationship between an individual’s distances from two cities as the individual travels back-and-forth along a road (see Figure 4(a)). Because the relationship is such that neither distance is a function of the other distance, we conjectured raising the idea of function after the students constructed their graphs might support them in reasoning about an explicitly defined parametric function.

Both students initially described the directional variation of each distance (e.g., as Homer moves from the beginning of his trip, the distance from each city decreases) (MA2). As Arya attempted to represent this relationship in her graph, she drew a segment from right to left getting closer to the horizontal and vertical axis (indicated by (1) in Figure 4(b)). After Arya re-described the directional relationship she conceived in the situation, she moved to her graph and marked points on each axis to confirm her graphed segment represented that Homer's distance from each city was decreasing (indicated by (2) and (3) in Figure 4(b)). As in previous situations, Arya conceived her graph as an emergent trace representing two projected covarying magnitudes, indicated by her careful attention to the axes when drawing this segment. Further, and similar to the students' activities addressing the Bottle Problem, Arya added an arrow to her completed graph (Figure 4(c)) to represent an additional aspect of the situation: how the graph was traced as Homer traveled along the road.

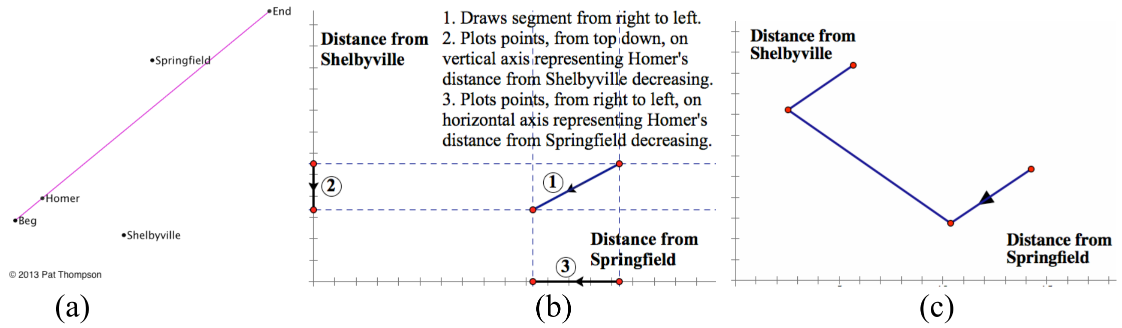


Figure 4: (a) The Car Problem applet, (b) a recreation of Arya's work, and (c) a recreation of Arya's final graph.

After Arya described that her graph did not represent distance from Springfield as a function of distance from Shelbyville or distance from Shelbyville as a function of distance from Springfield, and hoping to raise the idea of a parametrically defined function, a researcher asked, "What if your input was total distance traveled and your output was two-dimensional?" He then described the output as being composed of both the distance from Springfield and the distance from Shelbyville. Arya stated that this relationship represented a function as each total distance input corresponded to exactly one pair of distances.

Similarly, addressing whether the relationship with the same two-dimensional output but with 'distance on the path' as the input represented a function, Katlyn identified, "Well that's what [my graph] shows, right?" Katlyn stated that for any of Homer's distances on the path there was only one corresponding coordinate point on her graph, concluding that this relationship represented a function. Katlyn added, "I understand, like, what I've been drawing this whole time is like, how I'm traveling on like this purple path. But I don't, I never thought of that as my input, but it really is." Both students were able to assimilate a question concerning a one-dimensional input and two-dimensional output to consider a parametrically defined function after they had engaged in constructing the relationship via covariational reasoning and considered the graph as an emergent trace of this covariation.

Discussion

The students' activities here (and throughout the teaching experiment) provide examples of students who developed and maintained images of covariation we interpreted to be compatible with the descriptions of Thompson, Saldanha, and Moore. In addition, we conjecture the students' reasoning addressing the Bottle Problem raised an intellectual need for parametric functions, a need that we then capitalized on with the Car Problem. Harel

(2007) described, “The term *intellectual need* refers to a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate” (emphasis in original, p. 13).

When addressing water evaporating in the Bottle Problem, the students’ actions resulted in their encountering an intrinsic problem (i.e. experiencing an intellectual need). Specifically, the students came to understand one curve as corresponding to two different experiential situations, which resulted in them seeking to determine how to differentiate between the two situations while using one curve. We conjecture that this problem, which was supported by their thinking about graphs as emergent traces of covarying quantities, was critical to the students considering the parametric nature of the relationships they represented. That is, by understanding one curve as representing two different emergent traces, the students became explicitly aware of their thinking about the curve in terms of two related quantities *and* (experiential or conceptual) time.

When addressing the Car Problem, we interpreted the students’ initial activities to indicate their reasoning parametrically about the relationship between Homer’s distance from the two cities covarying as Homer’s total distance or ‘distance on the path’ varied. However, the students did not explicitly conceive their graph parametrically until we asked the students to consider a relationship with a one-dimensional input and two-dimensional output as representing a function. Addressing this question, the students brought to the surface a particular conception of the graph, a graph as an emergent trace of covarying quantities, in relation to “function” (i.e. the uniqueness of a mapping). Both students described such a parametrically defined relationship as representing a function with Katlyn explicitly addressing the novelty of this reasoning to her (e.g., “I never thought of that as my input, but it really is”).

In one of the few studies examining students’ understanding of parametric functions, and parameters more generally, Keene defined dynamic reasoning as “developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems” (2007, p. 231). The students’ reasoning was compatible with Keene’s (2007) definition of dynamic reasoning with their initial activities in each problem being compatible with Keene’s description of *implicitly* coordinating time with other quantities. Although the students engaged in reasoning that was parametric or dynamic in nature when responding to both tasks, the students did not exhibit activities to indicate they were *explicitly* aware of the parametric nature of their reasoning until they addressed later questions that we designed to focus in this area.

Unlike Keene (2007) and other researchers who have set out to examine students’ understandings of parameters and parametric function in differential equations or calculus settings (Stalvey & Vidakovic, 2015; Trigueros, 2004), in this study, we intended to examine students’ developing understandings of pre-calculus concepts through their quantitative and covariational reasoning; although this reasoning can be parametric in nature (e.g., emergent shape thinking) we did not expect to examine the students’ developing *parametric function* understandings. That fact that the students spontaneously engaged in reasoning that we interpreted as creating an intellectual need for parametric functions has both curricular and research implications. Future researchers and curriculum designers might examine how providing students with experiences in constructing graphs as emergent traces provide foundations for more explicit and formal introductions to parametric functions. For instance, and stemming from the current study ending before we could more extensively pursue the students’ reasoning about parametric relationships, researchers and educators should further explore how using different situations that result in students constructing and reasoning about the same displayed graph via different emergent traces has the potential to create an intellectual need for parametric relationships and functions.

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